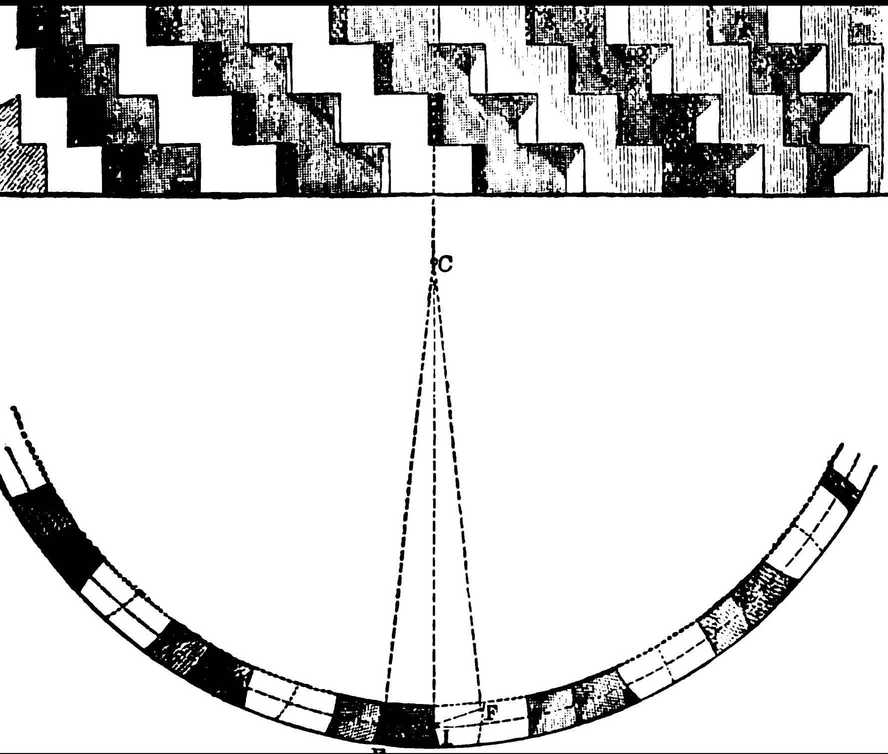

A manual of machinery and millwork

William John Macquorn Rankine,
Edward Fisher Bamber

A MANUAL

OF

MACHINERY AND MILLWORK.

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BY

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With Numerous Diagrams.

SECOND EDITION, THOROUGHLY REVISED,
BY
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PREFACE.

THIS book is divided into three parts: the first treats of the Geometry of Machinery; the second of the Dynamics of Machinery; and the third of the Materials, Strength, and Construction of Machinery.

Under the head of the Geometry of Machinery, machines are considered with reference to the comparative motions only of their moving parts; and rules are given for designing and arranging those parts so as to produce any given comparative motion.

Considering that the object of such rules is to adjust the dimensions of the parts of machines by processes of practical geometry, I have thought it advisable to solve every question by drawing, rather than by calculation, except in a few special cases where calculation is indispensable.

Many of the graphic rules thus obtained are made more easy and accurate, and some, indeed, are made possible which were not so before, by the aid of new methods of measuring and laying off the lengths of curved lines.

Two chapters of the first part are devoted to the detailed consideration of the movements of single pieces in machines. The remainder of the part relates to Pure Mechanism, as defined and reduced to a system by Professor WILLIS. The order in which the various combinations in mechanism are treated of is different from that adopted by him; but the principles are the same.

Several problems in mechanism are solved by methods which, so far as I know, have not hitherto been published; and which possess advantages in point of ease or of accuracy. I may specify, in particular, the drawing of rolling curves, and of some kinds of cams; the construction of the figures of teeth of skew-bevel wheels, and of threads of gearing screws, by the help of the normal section; and some improvements in the details of processes for designing intermittent gear, link-motions, and parallel motions.

Under the head of the Dynamics of Machinery are considered the forces exerted and the work done in machines; the means of measuring those quantities by indicators and dynamometers, of determining and balancing the reactions of moving masses in machines, and of regulating work and speed; and the efficiency, or proportion in which the useful work is less than the total work, in the different sorts of moving pieces, and in their various combinations.

Considering that a convenient single word is wanted to denote the proportion in which the total work in a machine is greater than the useful work, I have ventured to propose the word COUNTER-EFFICIENCY for that purpose.

Under the head of the Materials, Strength, and Construction of Machinery are considered, *first*, the properties of various materials, as affecting their treatment and use in the construction of machines; *secondly*, the general principles of the strength of materials; *thirdly*, the special application of those principles to questions relating to the strength and the construction of various parts of machines; and *fourthly*, the principles of the action of cutting tools.

Great care has been taken to ascertain the values of the factor of safety and of the working stress in successful examples of actual machinery; and some of the problems respecting the strength of special parts of machines have not been published previously except in scientific journals and in lectures.

Authorities for facts and information are cited where it is necessary to do so. The following works are so frequently referred to, that it may be desirable to mention them here specially:—

WILLIS *On Mechanism*, first edition, 1841; second edition, 1870.

FAIRBAIRN *On Millwork*.

HOLTZAPFFEL *On Mechanical Manipulation*.

BUCHANAN *On Millwork;* edited by TREDGOLD and G. RENNIE, with an *Essay on Tools* by NASMYTH.

W. J. M. R.

ADVERTISEMENT TO THE SECOND EDITION.

I have carefully examined this *Second* Edition, and the few errors which I have detected have been corrected.

E. F. B.

GLASGOW, *October*, 1873.

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ADDENDUM TO ARTICLE 262, PAGE 297.

Disengagements acting by Rolling Contact.—In fig. 213, the radii A D and B E of the two smooth wheels to be connected are equal; but those radii may, if required, be made unequal: the essential condition of the proper working of the combination being that the angle, A C B, made at the centre of the intermediate wheel by the two lines of centres, should be a little less than twice the complement of the angle of repose. (See page 298.)

A MANUAL OF MACHINERY AND MILLWORK.

INTRODUCTION

ART. 1. **Nature and Use of Machinery in General.**—The use of machinery is to transmit and modify motion and force. The parts of which it consists may be distinguished into two principal divisions,—the *Mechanism*, or moving parts; and the *Frame*, being the structure which supports the pieces of the mechanism, and to a certain extent determines the nature of their motions. In the action of a machine the following three things take place:—*First*, Some natural source of energy communicates motion and force to a part of the mechanism called the *Prime Mover; Secondly*, The motion and force are transmitted from the prime mover through the train of mechanism to the *working piece;* and during that transmission the motion and force are modified in amount and in direction, so as to be rendered suitable for the purpose to which they are to be applied; and, *Thirdly*, The working piece, by means of its motion, or of its motion and force combined, accomplishes some useful purpose.

2. **Distinction between the Geometry and the Dynamics of Machinery.**—The modification of motion in machinery depends on the figures and arrangement of the moving pieces, and the way in which they are connected with the frame and with each other; and almost all questions respecting it can be solved by the application of geometrical principles alone. The modification of force depends on the modification of motion; and those two phenomena always take place together; but in solving questions relating to the modification of force, the principles of dynamics have to be applied in addition to those of geometry. Hence, in treating of the art of designing machinery, arises a division into two departments,—the "*Geometry of Machinery*," or "*Science of Pure Mechanism*" (to use a term introduced by Professor Willis), which shows how the figure, arrangement, and mode of connection of the pieces of a machine are to be adapted to the modification of motion which they are to

produce; and the "*Dynamics of Machinery*," which shows what modifications of force accompany given modifications of motion, and what modifications of motion are required in order to produce given modifications of force.

3. **Strength of Machinery.**—In order that a machine may be fit for use, every part, both of the machinery and of the framework, must be capable of bearing the utmost straining action which can be exerted upon it during the working of the machine, without any risk of being broken or overstrained; and the dimensions required for that purpose are to be determined by the proper application of the principles of the strength of materials.

4. The **Art of the Construction of Machinery** consists of three departments,—the selecting and obtaining of suitable materials for the parts of the mechanism and framework; the shaping of those parts to the proper figures and dimensions by means of suitable tools; and the fitting-up of the machine, by putting its parts together.

5. **Division of the Subject.**—For the reasons explained in the preceding Articles, the subjects of this work are treated of under four principal heads,—Geometry of Machinery, or Pure Mechanism; Dynamics of Machinery; Materials, Construction, and Strength of Machinery.

PART I.

GEOMETRY OF MACHINERY.

CHAPTER I.

ELEMENTARY RULES IN DESCRIPTIVE GEOMETRY.

SECTION I.—*General Explanations—Projection of Points and Lines.*

6. **Descriptive Geometry** is the art of representing solid figures upon a plane surface. In the present chapter are given some general elementary rules in that art, whose application is of very frequent occurrence in designing mechanism. The more special and complex rules will be given in the ensuing chapters, in treating of the particular kinds of mechanism to which those rules belong.*

7. By the **Projection of a Point** upon a given plane is meant the foot of a perpendicular let fall from the point on the plane. For example, in fig. 1, X Z Z X represents a plane (called a plane of projection), A a point, and A B a perpendicular let fall from the point on the plane; the foot, B, of that perpendicular is the projection of the point A on the plane X Z Z X.

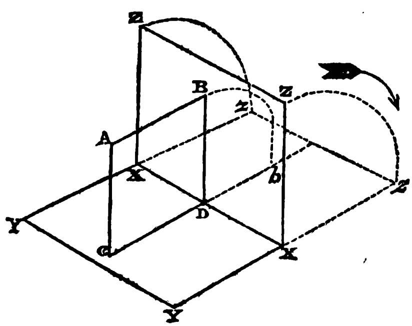

Fig. 1.

8. The **Position of a Point** is completely determined when its projections upon two planes not parallel to each other are known. In descriptive geometry a pair of planes of projection at right angles to each other are used; and in general one of these is vertical and the other horizontal. Thus, in fig. 1,

* For complete information on the subject of descriptive geometry, reference may be made to the works of Monge and Hachette in French, and of Dr. Woolley in English.

X Z Z X is the vertical plane of projection, and X Y Y X the horizontal plane of projection; B is the vertical projection, and C the horizontal projection of the point A; and those two projections completely determine the position of the point A; for no other point can have the same pair of projections.

9. The **Axis of Projection** is the line X X, in which the two planes of projection cut each other.

10. **Rabatment.**—When the two projections of an object are shown in one drawing, it is convenient to represent to the mind that the following process has been performed:—Suppose that the vertical plane of projection is hinged to the horizontal plane at the axis X X, and that after the projection of the object on the vertical plane has been made, that plane is turned about that axis until it lies flat in the position X *z z* X, so as to be continuous with the horizontal plane: thus bringing down the projection B to *b*. This process is called the *rabatment* of the vertical plane upon the horizontal plane (to use a term borrowed from the French "*rabattement*" by Dr. Woolley). The two points C and *b* are in one straight line perpendicular to X X. The process of rabatment may be conceived also to be performed upon a plane in any position when a figure contained in that plane is shown in its true dimensions on one of the planes of projection.

11. **Projections of Lines.**—The projection of a line is a line containing the projections of all the points of the projected line. The projection of a straight line perpendicular to the plane of projection is a point; for example, the projection on the vertical plane, X Z Z X (fig. 1), of the straight line A B, perpendicular to that plane, is the point B. The projection of a straight line in any other position relatively to the plane of projection is a straight line. If the projected line is parallel to the plane of projection, its projection is parallel and equal to the projected line itself; thus the projection on the horizontal plane, X Y Y X, of the horizontal straight line A B, is the parallel and equal line C D. If the projected line is oblique to the plane of projection, the projection is shorter than the original line.

The projections, on the same plane, of parallel and equal straight lines are parallel and equal. The projections, on the same plane, of parallel lines bearing given proportions to each other are parallel lines bearing the same proportions to each other. When the plane of a plane curved line is perpendicular to a plane of projection, the projection of the curve on this plane is a straight line, being the intersection of the plane of the curve with the plane of projection. When the plane of the projected curve is parallel to a plane of projection, the projection of the curve on this plane is similar and equal to the original curve. In all other cases, it follows from the preservation of the proportions of a set of parallel ordinates amongst their projections, that the projections of a plane curve of a given algebraical order are curves of the same algebraical order. The projections of a circle are ellipses; the projections of a parabola of a given order are parabolas of the same order. The projections of a straight tangent to a plane curve are straight tangents to the projections of that curve. The projections of a point of contrary flexure in a plane curve are points of contrary flexure in its projections.

12. **Drawings of a Machine.**—A third plane of projection, perpendicular to the first two, is often employed, not as being mathematically necessary, but as being more convenient for the representation of certain lines. Thus, for example, the drawings of a machine usually consist of three projections on three planes at right angles to each other; one horizontal (*the plan*), and the other two vertical (*the elevations*). Any two of those projections are mathematically sufficient to show the whole dimensions and figure of the machine; and from any two the third can be constructed; but it is convenient, for purposes of measurement, calculation, and construction, to have the whole three projections.

In the application of the rules about to be stated in the sequel of this Section, the two planes of projection may be held to represent any two of the three views of a machine; and the axis of projection will then have the directions stated in the following table:—

Views Represented by the Planes of Projection.	Direction of the Axis of Projection.
Longitudinal Elevation and Plan,............	Longitudinal.
Longitudinal and Transverse Elevations,....	Vertical.
Plan and Transverse Elevation,...............	Transverse.

Projections of figures upon planes oblique to the principal planes of projection may be used for special purposes.

SECTION II.—*Traces of Lines and Surfaces.*

13. By a **Trace** is meant the intersection of a line with a surface, or of one surface with another. The trace of a line upon a surface is a point; the trace of one surface upon another is a line.

In descriptive geometry the term *traces* is specially employed, when not otherwise specified, to denote the intersections of a line or surface with the planes of projection.

14. **Traces of a Straight Line.**—The position of a *straight line* is completely determined when its traces are known. For example, the straight line A C, in fig. 2, has its position completely determined by its traces, A and C, being the points where it cuts the

two planes of projection. The *rabatment* of the trace C is represented by *c*.

A straight line parallel to one of the planes of projection has only one trace, being the point where it cuts the other plane of projection.

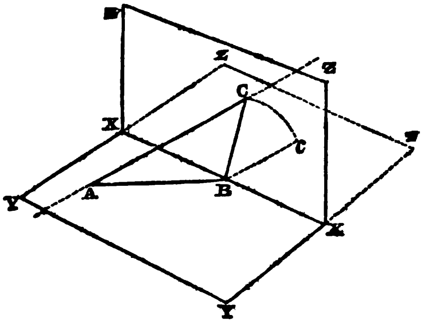

Fig. 2.

A straight line parallel to the axis of projection has no traces.

15. The **Traces of a Plane** are straight lines which (unless they are both parallel to the axis of projection) meet that axis in one point. The position of a plane is completely determined when its traces are known. For example, the plane A B C, in fig. 2, has its position completely determined by its traces, B A and B C.

A plane perpendicular to one of the planes of projection has its trace on the other plane of projection perpendicular to the axis of projection. A plane perpendicular to both planes of projection has for its traces two lines perpendicular to the axis. Thus, in fig. 1, page 3, the traces of the plane A B C D are D C and D B, both perpendicular to X X.

A plane parallel to one of the planes of projection has a trace on the other plane of projection only, being a straight line parallel to X X.

If a plane traverses a straight line, the traces of the plane traverse the traces of the line.

Section III.—*Rules Relating to Straight Lines.*

16. **General Explanations.**—In each of the figures illustrating the following rules the axis of projection is represented by X X; and in general the part of the figure above that line represents the rabatment of the vertical plane of projection, and the part below, the horizontal plane of projection. The projections of points on the horizontal plane are in general marked with capital letters, and the projections on the vertical plane with small letters.

17. **Given** (in fig. 3), **the Traces**, A, *b*, **of a Straight Line, to Draw its Projections.**—From A and *b* let fall A *a* and *b* B perpendicular to X X. Then *a* will be the vertical projection of the

trace A, and B the horizontal projection of the trace *b*. Join *a b*, A B; these will be the projections required.

(It may here be remarked, that *a* A and *a b* are the traces of a plane traversing the given line, and perpendicular to the vertical plane of projection; and that B A and B *b* are the traces of a plane traversing the given line, and perpendicular to the horizontal plane of projection.)

18. **Given** (in fig. 3), **the Projections**, A B, *a b*, **of a Straight Line, to find its Traces.**—From *a* and B, where the given projections meet the axis, draw *a* A and B *b* perpendicular to X X, cutting the given projections in A and *b* respectively. These points will be the required traces.

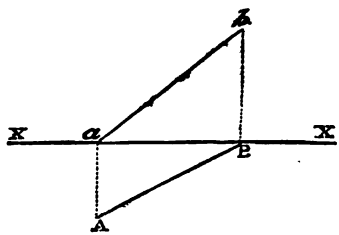

Fig. 3.

19. **Given, the Projections of two Points**, A, *a*, B, *b* (fig. 4), **to Measure the Distance between them.**—Join *a b*, A B; these will be the projections of the straight line to be measured. Through either end of either of those projections (as *b*) draw *d b e* parallel to X X; through the other end, *a*, of the same projection, draw *a d* perpendicular to X X, cutting *d b e* in *d*; make *d e* = the other projection, A B; join *a e*; this will be the length required.

The same operation may be performed on the other plane of projection.

20. **Given** (in fig. 4), **the Projections**, A, *a*, **of a Point, and the Projections**, A B, *a b*, **of a Straight Line through that Point, to Lay off a given Distance from the Point along the Line.** — In any convenient position, draw a straight line, B *b*, perpendicular to X X, meeting the projections of the given straight line in two points, B, *b*, which are the projections of one point; then perform the construction described in Article 19, so as to find *a e*. From the point *a*, in the line *a e*, lay off the given distance, *a f*. Through *f* draw *f h* parallel to X X, cutting *a b* in *g*; *a g* will be one of the projections of the given distance. Then draw *g* G perpen-

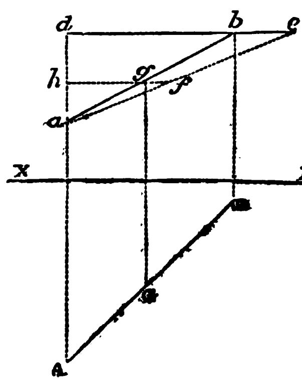

Fig. 4.

dicular to X X, cutting A B in G; A G will be the other projection of the given distance.

Another method of finding G is to lay off A G = *h f*.

21. **Given** (in fig. 4), **the Projections,** *a b*, A B, **of a Straight Line, to Find the Angle which it makes with One of the Planes of Projection** (for example, the horizontal plane).—Perform the construction described in Article 19; then *d e a* is the angle made by the given line with the horizontal plane. The same construction performed in the horizontal plane of projection will give the angle made by the given line with the vertical plane of projection.

22. **Given** (in fig. 5), **the Projections,** *a b* **and** A B, *a c* **and** A C, **of a Pair of Straight Lines which Intersect each other in the Point whose Projections are** *a*, A, **to find the Angle between these Lines.**—In either of the planes of projection (for example, the vertical

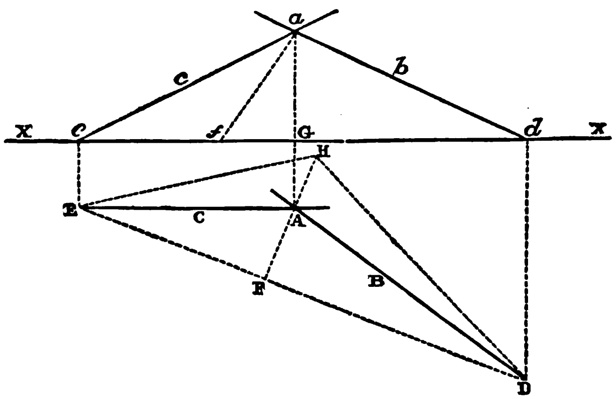

Fig. 5.

plane) find the points, *d*, *e*, where the projections of the given line cut the axis X X; these will be also the vertical projections of the horizontal traces of the lines. Through *e* and *d* draw *e* E, *d* D, perpendicular to X X, cutting A C and A B in E and D respectively; these points will be the horizontal traces of the lines. Join D E (which will be the horizontal trace of the plane containing the lines), and on it let fall the perpendicular F A. Join A *a* (which of course is perpendicular to X X); let it cut X X in G. Make G *f* = A F, and join *a f*. In F A produced, take F H = *a f*; join H E, H D; E H D will be the angle required.

Remark.—The triangle E H D is the *rabatment* upon the horizontal plane of the triangle whose projections are E A D and *e a d*.

22 A. **Given** (in fig. 5), **the Projections,** *a b* **and** A B, **of a Straight Line, and One Trace** (say D E) **of a Plane Traversing that Line, to Find the Projections of a Straight Line which shall, at a given Point,** *a*, A, **make a given Angle in the given Plane with the given Straight Line.**—Join A *a*, which will be perpendicular to X X. On D E let fall the perpendicular A F. In X X take G *f* = A F; join *a f*. In F A produced take F H = *a f*. Join H D; and draw H E, making D H E = the given angle, and cutting D E in E. From E let fall E *e* perpendicular to X X; join A E, *a e*; these will be the projections of the line required.

Section IV.—*Rules Relating to Planes.*

23. **Given, the Projections of Three Points, to draw the Traces of a Plane Passing through them.**—Draw straight lines from one of the points to the two others; find, by Article 18, the traces of those straight lines; through those traces, on the two planes of projection respectively, draw two straight lines; these will be the traces required.

23 A. **Given, the Projections of Two Points and of a Straight Line, to Draw the Traces of a Plane Traversing the Points and Parallel to the Line.**—Through the projections of either of the given points draw straight lines parallel respectively to the corresponding projections of the given line; these will be the projections of a straight line through the given point, parallel to the given straight line; then, by Article 23, find the traces of a plane traversing the new straight line and the other given point.

24. **Given** (in fig. 6), **the Traces of a Plane,** B A, B C, **to Find the Angle which it makes with one of the Planes of Projection** (for example, the vertical plane). —From any convenient point, A, in the horizontal trace let fall A D perpendicular to X X. From D let fall D *e* perpendicular to B C. In D B lay off D *f* = D *e*. Join *f* A (this will represent the perpendicular distance from B C of the point whose projections are D and A). A *f* D will be the angle required.

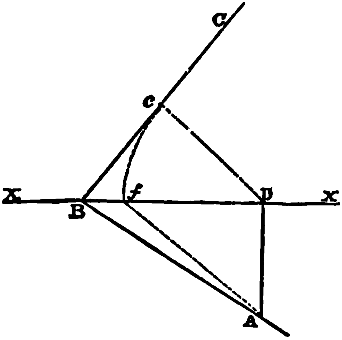

Fig. 6.

25. **Given** (in fig. 7), **the Traces of a Plane,** B A, B C, **to Find the Angle which it makes with the Axis of Projection,** X X.—In either of the two traces (for

example, B A) take any convenient point, A, from which let fall A D perpendicular to X X, and on B D as a diameter describe a circle. From D let fall perpendiculars, D *e*, D F, on the two given traces. From the point *e*, thus found on the opposite trace to that on which the point A was assumed, let fall *e* E perpendicular to X X; join E A, cutting D F in G. From G draw G H perpendicular to X X, cutting the circle in H; D B H will be the required angle.

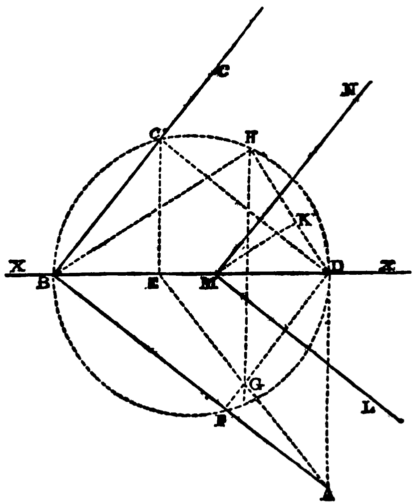

Fig. 7.

26. **Given (in fig. 7), the Traces of a Plane, B A, B C, to Draw the Traces of another Plane which shall be Parallel to the given Plane, and at a given Perpendicular Distance from it in either Direction.**—Complete the construction described in Article 25. Join D H (this represents the perpendicular distance of the point D in the axis from the given plane); then from H, along H D (or along D H produced, according to the direction in which the new plane is to lie), lay off the given perpendicular distance between the planes, H K. From K draw K M parallel to H B, cutting X X in M. From M draw M N parallel to B C, and M L parallel to B A; these will be the traces of the plane required.

Or otherwise:—Complete the construction described in Article 24 (see fig. 8). A *f* is the rabatment of the intersection of the given plane with a plane, A D *e*, perpendicular to the vertical trace B C. Through A draw A M perpendicular to A *f*, and make A M equal to the given distance between the planes; draw M N parallel to A *f*, cutting X X in N. In D *e* produced take D O equal to D N. O is a point in the trace of the plane required. Through O draw O P parallel to B C, cutting X X in P; and through P draw P Q parallel to B A. O P Q is the plane required.

27. **Given (in fig. 9), the Traces of Two Planes, C A *d* and C B *d*, to Draw the Projections of their Line of Intersection.**—The traces of

the required line are C and *d*, where the traces of the given planes intersect. From those points respectively let fall C *c* and

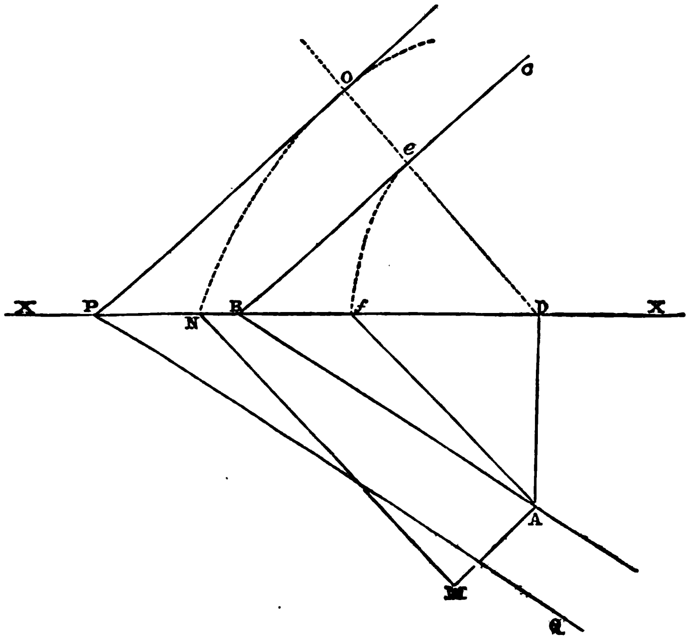

Fig. 8.

d D perpendicular to X X; join C D, *c d*; these will be the projections required.

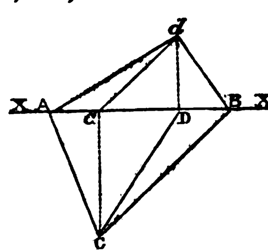

Fig. 9.

28. **To Find the Projections of the Point where a Straight Line Intersects a Plane** (the traces of the line and of the plane being given), it is only necessary to draw the traces of two planes traversing the given line in convenient directions, and find the projections of the lines in which those two planes cut the given plane; the intersections of those projections will be the projections of the point required.

29. **Given** (in fig. 10), **the Traces of Two Planes**, C A *d*, C B *d*,

to Find the Angle between them.—From either of the intersections of the traces (say *d*) let fall *d* D perpendicular to X X; draw D C, joining D with the other intersection of the traces. Through any convenient point, I, in D C, draw G I H perpendicular to D C,

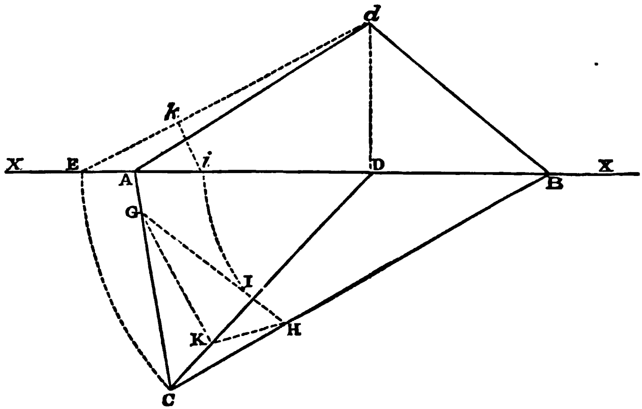

Fig. 10.

cutting A C in G and B C in H. Along X X lay off D E = D C, and D *i* = D I; join *d* E (this will be the length of the line of intersection of the planes). From *i* let fall *i k* perpendicular to *d* E; in I C take I K = *i k*; join K G, K H; G K H will be the angle required.

When the traces of the two given planes are inconveniently placed for the completion of the figure, we may substitute for either pair of traces another pair of traces parallel to them, and more conveniently placed.

30. **Given** (in fig. 10), **the Traces,** A *d* **and** A C, **of a Plane; also the Traces,** *d* **and** C, **of a Straight Line in that Plane; to Draw the Traces of a Plane which shall Cut the given Plane in that Line at a given Angle.**—From either of the traces of the straight line, as *d*, let fall *d* D perpendicular to X X; draw the straight line D C, joining D with the other trace, C, of the straight line. Through any convenient point, I, in D C, draw I G perpendicular to D C, cutting C A in G. In X X lay off D E = D C and D *i* = D I; join *d* E, and on it let fall the perpendicular *i k*. In I C take I K = *i k*; join K G. Then draw K H, making G K H = the given angle, and cutting G I, produced if necessary, in H. Draw C H, cutting X X in B, and join B *d*; these will be the traces of the plane required.

31. **Given** (in fig. 11), **the Traces of a Plane,** A B C, **and the Projections of a Point,** G, *g*, **to Draw the Traces of a Plane Traversing the given Point, and Parallel to the given Plane.**—Through either of the projections of the given point (say G) draw G H parallel to the corresponding trace of the given plane, and cutting X X in H. (This will be one of the projections of a line through the given point, parallel to the trace A B of the given plane.)

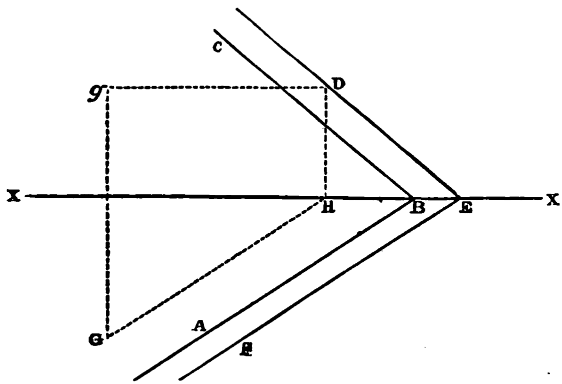

Fig. 11.

Through H draw H D perpendicular to X X; and through *g* draw *g* D parallel to X X, cutting H D in D (*g* D will be the projection and D one of the traces of the line before mentioned). Through D draw D E parallel to C B, cutting X X in E; and through E draw E F parallel to B A; D E F will be the traces of the required plane.

32. **Given, the Traces of a Plane,** E F, E D (in fig. 11), **and One Projection of a Point in that Plane, to Find the other Projection of that Point.**—Suppose *g*, the vertical projection of the point, to be given. Draw *g* D parallel to X X, cutting E D in D. From D let fall D H perpendicular to X X. From *g* draw *g* G perpendicular to X X, and from H draw H G parallel to E F; the intersection of those lines, G, will be the required horizontal projection of the given point.

33. **Given** (in fig. 12), **the Traces,** A B C, **of a Plane, and the Projections,** D, *d*, **of a Point, to Draw the Projections of a Perpendicular let Fall from the Point on the Plane.**—From one of the projections of the given point (say D) draw D E F perpendicular to the corresponding trace, B A, of the given plane, and cutting B A in E, and X X in F. From E let fall E *e* perpendicular to

X X; from F draw F *f* perpendicular to X X, cutting the trace B C in *f*; join *f e*; from *d* draw *d g* perpendicular to B C, cutting *f e* in *g*; and from *g* draw *g* G perpendicular to X X, cutting D F in G. D G and *d g* will be the projections of the perpendicular required.

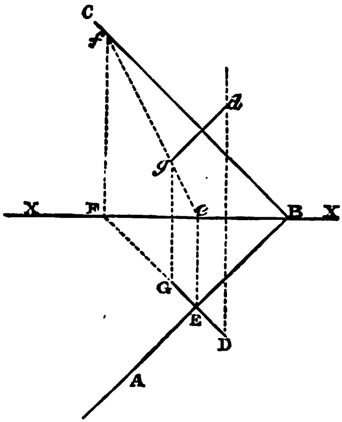

Fig. 12.

34. **Given** (in fig. 13), **the Projections of a Point,** D, *d*, **and those of a Straight Line,** A B, *a b*, **to Draw the Traces of a Plane which shall Traverse the Point, and be Perpendicular to the Line.**—Through one of the projections of the given point (say D) draw D G perpendicular to A B (the corresponding projection of the given line), cutting X X in G. Through G draw G *g* perpendicular to X X; through *d*, the other projection of the point, draw *d g* parallel to X X, cutting G *g* in *g*; through *g* draw E C perpendicular to *a b*, cutting X X in C; and through C draw C F perpendicular to A B. E C F will be the traces of the required plane.

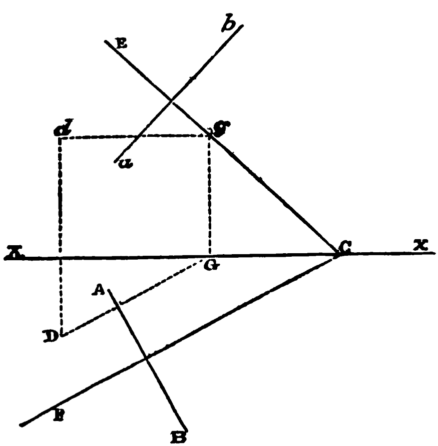

Fig. 13.

35. **Given, the Projections of a Point and of a Straight Line, to Draw the Projections of a Perpendicular let Fall from the Point upon the Straight Line.**—Find by the preceding rule the traces of a plane traversing the given point, and perpendicular to the given line; then, by Article 23, find the traces of a plane traversing the given point and line; and finally, by Article 27, find the projection of the line of intersection of those two planes.

36. **Given, the Projections of Two Straight Lines that are neither**

Parallel nor Intersecting, to Find the Projections of their Common Perpendicular.—By Article 23 A, find the traces of a plane traversing one of the lines and parallel to the other. Then, by Article 33, find the projections of a perpendicular let fall on that plane from any convenient point in the second line. Then through the projections of the foot of that perpendicular draw the projections of a straight line parallel to the second straight line; these will cut the projections of the first straight line at one end of the common perpendicular, whose projections will be parallel and equal to those of the perpendicular already found.

36 A. **Projections of a Circle.**—When an instrument which draws ellipses *accurately* is at hand, it may be used for the purpose of drawing the projections of a circle of a given radius, described about a given point in a given plane, and may thus facilitate much the solution of various problems. The following is the process for obtaining the projections of a circle:—

Given (in fig. 14), **the Traces of a Plane,** A B C, **and the Projections of a Point in that Plane,** D, *d*, **to Draw the Projections of a Circle of a given Radius, described in the given Plane and about the given Point.**—For the vertical projection, describe about *d* a circle of the given radius, *d f* = *d e*, and draw the diameter *e f* parallel to the trace C B; *e f* will itself be the vertical projection of one diameter of the circle. Draw *d g* perpendicular to *e f*. Find, by Article 24, the angle which the given plane makes with the vertical plane of projection, and lay off *g d h* equal to the angle so found. From *h*, in the circle, draw *h k* parallel to *f e*, and cutting *d g* in *k*; then *d k* will be the vertical projection of a radius of the circle perpendicular to *e f*. Then on the major axis, *e f*, and minor semi-axis, *d k*, describe an ellipse; that ellipse will be the required vertical projection of the circle.

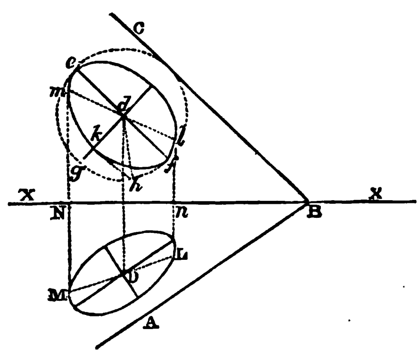

Fig. 14.

The horizontal projection is obtained by a precisely similar process, the rule of Article 24 being now used to find the angle which the given plane makes with the horizontal plane of projection.

The two ellipses are both touched by a pair of tangents, M *m*, L *l*, perpendicular to X X; and the diameters, *l m*, L M, are the projections of one diameter of the circle—viz., that diameter in which the plane A B C is cut at right angles by a plane parallel to X X. The perpendicular distance, N *n*, between the two tangents is equal to the diameter of the circle multiplied by the cosine of the angle which the given plane makes with X X, and is bisected by the line D *d*.

CHAPTER II.

OF THE MOTIONS OF PRIMARY MOVING PIECES IN MACHINES.

SECTION I.—*General Explanations.*

37. **Frame; Moving Pieces, Primary and Secondary.** (*A. M.*, 427.) —The *frame* of a machine is a structure which supports the *moving pieces*, and regulates the path or kind of motion of most of them directly. In considering the movements of machines mathematically, the frame is considered as fixed, and the motions of the moving pieces are referred to it. The frame itself may have (as in the case of a ship or of a locomotive engine) a motion relatively to the earth, and in that case the motions of the moving pieces relatively to the earth are the resultants of their motions relatively to the frame, and of the motion of the frame relatively to the earth; but in all problems of pure mechanism, and in many problems of the dynamics of machinery, the motion of the frame relatively to the earth does not require to be considered.

The *moving pieces* may be distinguished into *primary* and *secondary;* the former being those which are directly carried by the frame, and have their motion wholly guided by their connection with the frame; and the latter, those which are carried by other moving pieces, or which have their motion not wholly guided by their connection with the frame. For example, the crank-shaft and the piston-rod of a steam engine are primary moving pieces; the wheels of a locomotive are primary moving pieces; the connecting-rod of a steam engine is a secondary moving piece

Connectors are those secondary moving pieces, such as links, belts, cords, and chains, which transmit motion from one moving piece to another, when that transmission is not effected by immediate contact.

38. **Bearings** (*A. M.*, 428,) are the surfaces of contact of primary moving pieces with the frame, and of secondary moving pieces with the pieces which carry them. Bearings guide the motions of the pieces which they support, and their figures depend on the nature of those motions. The bearings of a piece which has a motion of translation in a straight line must have plane or cylindrical *

* The word "cylindrical" is here used in the comprehensive sense, which denotes any surface generated by the motion of a straight line parallel to itself.

surfaces, *exactly straight* in the direction of motion. The bearings of rotating pieces must have surfaces accurately turned to *figures of revolution*, such as circular cylinders, spheres, cones, conoids, and flat discs. The bearing of a piece whose motion is helical, must be an exact screw. Those parts of moving pieces which touch the bearings should have surfaces accurately fitting those of the bearings. They may be distinguished into *slides*, for pieces which move in straight lines, *gudgeons, journals, bushes*, and *pivots*, for those which rotate, and *screws* for those which move helically.

The accurate formation and fitting of bearing surfaces is of primary importance to the correct and efficient working of machines.

39. **The Motions of Primary Moving Pieces** (*A. M.*, 429,) **are Limited** by the fact, that in order that different portions of a pair of bearing surfaces may accurately fit each other during their relative motion, those surfaces must be either straight, circular, or helical; from which it follows, that the motions in question can be of three kinds only, viz.:—

I. *Straight translation*, or *shifting*, which is necessarily of limited extent, and which, if the motion of the machine is of indefinite duration, must be *reciprocating;* that is to say, must take place alternately in opposite directions: for example, the piston-rod of a steam engine.

II. *Simple rotation*, or *turning* about a fixed axis, which motion may be either continuous or reciprocating, being called in the latter case *swinging, rocking*, or *oscillation*. Continuous rotation is exemplified by the shaft of a steam engine; reciprocating rotation by various beams or levers.

III. *Helical* or *screw-like motion*, compounded of rotation about a fixed axis, and translation along that axis.

Section II.—*Straight Motion of Primary Pieces.*

40. **Straight Translation** is the motion of a primary piece sliding along a straight guiding surface. All the particles of the piece move through equal distances in a given time, along parallel straight lines; and the line joining any two particles remains unaltered in length and in direction.

41. **Resolution and Composition of Motions.**—The *resultant* of two or more *component* motions is the motion which results from putting them together. If the component motions are represented by straight lines, their resultant is found geometrically by joining together, end to end, a series of straight lines respectively equal and parallel to the given straight lines, and pointing in the same directions, and then drawing a straight line from the starting point to the further end of the series. For example:—

I. (See fig. 15.) *To find the resultant of two component motions,* A B *and* A C. Let the paper represent the plane of those motions. From B draw B D parallel and equal to A C, and pointing in the same direction; join A D; this will be the required resultant motion; or, in other words, complete the parallelogram, A B D C; its diagonal, A D, will be the required resultant.

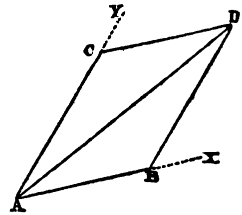

Fig. 15.

A motion may, if required, be *resolved into components.* The following are the cases most useful in mechanism:—

II. (Fig. 15.) *To resolve a given motion,* A D, *into components in two given directions in the same plane,* A X *and* A Y. Through D draw D C parallel to X A, cutting A Y in C, and D B parallel to Y A, cutting A X in B; A B and A C will be the required components.

III. (Fig. 16.) *To resolve a given motion,* A D, *into one component parallel and another component perpendicular to a given direction.* Through A, parallel to the given direction, draw A X, upon which let fall the perpendicular D B; then A B will be the first of the required components, and A C parallel and equal to B D will be the second.

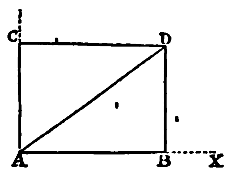

Fig. 16.

IV. *Given, the traces of a plane* (Article 15, page 6) *and the projections of a straight line representing a motion* (Article 11, page 4), *to find the projections of two component motions, one perpendicular and the other parallel to the plane.* By the rule of Article 31, page 13, draw the traces of a second plane parallel to the given plane, and traversing the point which represents one end of the given motion. Then by the rule of Article 33, page 13, find the projections of the perpendicular let fall on the second plane from the point representing the other end of the given motion. That perpendicular will be one of the required components; and the straight line from the first-mentioned point to the foot of the perpendicular will be the other. The lengths of the lines representing the component motions may be found, if required, by Article 19, page 7.

The component of the motion parallel to the given plane is obviously its projection on that plane. It is sometimes called the *tangential component,* and the component perpendicular to the given plane the *normal component* of the given motion.

V. *Given, a resultant motion and one of two component motions, to find the other component motion.* Combine the given resultant motion with a motion equal and opposite to the given component motion; the resultant of these two will be the required other component motion. For example, in fig. 15, let A D be the given resultant motion, and A B the given component; draw D C equal and parallel to A B, and pointing the opposite way; join A C; this will be the required other component: or otherwise, join B D and draw A C equal and parallel to it.

VI. (Fig. 17.) *Given, the vertical projection,* A B, *and the horizontal projection,* A′ B′, *of a straight line representing a motion, to resolve that motion into three rectangular components parallel and perpendicular to the planes of projection.* Let O X be the axis of projection (Article 9, page 4). Draw the straight lines A A′, B B′,

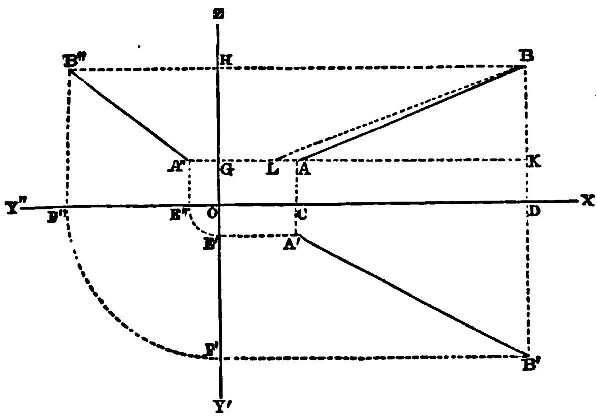

Fig. 17.

cutting the axis of projection (of course at right angles) in C and D. Then through any convenient point, O, in the axis of projection, draw the straight line Z O Y′ at right angles to that axis; and take O Y′ to represent a transverse horizontal axis, and O Z to represent a vertical axis. (The point O is called the *origin.*) Then parallel to X O draw A′ E′ and B′ F′ to meet O Y′, and A G and B H to meet O Z. The three components required will be represented by C D, E′ F′, and G H.

VII. *Given* (in fig. 17), *the vertical projection,* A B, *and the horizontal projection,* A′ B′, *of a straight line representing a motion, to draw a third projection of the same straight line on a vertical transverse plane of projection perpendicular to the first two planes of projection.* Construct fig. 17 as described in the preceding Rule. O Z and O Y′ will be the traces of the third plane of projection. Produce X O towards Y″, then O Y″ will represent the rabatment of O Y′, and Z O Y″ the rabatment of the vertical transverse plane upon the vertical longitudinal plane of projection. In O Y″ take O E″ = O E′, and O F″ = O F′; draw E″ A″ and F″B″ parallel to O Z, to meet A G and B H produced in A″ and B″ respectively; join A″ B″; this will be the projection required.

According to the rule already stated in Article 19, page 7, the motion of which A B and A′ B′ are the projections is to be found by making K L = A′ B′, and joining L B, which line will represent the extent of the resultant motion.

The following are the relations between a resultant motion and its components as expressed by calculation. In fig. 15,—

$$\sin C A B : \sin C A D : \sin D A B :: A D : A B : A C;$$

also, $$A D^2 = A B^2 + A C^2 + 2 A B \cdot A C \cdot \cos C A B.$$

In fig. 16,

$$A B = A D \cdot \cos B A D; \; A C = A D \cdot \sin B A D;$$

$$A D^2 = A B^2 + A C^2.$$

In fig. 17,

$$L B^2 = C D^2 + E' F'^2 + G H^2.$$

42. **Relative Motion of Two Moving Pieces.**—All motion is relative: that is to say, every conceivable motion consists in a change of the relative position of two or more points. In speaking of the motions of the moving pieces of machines, *motions relatively to the frame* are always to be understood, unless it is otherwise specified. It is often requisite, however, to express the motion of a point in a moving piece relatively to a point in the same or in another moving piece.

In the case considered in the present section, where the relative position of two points in the same moving piece remains unaltered, not only as to distance but as to direction, the relative motion of such a pair of points is *nothing*. The motion of one moving piece relatively to another is determined by the following principle:—Let P, Q, and R denote any three points; then the motion of R relatively to P is the resultant of the motion of R relatively to Q, combined with the motion of Q relatively to P; so that if the motions of Q relatively to P, and of R relatively to P are given, the motion of R relatively to Q is to be found according to Rule V. of the preceding Article, by compounding with the motion of R relatively to P a motion equal and opposite to that of Q relatively to P. For example, let P stand for the frame of a machine, and Q and R for two moving pieces which slide along straight guides;

and in a given interval of time let A B, in fig. 15, page 19, represent the motion of Q relatively to P, and A D the motion of R relatively to P; then A C, found by Rule V. of Article 41, will represent the motion of R relatively to Q.

In all cases whatsoever of relative motion of two bodies, the motion of one relatively to the other is exactly equal and contrary to that of the second relatively to the first. For example, let P and Q be two points; and when P is treated as fixed, let Q move through a given distance in a given direction relatively to P; then if Q is treated as fixed, P moves through the same distance in the contrary direction relatively to Q.

43. **Comparative Motion** (*A. M.*, 358,) is the relation borne to each other by the simultaneous motions of two points, either in the same body or in different bodies, relatively to one and the same fixed point or body. It consists of two elements: the *velocity-ratio*, which is the proportion borne to each other by the distances moved through by the two points in the same interval of time; and the *directional relation*, which is the relation between the directions in which the two points are moving at the same instant.

In the case of two points in a primary piece whose motion is one of translation, the velocity-ratio is that of *equality*, and the directional relation that of *identity;* for all points in such a piece are moving with equal speed in parallel directions at the same instant.

When two points in two different pieces are compared, the comparison may give a different result. For example, let P, as before, stand for the frame of a machine, and Q and R for two moving pieces; and while Q performs relatively to P the motion represented by A B (fig. 15, page 19), let R perform relatively to P the motion represented by A D. Then the *comparative motion* of R and Q consists of the following elements:—

$$\text{the velocity-ratio, } \frac{\text{A D}}{\text{A B}};$$

and the directional relation, represented by the angle B A D.

In most of the cases which occur in mechanism the motion of each point is limited to two directions—forward or backward—in a fixed path; so that the directional relation of two points may often be sufficiently expressed by prefixing the sign + or − to their velocity-ratio, according as their motions are similar or contrary; that is, the sign + denotes that those motions are both forward or both backward; and the sign − that one is forward and the other backward.

We may compare together the different components of the motion of one point, and the resultant motion. For example, in figs. 15 and 16, page 19, the velocity-ratios of two component motions, as compared with their resultant, are expressed by

$$\frac{AB}{AD} \text{ and } \frac{AC}{AD};$$

and in fig. 17, page 20, the velocity-ratios of three rectangular component motions, as compared with their resultant, are expressed by

$$\frac{CD}{LB}, \frac{EF}{LB}, \text{ and } \frac{GH}{LB}.$$

Strictly speaking, the principles of the geometry of machines, or of pure mechanism, are concerned with comparative motions only, and not with absolute velocities: or, in other words, those principles relate to the motions which different moving points perform in the course of the same interval of time, but not to the length of the interval of time in which such motions are performed. For example, in the case of a direct-acting steam engine, the principles of pure mechanism show that the piston makes one double stroke for each revolution of the crank; that the directional relation of the piston and crank-pin varies periodically, the piston moving to and fro, while the crank-pin moves continuously round in a circle; and that in particular positions of those pieces their velocity-ratio takes particular values; but the question of what interval of time is occupied by a revolution, or of how many revolutions are performed in a minute, belongs not to the geometry, but to the dynamics of machines. Further, in the case of a pair of spur wheels gearing into each other, the principles of pure mechanism show that in any given interval of time the numbers of revolutions performed by those wheels respectively are inversely as their numbers of teeth, and that the directions in which they turn are contrary; but those principles do not inform us how many revolutions either wheel makes in a minute.

44. **Driving Point and Working Point.**—The term *driving point* is used to denote that point, either in a whole machine or in a given moving piece of a machine, where the force is applied that causes the motion; and the term *working point* is used to denote the point where the useful work is done. These explanations contain references to the dynamics of machines; but it is to be understood that in the geometry of machines, or pure mechanism, it is the *comparative motion* only of the driving point and working point that is taken into consideration. It is to be observed, too, that the word "*point*" is here taken in an extended meaning; for the exertion of force or communication of motion at a mathematical point, of no sensible magnitude, is purely ideal; and when the word *point* is used with reference to the driving or the work of machines,

it is to be held to mean the *place* where the action that drives or that resists a machine is exerted, of what magnitude soever that place may be, whether a surface or a volume. Thus, the driving point in a steam engine comprehends the whole surface of the piston that is pressed upon by the steam which drives the engine; and the working point, where friction is overcome, comprehends the whole of the rubbing surface, and where a heavy body is lifted, the whole volume of that body. Nevertheless, for the sake of convenience in mathematical investigation, such places of the action of driving or resisting forces are often treated on the supposition that they may be represented by single points; for when such points are properly chosen, no error is incurred by making that supposition.

SECTION III.—*Rotation of Primary Pieces.*

45. **Rotation of a Primary Piece.** (*A. M.*, 370-372.)—*Rotation* or *Turning* is the motion of a rigid body when lines in it change their directions; and it is the only kind of motion involving change of the relative positions of the particles of a body that is possible consistently with rigidity; that is to say, with the maintenance of the distance between every pair of particles in the body unchanged. An *axis* of *rotation* is a line in a rigid body whose direction is unchanged by the rotation; and a *fixed axis of rotation* is a line whose position, as well as its direction, is unchanged by the rotation. Every line in a rotating body which is parallel to the axis has its direction unchanged by the rotation. The rotation of a primary piece in a machine always takes place about an axis that is fixed relatively to the frame of the machine; that axis being the geometrical axis, or centre line, of a bearing surface (such as that of the journals or gudgeons of a shaft), whose form is that either of a circular cylinder or of some other surface of revolution The *plane of rotation* is any plane perpendicular to the axis. Every such plane in a rotating body has its position unchanged by the rotation; and straight lines in such a plane—that is, straight lines perpendicular to the axis of rotation—change their directions more rapidly than any other straight lines in the same body.

46. **Speed of Rotation.** (*A. M.*, 373.)—Although in the case of rotation, as well as in that of translation, the principles of pure mechanism are concerned with comparative velocities only, still it is desirable here to state, that the speed with which a rotating body turns is expressed in two different ways. For most practical purposes it is usually stated in turns and fractions of a turn in some convenient unit of time; such as a second, or (more commonly) a minute. For scientific purposes, and for some practical purposes also, it is expressed in *angular velocity;* which means, the angle swept through in a second by a line perpendicular to the axis of

rotation: that angle being stated in *circular measure;* which means the ratio of the length of the arc subtended by an angle to the radius of that arc. The following are examples of the values of angles in circular measure:—

One degree,.................	0·0174533 nearly;
A right angle, or quarter revolution,................	1·5708 nearly;
Two right angles, or half a revolution, ...	3·1416 nearly $= \frac{355}{113}$ very nearly.
Four right angles, or a revolution,................	6·2832 nearly $= \frac{710}{113}$ very nearly.

Hence, *to convert turns per second into angular velocity,* multiply by $\frac{710}{113} = 6{\cdot}2832$ nearly; and *to convert angular velocity into turns per second,* multiply by $\frac{113}{710} = 0{\cdot}159155$ nearly. The *time of revolution* in seconds is the reciprocal of the speed expressed in turns per second. The *comparative speed* or *angular velocity-ratio* of two rotating pieces is independent of the kind of unit in which their absolute speeds may be expressed; it is the reciprocal of the ratio of their times of revolution.

47. **Rotation is Common to all Parts of the Turning Body.** (*A. M.*, 375.)—Since the angular motion of rotation consists in the change of direction of a line in a plane of rotation, and since that change of direction is the same how short soever the line may be, it is evident that the condition of rotation, like that of translation, is common to every particle, how small soever, of the turning rigid body, and that the angular velocity of turning of each particle, how small soever, is the same with that of the entire body. This is otherwise evident, by considering that each part into which a rigid body can be divided turns completely about in the same time with every other part, and with the entire body, and makes the same number of turns in a second, or a minute, or any other interval of time.

48. **Right and Left-Handed Rotation.** (*A. M.*, 376.)—The *direction* of rotation round a given axis is distinguished in an arbitrary manner into *right-handed* and *left-handed.* One end of the axis is chosen as that from which an observer is supposed to look along the direction of the axis towards the rotating body. Then if the body seems to the observer to turn in the same direction in which the sun seems to revolve to an observer north of the tropics, the rotation is said to be *right-handed;* if in the contrary direction, *left-handed;* and it is usual to consider the angular velocity of right-handed rotation to be positive, and that of left-handed rota

tion to be negative; but this is a matter of convenience. It is obvious that the same rotation which seems right-handed when looked at from one end of the axis, seems left-handed when looked at from the other end. In fig 18, the arrow R represents right-handed rotation, and the arrow L left-handed rotation. When a body *oscillates* about an axis its rotation is alternately right-handed and left-handed.

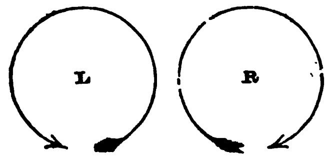

Fig. 18.

49. **Translation of a Point in a Rotating Piece.** (*A. M.*, 377.)—Each point in a rotating piece (except those situated in the axis) has a motion of *revolution*—that is, *translation in a circular path*, round the axis of rotation; and the velocity of that translation is the product of the perpendicular distance of the point from the axis—that is, the radius of the circular path, into the angular velocity of rotation (Article 46, page 24). Thus, in fig. 19, let the

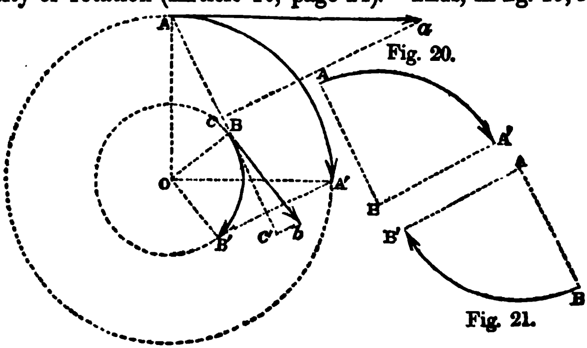

Fig. 20.

Fig. 21.

Fig. 19.

surface of the paper represent a plane of rotation; let O be at once the trace and the projection of the axis of rotation on that plane, and A the projection of a point in the rotating piece under consideration. Then the motion of that point (and of its projection A), takes place in a circle of the radius O A; and if A A′ be the arc described in a second, then

$$\text{A A}' = \text{O A} \times \text{angular velocity};$$

$$= \text{O A} \times \frac{710}{113} \times \text{number of turns per second};$$

also,

$$\text{angular velocity} = \frac{\text{A A}'}{\text{O A}}.$$

The *velocity at a given instant* of a point which moves in a curve, as distinguished from the *arc traced in a second by that point*, is represented by a straight line equal in length to that arc, and pointing in the direction in which the point is moving at the given instant; that is to say, being a tangent to the path of the point at that instant. Therefore, to represent by a straight line the velocity of the point now in question at the instant when its projection is at A, draw A *a* perpendicular to O A, and equal in length to A A′ (= O A × angular velocity).

50. **Motion of a Part of a Rotating Piece.**—When what has just been explained is considered together with the statement in Article 47, it is easily seen, that if the centre of any part of a moving piece rotating about a fixed axis is situated in that axis, then the motion of that part is simply a rotation similar and equal to that of the whole piece; but if the centre of the part is situated at a distance, O A, from that axis, the motion of that part consists of a *translation of its centre*, with the velocity O A × the angular velocity, in a circle described about the fixed axis with the radius O A, *combined with a rotation* similar and equal to that of the whole piece, about a *moving axis* traversing A, and parallel to the fixed axis which traverses O. Consider, for example, that rotating piece in a steam engine which consists of the shaft, crank, and crank-pin, and which turns about the axis of the shaft, as a fixed axis of rotation, to which the axis of the crank-pin is parallel. Then the motion of the shaft consists simply in a rotation about its own axis; while the motion of the crank-pin consists in a translation of its centre, and of each point in its axis, in a circular path described about the axis of the shaft, combined with a rotation about its own moving axis similar and equal to that of the shaft. As an additional illustration, suppose one end of a cord to be held still, and the other to be attached to a hook which is fixed at the centre of a rotating wheel, and which therefore rotates along with and as part of the wheel. The cord will undergo one twist for each turn of the wheel. Now let the hook be removed from the centre, and fixed at any point in an arm of the wheel, or in its rim; the cord will still undergo one twist, neither more nor less, for each turn of the wheel; thus showing, as before, the effect of the rotation of the hook along with the wheel; and the only difference in the motion will be that the end of the cord attached to the hook will be carried round in a circle, at the same time that the whole cord is twisted. A *secondary* piece in a machine may be so contrived as to have translation in a circle or some other curved path without rotation; this will be considered in a later chapter.

51. **Rules as to Lengths of Circular Arcs.**—In connection with the motion of points in rotating pieces, and with various other

questions in mechanism, there is frequent occasion to measure the lengths of circular arcs, and to lay off circular arcs of given lengths. These processes may be performed by the help of calculation, and of the well-known approximate values of the ratio which the radius and the circumference of a circle bear to each other, viz.:—

$$\frac{\text{circumference}}{\text{radius}} = \frac{710}{113} \text{ nearly} = 6{\cdot}283185 \text{ nearly};$$

$$\frac{\text{radius}}{\text{circumference}} = \frac{113}{710} \text{ nearly} = 0{\cdot}159155 \text{ nearly};$$

but it is often much more convenient in practice to proceed by drawing; and then the following rules are the most accurate yet known:—*

I. (Fig. 22.) *To draw a straight line approximately equal to a given circular arc,* A B. Draw the straight chord B A; produce A to C, making A C = $\frac{1}{2}$ B A; about C, with the radius C B = $\frac{3}{2}$ B A, draw a circle; then draw the straight line A D, touching the given arc in A, and meeting the last-mentioned circle in D; A D will be the straight line required.

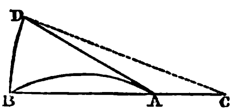

Fig. 22.

The *error* of this rule consists in the straight line being a little shorter than the arc: in fractions of the length of the arc, it is about $\frac{1}{1000}$ for an arc equal in length to its own radius; and it varies as the fourth power of the angle subtended by the arc; so that it may be diminished to any required extent by subdividing the arc to be measured by means of bisections. For example, in drawing a straight line approximately equal to an arc subtending 60°, the error is about $\frac{1}{900}$ of the length of the arc; divide the arc into two arcs, each subtending 30°; draw a straight line approximately equal to one of these, and double it; the error will be reduced to *one-sixteenth* of its former amount; that is, to about $\frac{1}{14400}$ of the length of the arc. The greatest angular extent of the arcs to which the rule is applied should be limited in each case according to the degree of precision required in the drawing.

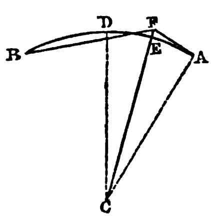

Fig. 23.

II. (Fig. 23.) *To draw a straight line approximately equal to a given circular arc,* A B. (Another Method.) Let C be the centre of the arc. Bisect the arc A B in D, and the arc A D in E; draw the straight

* These rules are extracted from Papers read to the British Association in 1867, and published in the *Philosophical Magazine* for September and October of that year.

secant C E F, and the straight tangent A F, meeting each other in F; draw the straight line F B; then a straight line of the length A F + F B will be approximately equal in length to the arc A B.

The error of this rule, in fractions of the length of the arc, is just one-fourth of the error of Rule I., but in the contrary direction; and it varies as the fourth power of the angle subtended by the arc.

III. *To lay off upon a given circle an arc approximately equal in length to a given straight line.* In fig. 24, let A D be part of the circumference of the given circle, A one end of the required arc, and A B a straight line of the given length, drawn so as to touch the circle at the point A. In A B take A C = $\frac{1}{4}$ A B, and about C, with the radius C B = $\frac{3}{4}$ A B, draw a circular arc B D, meeting the given circle in D. A D will be the arc required.

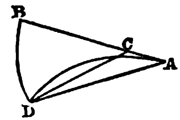

Fig. 24.

The error of this rule, in fractions of the given length, is the same as that of Rule I., and follows the same law.

IV. (Fig. 24.) *To draw a circular arc which shall be approximately equal in length to the straight line* A B, *shall with one of its ends touch that straight line at* A, *and shall subtend a given angle.* In A B take A C = $\frac{1}{4}$ A B; and about C, with the radius C B = $\frac{3}{4}$ A B, draw a circle, B D. Draw the straight line A D, making the angle B A D = one-half of the given angle, and meeting the circle B D in D. Then D will be the other end of the required arc, which may be drawn by well-known rules.

The error of this rule, in fractions of the given length, is the same with that of Rules I. and III., and follows the same law.

V. *To divide a circular arc, approximately, into any required number of equal parts.* By Rule I. or II., draw a straight line approximately equal in length to the given arc; divide that straight line into the required number of equal parts, and then lay off upon the given arc, by Rule III., an arc approximately equal in length to one of the parts of the straight line.

Rule V. becomes unnecessary when the number of parts is 2, 4, 8, or any other power of 2; for then the required division can be performed *exactly* by plane geometry.

VI. *To divide the whole circumference of a circle approximately into any required number of equal arcs.* When the required number of equal arcs is any one of the following numbers, the division can be made exactly by plane geometry, and the present rule is not needed:—any power of 2; 3; 3 × any power of 2; 5; 5 × any power of 2; 15; 15 × any power of 2.* In other

* It may be convenient here to state the methods of subdividing arcs and whole circles by plane geometry. (1.) *To bisect any circular arc.* On the chord of the arc as a base, construct any convenient isosceles triangle,

cases proceed as follows:—Divide the circumference exactly, by plane geometry, into such a number of equal arcs as may be required in order to give sufficient precision to the approximative part of the process. Let the number of equal arcs in that preliminary division be called n. Divide one of them, by means of Rule V., into the required number of equal parts; n times one of those parts will be one of the required equal arcs into which the whole circumference is to be divided.

Rules I., III., and V., are applicable to arcs of other curves besides the circle, provided the changes of curvature in such arcs are small and gradual.

52. **Relative Translation of a Pair of Points in a Rotating Piece.**— In fig. 19, page 26 (where O, as already explained, is at once the projection and the trace of a fixed axis of rotation on a plane perpendicular to it, and A the projection of a point in the rotating piece), let B be the projection of another point in the rotating piece, and A B the projection of the straight line connecting those two points. The point B describes a circle of the radius O B about the fixed axis; and the radii O A and O B sweep round with the angular velocity common to all parts of the rotating piece, so that by the time that A has moved to the position A′, B has moved to the position B′, such that the angles A O A′ and B O B′ are equal. In order to determine the motion of one of those moving points (as A) relatively to the other (as B), it is to be considered that, owing to the rigidity of the body, the length of A B is invariable, and that the change of direction of that line (as projected on the plane of rotation), consists in turning in a given time through an angle equal to that through which the whole piece turns. In fig. 20, take B to represent at once the trace and the projection, on a plane of rotation, of an axis parallel to the fixed axis, and traversing the point B. Draw B A in fig. 20 parallel and equal to B A in fig. 19; and B A′ in fig. 20 parallel and equal to B′ A′ in fig. 19. Then A and A′ in fig. 20 represent two successive positions of A

with the summit pointing away from the centre of the arc; a straight line from the centre of the arc to that summit will bisect the arc. (2.) *To mark the sixth part of the circumference of a circle.* Lay off a chord equal to the radius. (3.) *To mark the tenth part of the circumference of a circle.* In fig. 24 A, draw the straight line A B = the radius of the circle; and perpendicular to A B, draw B C = ½ A B. Join A C, and from it cut off C D = C B. A D will be the chord of one-tenth part of the circumference of the circle. (4.) *For the fifteenth part,* take the difference between one-sixth and one-tenth. It may be added, that Gauss discovered a method of dividing the circumference of a circle by geometry exactly, when the number of equal parts is any prime number that is equal to 1 + a power of 2; such as $1 + 2^4 = 17$; $1 + 2^8 = 257$, &c.; but the method is too laborious for use in designing mechanism.

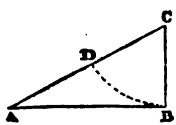

Fig. 24 A.

relatively to the axis traversing B, at the beginning and end respectively of the interval of time in which the rotating piece turns through the angle A O A′ (fig. 19) = A B A′ (fig. 20). The translation of A relatively to this new axis consists in revolution in a circle of the radius B A, in the same direction with the rotation (that is, in the present example, right-handed); and the velocity of that relative translation is B A × the angular velocity of rotation. Fig. 21 shows how, by a similar construction, the motion of B relatively to an axis traversing A is represented. Take A in fig. 21 to represent at once the trace and the projection, on a plane of rotation, of an axis parallel to the fixed axis, and traversing A. Draw A B and A B′ in fig. 21 parallel and equal respectively to A B and A′ B′ in fig. 19. Then B and B′ in fig. 21 represent two successive positions of B relatively to the axis traversing A, at the beginning and end respectively of the interval of time in which the rotating piece turns through the angle A O A′ (fig. 19) = B A B′ (fig. 21); the translation of B relatively to this new axis consists in revolution in a circle of the radius A B, in the same direction with the rotation (that is, in the present example, right-handed); and the velocity of that relative translation is A B × the angular velocity, and is at each instant equal, parallel, and contrary to the velocity of translation of A relatively to B, agreeably to the general principle stated at the end of Article 42, page 21.

53. **Comparative Motion of Points in a Rotating Piece.**—In fig. 19, page 26, as before, let A and B be the projections at a given instant, on a plane of rotation, of two points whose motions are to be compared. The directions of motion of those points at that instant are represented by the straight lines A *a*, B *b*, tangents to the circles in which the points revolve about the axis O; and the *directional relation* of the points is expressed by the fact, that the angle between those directions of motion is equal to the angle A O B, between the perpendiculars let fall from the two points on the axis O; or, in other words, the angle between the planes traversing that axis and the two points respectively; of which planes O A and O B are the traces upon the plane of rotation; for the directions of motion, A *a*, B *b*, are respectively perpendicular to those two planes.

The *velocity-ratio* of the two points is equal to the ratio O B : O A borne to each other by the radii of their circular paths. In other words, if A *a* = A A′ be taken, as before, to represent the velocity of A, and B *b* = B B′ to represent the velocity of B, then

$$O A : O B :: A a : B b;$$

and if the velocities of any number of points in a rotating piece are compared together, they are all proportional respectively to the perpendicular distances of those points from the axis of rotation.

It is obvious that all points in a circular cylindrical surface described about the axis of rotation have equal velocities. The dotted circles in fig. 19, page 26, represent the traces of two such surfaces.

The relative motions of any two pairs of points in a rotating piece may be compared together. For example, let it be proposed to compare the motion of A relatively to B with the motion of B relatively to O. Then, because the velocity of the motion of A relatively to B is proportional to B A, and its direction perpendicular to the plane whose trace is B A, while the velocity of the motion of B relatively to O is proportional to O B, and its direction perpendicular to the plane of which O B is the trace, the directional relation is expressed by the angle made by those planes with each other, and the velocity-ratio by the ratio B A : O B borne to each other by the projections on the plane of rotation of the two lines of connection of the two pairs of points.

54. **Relative and Comparative Translation of a Pair of Rigidly Connected Points.**—The following proposition is applicable to all motions whatsoever of a pair of points so connected that the distance between them is invariable. It forms the basis of nearly the whole theory of combinations in mechanism, and many of its consequences will be explained in the ensuing chapters of this Part. At present it is introduced with a view to its application to pairs of points in a rotating piece.

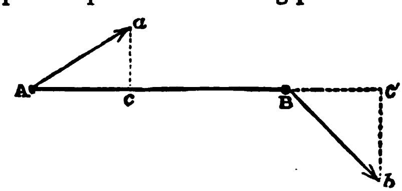

Fig. 25.

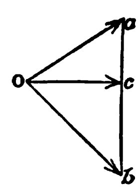

Fig. 26.

THEOREM.—*If two points are so connected that their distance apart is invariable, the components of their velocities along the straight line which traverses them both must be equal;* for if those component velocities are unequal, the distance between the points must necessarily change.

The straight line which traverses the points is called their *Line of Connection.*

For example, in fig. 25, let A and B represent two points in the plane of the paper, whose distance apart, A B, is invariable. At a given instant let the velocities of those points be represented by straight lines, which may be in the same plane, or in different planes, according to circumstances; and let A *a* and B *b* be the projections of those lines. From *a* and *b* let fall *a c* and *b c′* perpendicular to the line of connection, A B; these will be the traces of two planes perpendicular to the line of connection, and traversing respectively the points of which *a* and *b* are the projections; the parts A *c* and B *c′*, cut off by those planes from the line of connection (produced where necessary), will be the components along that line of the velocities of A and B respectively; and those components must necessarily be equal—that is, B *c′* = A *c*. The component velocities transverse to the line of connection are represented by the lines whose projections are *c a* and *c′ b*, and may bear to each other any proportion whatsoever.

The same principle is illustrated in fig. 19, page 26. In that figure A *a* and B *b* represent the velocities of two points, A and B, whose line of connection is A B, and is of invariable length; *a c* and *b c′* are perpendiculars let fall from *a* and *b* upon A B, produced where necessary; and A *c* and B *c′* represent the component velocities of A and B along the line of connection, which are equal to each other.

RULE.—*Given (in fig. 25), a pair of rigidly connected points,* A *and* B, *and the* directions *of the projections* A a *and* B b *upon a plane traversing* A B, *of their velocities at a given instant, to find the ratio of those projections or component velocities to each other.* In fig. 26, draw O *c* of any convenient length parallel to A B, and *a c b* perpendicular to it; through O draw O *a* in fig. 26 parallel to A *a* in fig. 25, and O *b* in fig. 26 parallel to B *b* in fig. 25; then the required ratio is

$$\frac{\mathrm{B}\,b}{\mathrm{A}\,a} = \frac{\mathrm{O}\,b}{\mathrm{O}\,a}.$$

55. **Components of Velocity of a Point in a Rotating Piece—Periodical Motion.** (*A. M.*, 380.)—The component parallel to an axis of rotation, of the velocity of a point in a rotating body relatively to that axis, is nothing. That velocity may be resolved into rectangular components parallel to the plane of rotation. Thus let O in fig. 27 represent the projection and trace of the axis of rotation of a body whose plane of rotation is that of the figure; and let A be the projection of a point in the body, the radius of whose circular path is O A. The velocity of that point being = O A × angular velocity, let it be represented by the straight line A V perpendicular to O A. Let B A be any direction in the plane of rotation parallel to which it is desired to find the component of the velocity of A. From V

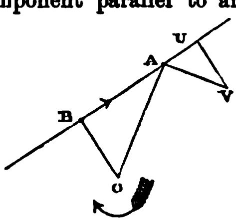

Fig. 27.

let fall V U perpendicular to B A; then A U represents the component in question. Sometimes the more convenient way of finding that component is the following:—

From O let fall O B perpendicular to B A. Then A and B represent a pair of rigidly connected points; therefore, according to Article 24, the component velocities of A and B along A B are equal. But B A, being perpendicular to O B, is the direction of the whole velocity of B; therefore *the component, along a given straight line in the plane of rotation, of the velocity of any point whose projection is in that line, is equal to the whole velocity of the point where a perpendicular from the axis meets that line.*

The whole velocity of B is = O B × the angular velocity; and the velocity-ratio of B to A, or, in other words, the ratio of the component velocity of A along B A to the whole-velocity of A, is O B : O A.

The velocity of a point such as A in a rotating piece may be resolved into components, oblique (see fig. 19) or rectangular (see fig. 27) as the case may be, by regarding the velocity of A relatively to O as the resultant of the velocity of A relatively to B, and of that of B relatively to O. The directions of that resultant velocity and its two components are respectively perpendicular to O A, B A, and O B, and their ratios to each other are equal to those of the lengths of the same three lines. This is a particular case of a more general proposition, viz.,—that *the velocities of three points relatively to each other are proportional to the three sides of a triangle which make with each other the same angles that the directions of those three relative velocities do* (*A. M.*, 355).

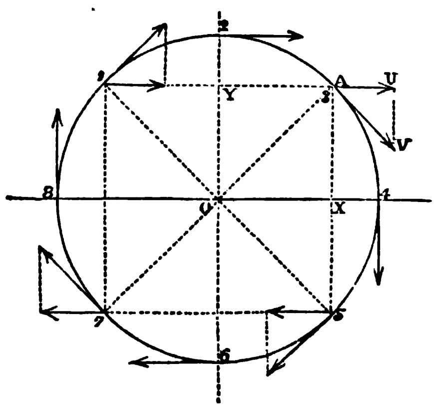

Fig. 28.

In fig. 28, let O be the trace of the axis on a plane of rotation, and A a point in the rotating piece, revolving in the circle O A, so as to assume successively a series of positions such as 1, 2, 3, 4, 5, 6, 7, 8; and in each position of A, let the component velocity A U, parallel to a fixed plane whose trace is the diameter 8 O 4, be compared with the whole velocity of revolution, A V.

Let 2 O 6 be a diameter perpendicular to 8 O 4; and through A draw A Y parallel to 4 O 8 and A X parallel to 2 O 6. Then, according to the principles already explained, the value of the velocity ratio in question is,

$$\frac{\mathrm{A\,U}}{\mathrm{A\,V}} = \frac{\mathrm{X\,A}}{\mathrm{O\,A}} = \frac{\mathrm{O\,Y}}{\mathrm{O\,A}};$$

and it is evident that this ratio is equal to *nothing* when A is at the points 4 and 8, and to *unity* when A is at the points 2 and 6. Further, if the velocity of revolution be considered as always positive, and if the component velocity A U be considered as positive when from left to right, and negative when from right to left, the ratio $\frac{\mathrm{A\,U}}{\mathrm{A\,V}}$ is,

in the quadrant 812, positive and increasing;
in the quadrant 234, positive and diminishing;
in the quadrant 456, negative and increasing;
in the quadrant 678, negative and diminishing.

It thus undergoes a series of *periodical* variations. All this is expressed in symbols by the formula,

$$\frac{\mathrm{A\,U}}{\mathrm{A\,V}} = \sin \theta;$$

where θ denotes the angle that the radius O A makes at any instant with the radius O 8.

Other and more complex ways of resolving the motions of points in rotating bodies into components will be considered in the next chapter.

56. **Comparative Motion of Two Rotating Pieces, and of Points in them.**—In comparing together the rotations of two rotating pieces without reference to the translations of points in them, their comparative speed is expressed (as already stated in Article 46) by the *angular-velocity ratio*, or ratio of the numbers of turns in a given time; which is also the reciprocal of the ratio of the periodic times of revolution. When the axes are parallel, or nearly parallel, the directional relation may be expressed simply by prefixing + or − to the velocity-ratio, according as the directions of rotation are similar or contrary; but there are cases to be considered further on, where the relative angular positions of the axes have to be considered with precision.

When the translations of two points in two different rotating pieces are compared, the directional relation is determined by the fact, that each point moves in a direction normal to a plane traversing itself and the axis about which it revolves; and that the velocity of each point is proportional to its perpendicular distance from that axis, and the speed of rotation about that axis, jointly. Hence, let a and a' denote the angular velocities of two rotating pieces, or a pair of numbers proportional to those angular velocities; r and r', the perpendicular distances of a pair of points in those two pieces from their respective axes, or a pair of numbers proportional to those distances; and v and v', the respective velocities of those two points, or a pair of numbers proportional to those velocities; then the velocity-ratio of the points is,

$$\frac{v'}{v} = \frac{a' r'}{a r}.$$

In order that a pair of points in a pair of rotating pieces may have equal velocities—that is, in order that $\frac{v'}{v}$ may be $= 1$, we must make the radii inversely proportional to the angular velocities—that is, $a' r' = a r$, or $\frac{r'}{r} = \frac{a}{a'}$.

Section IV.—*Screw-like Motion of Primary Pieces.*

57. **Helical or Screw-like Motion** (*A. M.*, 382,) is compounded of rotation about a fixed axis, and of translation along that axis: the *advance* (as the translation in a given time is called) bearing a constant proportion to the rotation in the same time; in other words, the moving piece advances along the axis of rotation through an uniform length during each turn.

The subject of the resolution of screw-like motion into components in other and more complex ways will be considered in the next chapter.

58. **General Figure of a Screw—Pitch.** (*A. M.*, 471.)—In order that a primary moving piece may have screw-like motion, its figure ought to be that of a true screw; and it ought to turn in a bearing of the same figure, fitting it accurately. The figure of a screw may be described in general terms as consisting of a projection of uniform cross-section called the *thread*, winding in successive coils round a circular cylinder. The best form of section for the thread of a screw that is to be used as a primary moving piece for producing helical motion only, and not as a fastening, nor in "screw gearing," is rectangular. The forms suited for other purposes will be considered later. There are two sorts of screws, convex, or *external*, and concave, or *internal;* in the former the thread winds round the outside of a cylindrical spindle; in the latter it winds round the inside of a hollow cylinder. When the word "screw" is used without qualification, an external screw is usually meant; an internal screw is called a "*nut.*" When a primary moving piece is an external screw, its bearing is an internal screw; when the primary moving piece is an internal screw, the bearing is an external screw. The *truth* or accuracy of

figure of a screw depends mainly on the perfect uniformity of the *pitch*; that is to say, the distance, measured parallel to the axis, from any point in one coil of the thread to the corresponding point in the next coil. For example, the pitch of the screw R in fig. 29, so long as it is measured parallel to the axis, may be measured either from D to F, from E to G, from F to H, or from G to K, or between any pair of corresponding points in two successive coils; and it ought to be exactly the same wheresoever it is measured. The pitch is also the uniform distance through which the screw advances at each turn.

59. **Right-Handed and Left-Handed Screws.**—A screw is said to be right-handed or left-handed according as right-handed or left-handed rotation is required in order to make it advance; and this is a permanent distinction, and not dependent on the position of the spectator, as the distinction between right-handed and left-handed rotation is (Article 48, page 25). For example, in fig. 29, L is a left-handed screw, and R a right-handed screw. Most screws used in the arts are right-handed; left-handed screws are made for special purposes only.

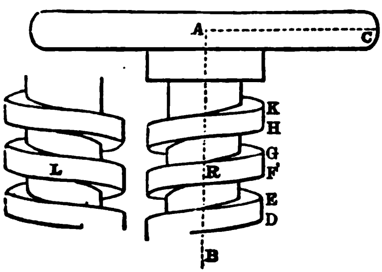

Fig. 29.

60. **Comparative Motion of a Point in a Screw.**—The principles of the present Article apply not only to any point in the thread or in the spindle of a screw, but to any point in a body that is rigidly attached to the screw, so as to move along with the screw as one piece; such as a wheel or a lever fixed to and turning with the screw. In fig. 29, let A B be the axis of the screw, and C a point rigidly attached to it at the perpendicular distance C A from the axis. Then, while the screw makes one turn the motion of the point C is the resultant of two components at right angles to each other: an *advance*, along with the whole screw, in a direction parallel to the axis, through a distance equal to the pitch of the screw; and a *revolution*, round a circle described about the axis with the radius A C, and having, therefore, the circumference 6·2832 A C. In most questions of comparative motion connected with screws, the quantity of most importance is the velocity-ratio of those two components of the motion of a given point, and it is expressed as follows:—

$$\frac{\text{velocity of revolution}}{\text{velocity of advance}} = \frac{\text{circumference}}{\text{pitch}} = \frac{6{\cdot}2832 \text{ A C}}{\text{D F}}.$$

61. **Path of a Point in a Screw—Linear Screw or Helix.**—A point in, or rigidly attached to, a screw, traces a path which may be called a screw-shaped line or *linear screw*. By mathematicians it is called a *helix*. A helix winds round in successive similar coils upon a cylindrical surface described about the axis of rotation with a radius equal to the perpendicular distance of the tracing point from the axis. The distance, measured parallel to the axis, between any two successive coils is everywhere the same, and is identical with the *pitch* of the screw; and the angle of inclination of the linear screw to the axis is everywhere the same.

Points in, or rigidly attached to, a screw, at equal distances from the axis, trace by their motion equal and similar linear screws on one and the same cylindrical surface. Points at unequal distances from the axis trace different linear screws, inclined to the axis at different angles, and situated on cylindrical surfaces of unequal radii; but the *pitch* of all those linear screws is the same. All the edges, whether projecting or re-entering, of a screw-thread are linear screws.

A linear screw may be traced on a cylindrical surface by any mechanical contrivance which ensures that, while the cylinder rotates, the tracing point shall advance along a line parallel to the axis at a rate bearing a constant proportion to the rate of rotation. This will be further considered in that part of this treatise which relates to the construction of machinery.

A linear screw is the shortest line on the surface of a cylinder between two points that are neither in one plane traversing the axis nor in one plane perpendicular to the axis; and a cord or a flexible wire stretched on a cylindrical surface between two such points tends to assume of itself the figure of a linear screw.

62. **Projection of a Linear Screw.**—The most useful projection of a linear screw is that upon a plane traversing the axis, and is drawn as follows:—In fig. 30, let A B represent the axis of the screw. Draw D A C perpendicular to A B, making A C = A D = the radius of the cylindrical surface in which the helix is to be situated. Draw D I and C F parallel to A B; those two lines will be the traces of the cylindrical surface. About A, with the radius A C, draw the semicircle C K D; this represents the trace of one-half of the cylindrical surface on a plane perpendicular to its axis, "rabatted" upon the plane of projection. Divide the semicircle into any convenient number of equal arcs (Article 51, page 27); the greater the number of those divisions, the greater will be the accuracy of the projection. In fig. 29 the semicircle is divided into six equal arcs only; in practice a greater number will in general be required.

On C F, or any other line parallel to the axis, lay off C E = the intended pitch of the screw, and divide it into *twice* as many equal parts as there are equal divisions in the semicircle C K D.

Through the points of division of the semicircle draw straight

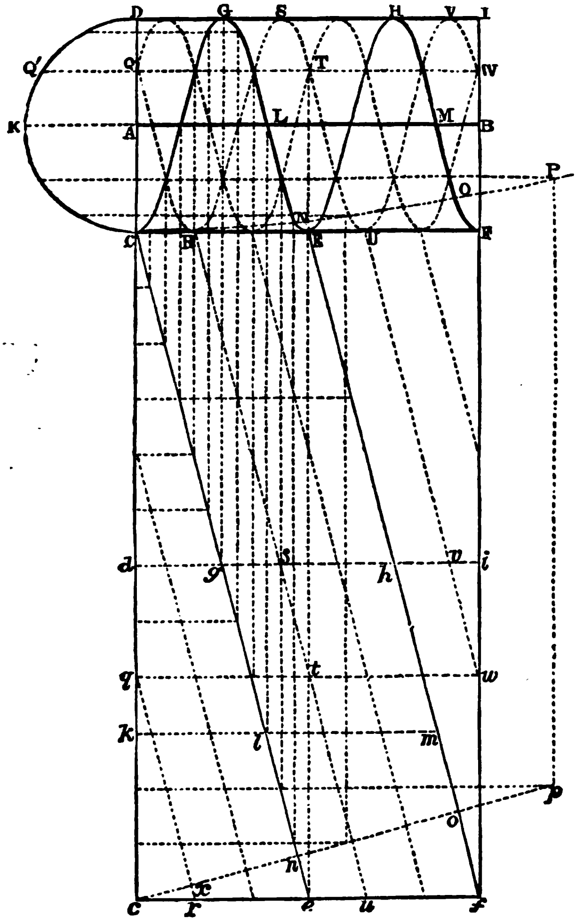

Fig. 30.

lines parallel to A B; and through the points of division of the pitch draw straight lines perpendicular to A B; the points of intersection of successive pairs of those two sets of lines will be points in the required projection of the linear screw, C G L E, which can then be drawn through those points. This is the projection of one coil; and as many successive coils as may be required may be projected by simply repeating the same curve. In fig. 30 the projection, E H M F, of a second coil is shown; and it has been constructed by laying off an uniform distance parallel to the axis and equal to the pitch from each projected point of the first coil; for example, G H, L M, E F.

The half coils on the nearer side of the cylinder, viz., G L E and H M F, are drawn in thicker lines than the half coils, C G, E H, on the farther side of the cylinder. The screw is right-handed. Had it been left-handed, C G and E H would have been on the nearer side, and G E and H F on the farther side of the cylinder.

63. **Development of a Linear Screw.**—The *development* of any figure drawn on a curved surface consists in supposing the surface to be a flexible sheet, and drawing the appearance which the figure would present if that sheet were spread out flat on a plane. Some curved surfaces only are capable of being developed, such as a cylinder, and a cone; others, such as a sphere, are not. To draw the development of the cylindrical surface in fig. 30, as bounded by the two circles whose projections are C D and F I, draw C c perpendicular to A B, and equal in length to the circumference of the cylinder (see Article 51, page 27), and complete the rectangular parallelogram C c f F; this will be the required development of the cylindrical surface.

To draw the development of one coil, C G E, of the linear screw, take c e = the pitch C E; draw the straight line C e; this will be the required development. To draw the development of the second coil, E H F, take e f = the pitch, and draw the straight line E f; and so on for any required number of coils.

The uniform *angle of inclination* of the linear screw to the axis is E C e = F E f.

One method of drawing a screw on a cylindrical surface is first to draw its development on a sheet of some flexible substance, and then to roll that sheet round a cylinder of the proper radius.

The several points in the development marked with small letters are the respective developments of the points marked with the corresponding capital letters in the projection.

To draw the development of the series of lines parallel to the axis which pass through the points of division of the circumference, divide C c into twice as many equal parts as the semicircle C D is divided into, and draw straight lines parallel to C F through the points of division, such as d i, q w, &c.

The *length of one coil of the screw* is

$$C\,e = \sqrt{(\text{circumference}^2 + \text{pitch}^2)}.$$

64. The **Radius of Curvature** of a linear screw is found by the following construction:—In fig. 31, let A C be the radius of the cylindrical surface in which the screw is situated. Draw A Y, making the angle C A Y equal to the angle which the screw makes with a plane perpendicular to its axis. Draw C Y perpendicular to A C, cutting A Y in Y, and Y Z perpendicular to A Y, cutting A C produced in Z. A Z will be the required radius of curvature. Its length may also be found by calculation, as follows:—

$$AZ = AC\left(1 + \frac{\text{pitch}^2}{\text{circumference}^2}\right).$$

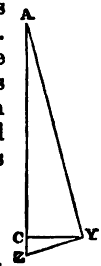

Fig. 31.

65. **Normal Pitch.** (*A. M.*, 472.)—By the *normal pitch* of a linear screw is to be understood the distance from one coil to the next, measured, not parallel to the axis, but along the shortest line on the cylindrical surface between the two coils; that is to say, along another helix or linear screw which cuts all the coils of the original linear screw at right angles. The normal pitch is to be determined from the development of the screw, as follows:—In fig. 30, from c let fall $c\,n$ perpendicular to C e; $c\,n$ will be the normal pitch. The straight line $c\,n$ is part of the development of the *normal helix*, as it may be called. When produced, it cuts E f, the development of the next coil, in o, and $n\,o = c\,n$ is also the normal pitch. By finding the intersections, such as p, of the development of the normal helix with the series of straight lines parallel to the axis, any number of points, such as P, in the projection of the normal helix, C N O P, may be found if required. The normal pitch may be calculated by the following formula:—

$$\overline{cn} = \frac{\overline{Cc}\;\overline{ce}}{Ce} = \frac{\text{circumference} \times \text{pitch}}{\text{length of one coil}}.$$

The *pitch of the normal helix*, if required, may be found by producing $c\,p$ in fig. 30 till it cuts C F produced, and measuring the distance of the point of intersection from C; and then its radius of curvature may be found by a construction like that in fig. 31; but in general it is more convenient to find these quantities by calculation, as follows:—

$$\text{pitch of normal helix} = \frac{\text{circumference}^2}{\text{pitch of original helix}}$$

$$\text{radius of curvature of normal helix} = AC \cdot \left(1 + \frac{\text{circumference}^2}{\text{pitch of orig. helix}^2}\right).$$

The sum of the reciprocals of the radii of curvature of the original helix and the normal helix is equal to the reciprocal of the radius of the cylinder; that is, to $\frac{1}{AC}$.

The pitch of a screw as measured parallel to the axis may be called the *axial pitch*, in order to distinguish it from the normal pitch; but when the word "pitch" is used without qualification, axial pitch is always to be understood.

The several linear screws which exist in the figure of an actual solid screw, or which are described by points in it or rigidly attached to it, have all the same axial pitch; but they have not the same normal pitch except when they are situated on the same cylindrical surface.

66. **Divided Pitch.**—A screw sometimes has more than one thread, in which case the distance between any coil of any one thread and the next coil of the same thread is divided by the other threads into as many parts as the total number of threads. In that case the distance from a point in one thread to the corresponding point in the next thread may be called the *divided pitch*, to distinguish it from the distance between two successive coils of the same thread, or pitch proper, which may, when required, be designated as the *total pitch*. The advance of the screw at each revolution depends on the total pitch only, in the manner already explained, and is wholly independent of the number of threads and of the divided pitch; so that division of the pitch does not affect the motion of a screw as a primary piece. Its use in combinations of pieces will be afterwards explained.

Division of the pitch of a linear screw is illustrated in fig. 30, where two additional linear screw threads, marked by dotted lines, are shown dividing the pitch of the original screw into three equal parts. To avoid confusion, one only of the additional screw-lines is lettered, viz., that marked Q R S T U V W in the projection, and $q\,r$, R $s\,t\,u$, U $v\,w$, in the development. The other is unlettered. The *divided axial pitch* is C R = $\frac{1}{3}$ C E, and the divided normal pitch $c\,x = \frac{1}{3}\,c\,n$.

The several linear screw threads in a case of this kind are all parallel and similar to each other; and in the development they are represented by parallel straight lines. They divide the circumference into as many equal parts as there are threads; and the length of one of those parts may be called the *circular* or *circumferential pitch*. In fig. 30, the circular pitch is represented by the arc C Q′, and by its development $c\,q$.

CHAPTER III.

OF THE MOTIONS OF SECONDARY MOVING PIECES.

67. **General Principles.** (*A. M.*, 383, 384, 503.)—In the present chapter the general principles only of the motions of secondary moving pieces in machines can be given, many of their most important applications being reserved for that chapter which will treat of "Aggregate Combinations in Mechanism," and some for the chapter on "Elementary Combinations." The mechanism for producing the motions of secondary moving pieces belongs wholly to those later chapters.

Secondary moving pieces have already been defined (in Article 37, page 17) as those which are carried by other moving pieces, or which have their motions not wholly guided by their connection with the frame. Their motions, therefore, are not restricted, like those of primary pieces, to translation in a straight line, rotation about a fixed axis, and that combination of those two motions which constitutes the motion of a screw with a fixed axis; they comprehend translations along curved lines of various figures, rotations about shifting axes, and various combinations of translations and rotations. The paths of points, too, in secondary pieces are not restricted to three forms—the straight line, the circle, and the helix; they comprehend a great variety of curved lines, both plane and of double curvature. The comparative motions of any two points in a primary piece are constant. The comparative motions of two points in a secondary piece very often vary from instant to instant as the piece changes its position.

In many cases the motions of secondary pieces are partially guided or restricted. For example, a secondary piece may be so guided that all its movements take place parallel to a fixed plane; in which case its motions are restricted to translations parallel to the fixed plane, and rotations about axes perpendicular to it; and the paths of its points are restricted to lines, straight or curved, in or parallel to that plane; and this restricted case is by far the most common in mechanism. Another kind of restriction on the movements of a secondary piece is when it turns about a ball and socket joint, or some equivalent contrivance, so that one point at the centre of the joint is kept fixed: in this case its motions are restricted to rotations about axes traversing that fixed point; and the motions of points in it are restricted to curves situated in spherical surfaces described about the fixed point. Cases in which the movements of secondary moving pieces are not restricted in one or other of those ways are comparatively rare.

The geometrical problems relating to the motions of secondary moving pieces may be divided into the two following classes:—

I. When the motions, in most cases, of two, and at furthest of three, points in a secondary moving piece are given, and it is required to find the motion of any other point in the piece, or of the piece as a whole. All problems of this class depend for their solution on the principle of Article 54, page 32.

II. When there are two moving pieces or moving points, C and B, the frame of the machine being denoted by A, and two out of the three motions of A, B, and C relatively to each other being given, it is required to find the third of those motions. All problems of this class depend for their solution on the principle (already stated in Article 42, page 21) that the motion of C relatively to A is the resultant of the motions of B relatively to A, and of C relatively to B.

68. **Translation of Secondary Moving Pieces.** (*A. M.*, 369.)—If, in a moving piece whose movements are not restricted, the directions of motion of three points not in the same straight line are parallel to each other and oblique to the plane of the three points; or if, in a moving piece restricted to movements parallel to one plane, the motions of two points are parallel to each other and oblique to the line of connection of the points; then the motion of the whole piece is a translation. All the points in the piece describe equal and similar paths, straight or curved; and all, at a given instant, move with equal velocities in parallel directions. The motion of any pair of points in the moving piece relatively to each other is nothing; and their comparative motion consists in the directional relation of parallelism and the velocity-ratio of equality.

To exemplify the translation of all the points of a moving piece in equal and similar curved paths, we may take the case of a coupling-rod (fig. 32) which connects together a pair of equal

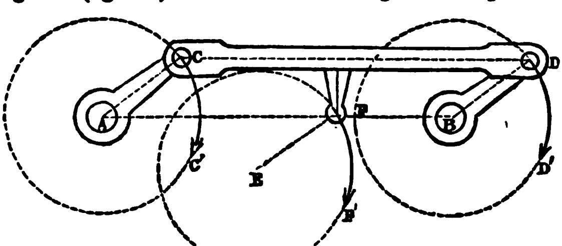

Fig. 32.

cranks, A C, B D, and has its effective length, C D, equal to the perpendicular distance, A B, between the axes of rotation of the two cranks. The motion of that coupling-rod is one of translation, in which all the particles describe with equal speed equal and similar circles of the radius A C = B D, in planes perpendicular to the axes A and B. The same is the case with any particle rigidly attached to the coupling-rod; such as F, which revolves in a circle of the radius E F = A C; so that, for example, the points C, D, and F move simultaneously through the equal and similar arcs C C′, D D′, F F′.

69. **Rotation Parallel to a Fixed Plane — Temporary Axis — Instantaneous Axis.**—The cases next in order as to complexity are those in which all the movements of the piece are parallel to a fixed plane; and the following are the problems which present themselves:—

I. *Given, the paths of two points in a moving piece, the distance between their projections on the plane of motion, and two successive positions of one of them, to find the* temporary axis of motion *of the piece.*

In fig. 33, let the plane of projection and of motion be that of

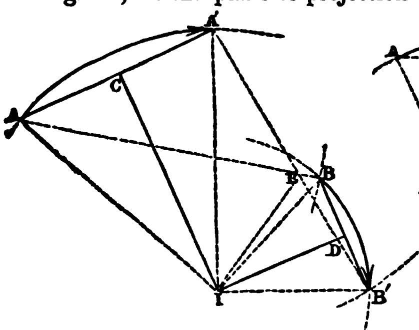

Fig. 33.

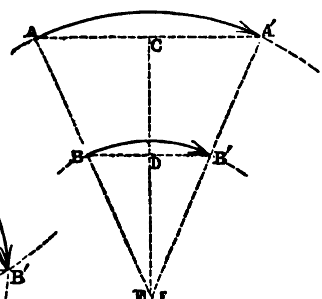

Fig. 34.

the paper, and let the partly dotted lines A A′ and B B′ be the projections of the paths of the two points, which may be straight lines or plane curves of any figure, subject only to the limitation that the distance between the points is invariable. Let A and A′ be the given two successive positions of one of the points. About A and A′ respectively, with the projection of the line of connection as a radius, draw circular arcs cutting the projected path of the other point in B and B′; these will be the projections of the two successive positions of the second point; and the straight lines A B and A′ B′ will be the projections of the line of connection in the two successive positions of the moving piece. Draw the straight lines A A′ and B B′; bisect them in C and D, through which points draw C I perpendicular to A A′ and D I perpendicular to B B′, meeting each other in I. Then, because A′ I = A I and B′ I = B I, I represents the same point in the two positions of the piece; and therefore I is the projection and the trace of a line perpendicular to the plane of motion, whose position is the same after the motion of A to A′ and of B to B′ that it was before. That line may be called the *Temporary Axis of Motion* of the moving piece, because the change of position of the piece is the same as if it had been turned through an angle A I A′ = B I B′ about that line.

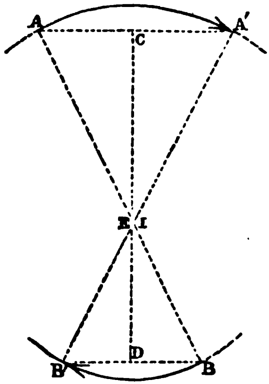

Fig. 35.

Let E be the point of intersection of A B and A′ B′. Then the straight line E I bisecting the angle A E B′ traverses the temporary axis I; and this affords a means of finding that axis when C I and D I cut each other at an angle so oblique as to make it difficult to determine precisely their point of intersection.

When B B′ is parallel to A A′, as in figs. 34 and 35, C I and D I become parts of one straight line, and have no intersection; and then the point I is determined by its coinciding with E. In most cases of this kind it is necessary that the two successive positions of B should be given as well as those of A.

II. *Given* (in fig. 36), *the projections* A *and* B, *at a given instant, of two points in a moving piece on the plane of motion, and the simultaneous directions of motion of those points,* A a *and* B b, *to find the* instantaneous axis *of the moving piece; and thence to deduce the comparative motions, at the given instant, of the given points, and of any other points in the moving piece.*

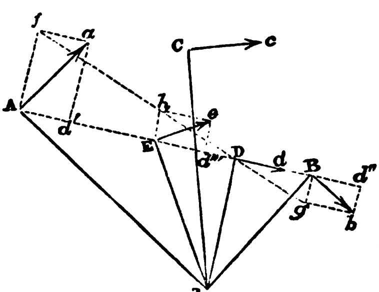

Fig. 36.

If the simultaneous directions of motion of the given points are perpendicular to their line of connection, the problem requires additional data for its solution, which will be stated in Rule III. If those directions are parallel to each other, and not perpendicular to the line of connection, the motion of the piece is one of translation, like that referred to in Article 68, page 44. The present rule comprehends all cases in which the given directions are not parallel to each other.

Through A and B draw A I and B I perpendicular respectively to A a and B b, and cutting each other in I. Then I will be the projection and the trace on the plane of motion of the required INSTANTANEOUS AXIS: that is to say, of a line such that the motion of the piece *at the instant in question* is one of rotation about that axis.

An instantaneous axis is so called because it is an imaginary line which is continually changing its position, both relatively to the frame of the machine and relatively to the secondary piece to which it belongs; so that it occupies any particular position, whether relatively to the frame or relatively to the secondary piece, at a particular instant only.

The comparative motions at the given instant of points in the secondary piece are deduced from the principle that the velocities of those points are proportional in magnitude and perpendicular in direction to the perpendiculars let fall from the points upon the instantaneous axis. For example, let A a, B b, C c, D d, E e, represent the directions and velocities of the points whose projections are A, B, C, D, E; then

$$\mathrm{A}a : \mathrm{B}b : \mathrm{C}c : \mathrm{D}d : \mathrm{E}e$$

are respectively proportional and perpendicular to

$$:: \mathrm{AI} : \mathrm{BI} : \mathrm{CI} : \mathrm{DI} : \mathrm{EI}.$$

From I let fall I D perpendicular to the projection, A B, of the line of connection of the given points. Then all points whose projections are at D are at the given instant in the act of moving parallel to A B; and all points whose projections are in A B, or in A B produced, such as A, B, and E, have for their component velocities along A B velocities equal to the velocity of D; that is to say,

$$\mathrm{D}d = \mathrm{A}d' = \mathrm{B}d'' = \mathrm{E}d''';$$

a consequence which follows also from the principle of Article 53, page 31.

The components perpendicular to A B of the velocities of points whose projections are in that line, such as A, B, and E, are proportional to the distances of those projections from D; that is to say, if A f, B g, and E h represent those transverse component velocities, we have the proportions,

$$\mathrm{DA} : \mathrm{DB} : \mathrm{DE}$$
$$:: \mathrm{A}f : \mathrm{B}g : \mathrm{E}h;$$

and the points f, h, D, g are in one straight line.

Hence, when A I and B I form an angle with each other so oblique as to make it difficult to determine precisely their point of intersection, we may proceed as follows to increase the precision of that determination:—Lay off any convenient equal distances, A d' = B d'', along A B from A and B respectively, to represent the longitudinal component of their velocities. Then complete the rectangular parallelograms A $d' a f$, B $d'' b g$; draw the straight line $f g$, cutting A B in D. Then from D perpendicular to A B draw D I; this line will traverse the instantaneous axis, and will increase the precision with which it is determined.

This last way of considering the motion of the piece is equivalent to regarding that motion as compounded of a rotation about an axis at D and a translation of that axis, and of the whole body along with it, with the velocity represented by D d.

III. *Given* (in fig. 37 or fig. 38), *the projections* A *and* B, *at a given instant, of two points in a moving piece on the plane of motion, and the ratio of their velocities, which are both perpendicular to the*

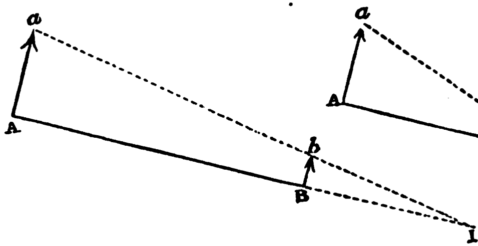

Fig. 37.

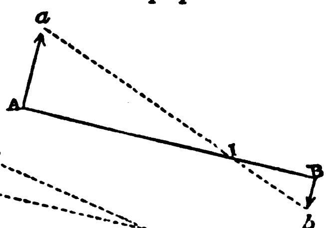

Fig. 38.

projection, A B, *of their line of connection, to find the* instantaneous axis *of motion of the piece.* Perpendicular to A B draw the straight lines A a, B b, bearing to each other the given proportion of the velocities of the two points: draw the straight line $a b$; the point of intersection, I, of A B and $a b$ (produced if necessary) will be the projection and trace on the plane of motion of the required instantaneous axis.

That axis may then be used as in the preceding Rule to determine the comparative motions of any set of points in the moving piece.

70. **Rotation about a Fixed Point.**—Every possible motion of a rigid body relatively to a point in the body is reducible to rotation about an axis, permanent, temporary, or instantaneous, as the case may be, which traverses that point. This is proved by showing that the following problem is always capable of solution:—

I. *Given, at any instant, the directions of motion of any two points,* B, C (fig. 39), *in a rigid body relatively to a point,* A, *in the* *body, to find the* instantaneous axis *of the motion of the whole body relatively to* A. In the first place, it is to be observed that when the motions of three points in a rigid body are determined, the

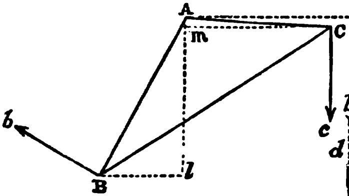

Fig. 40.

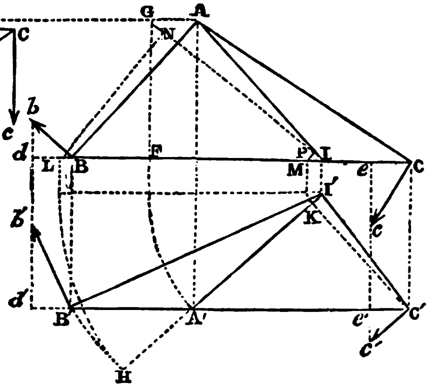

Fig. 39.

motion of the whole body is determined; for the distances of any fourth point in the body from those three points being invariable, the position of that fourth point at every instant is completely determined by the positions of the three points.

In order that the solution may be put in the simplest possible form, let the plane of the three points themselves, or a plane parallel to it, be taken for one plane of projection; and in fig. 39 let A, B, C be the projections of the three points on that plane. For a second plane of projection, take a plane perpendicular to the first plane, and traversing B C, and let A′, B′, and C′ (which are in one straight line) be the projections of the three points on that second plane; so that B′ C′ is parallel to B C, and A A′, B B′, and C C′ are perpendicular to B C.

Because the instantaneous axis must traverse A, it is obvious that A B and A C are the traces on the first plane of projection of two planes traversing the instantaneous axis and the points B and C respectively; and also, that if B *b* and C *c* are the projections on the first plane of projection of the directions of motion of B and C at the given instant, those projections must be perpendicular to A B and A C. Let B′ *b*′ and C′ *c*′ represent the projections of the directions of motion of B and C on the second plane of projection. Draw B′ I′ and C′ I′ perpendicular respectively to B′ *b*′ and C′ *c*′, and meeting each other in I′; then B′ I′ and C′ I′ are the traces, on the second plane of projection, of two planes perpendicular respectively to the instantaneous directions of motion of B and C; that is to say, of the two planes already mentioned, which traverse the instantaneous axis and the points B and C respectively; and I′ is *the trace of the instantaneous axis on the second plane of projection.* From I′ let fall I′ I perpendicular to B C; then I is the projection of I′ on the first plane of projection. Draw the straight lines A I, A′ I′: those are *the projections of the instantaneous axis.*

II. *To draw the projections of the points* B *and* C *on a plane perpendicular to the instantaneous axis, and to find the comparative motion of those points.* In B C, fig. 39, take I F = I′ A′; draw A G parallel and F G perpendicular to B C, cutting each other in G; join I G: this line will be the rabatment of I A. From B′ and C′ let fall B′ H and C′ K perpendicular to I′ A′ (produced if required). In B C take I L = I′ H, and I M = I′ K; then G, L, and M will represent the respective projections of A, B, and C upon a plane which traverses the instantaneous axis, and is perpendicular to the second plane of projection. From L and M let fall L N and M P perpendicular to I G. Then, in fig. 40, let the paper represent a plane of projection perpendicular to the instantaneous axis: let A be the trace and projection of that axis, and A *l* the trace of the plane already mentioned as being perpendicular to the second plane of projection in fig. 39. Make A *l* in fig. 40 = N L in fig. 39, and A *m* in fig. 40 = P M in fig. 39. Draw *l* B in fig. 40 perpendicular to A *l* in fig. 40 and = H B′ in fig. 39; also *m* C in fig. 40 perpendicular to A *l* in fig. 40 and = K C′ in fig. 39. Join A B, A C. Then B and C in fig. 40 will be *the projections required;* and the velocities of B and C relatively to A will be *perpendicular in direction and proportional in magnitude to* A B *and* A C *respectively.*

Another mode of finding the comparative motion of A and B is the following:—According to the principle of Article 54, page 32, the component velocities of B and C along their line of connection, B C, are equal. Therefore, in fig. 39, lay off along B C and B′ C′ the equal distances B *d*, C *e*, B′ *d′*, C′ *e′*, to represent that component; then draw *d′ b′ d b*, *c′ e′ c e* perpendicular to B C, cutting B *b* in *b*, B *b′* in *b′*, C *c* in *c*, and C′ *c′* in *c′*; then B *b* and B′ *b′* will be the projections of the velocity of B relatively to A; and C *e* and C′ *e′* will be the projections of the velocity of C relatively to A. Then, by the rule of Article 19, page 7, find the lengths of the lines of which B *b* and B′ *b′*, C *c* and C′ *c′* are the projections; the ratio of those lengths to each other will be the velocity-ratio of the two points.

71. **Unrestricted Motion of a Rigid Body.**—How complicated soever the motion of a rigid body may be, it may always be regarded as made up of a change of position of the body as a whole—that is, a translation of the body, and a change of position of

the body relatively to some point in it; that is to say, as has been shown in the preceding Article, a rotation about an axis traversing that point, which axis may be either permanent, temporary, or instantaneous. It will be shown further on that a rotation and a translation parallel to the plane of rotation, when compounded, are equivalent to a motion of rotation about an instantaneous axis perpendicular to that plane. Hence it follows, that if a rigid body has any translation combined with any rotation, the translation is to be resolved into two components, one parallel and one normal to the plane of rotation; when the parallel component of the translation, being combined with the rotation, will be equivalent to a rotation about a new instantaneous axis perpendicular to the plane of rotation; and the whole motion of the body will be equivalent to this new rotation combined with the normal component of the translation, the direction of which component is parallel to the axis. In short, how complex soever the motion of a rigid body may be, it is at each instant equivalent to a helical or screw-like motion about an instantaneous axis. The most comprehensive problem that can occur respecting the unrestricted motion of a rigid body at a given instant is this: given the simultaneous velocities and directions of motion of three points (say A, B, and C), to find the motion of the whole body at that instant. This is to be solved by choosing one of the points (say A), and regarding it for the time as fixed, and determining the motions of B and C relatively to A. Then, by means of the rules of Article 70, the position is to be determined of an instantaneous axis traversing A; when the motion of the whole body will be compounded of a rotation about that axis and a translation of the point A, carrying the instantaneous axis and the whole body along with it.

72. **Instantaneous Axis of a Rolling Body.**—The following proposition has many useful applications in mechanism:—*The instantaneous axis of a rigid body, which rolls without slipping upon the surface of another rigid body, passes through all the points in which those bodies touch each other;* for the particles in the rolling body which at any given instant touch the fixed body without slipping must have, at that instant, no velocity, and must therefore be, at that instant, in the instantaneous axis.

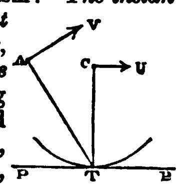

Fig. 41.

Moreover, because an instantaneous axis is a straight line, it follows that, in order that two surfaces which touch each other at more than one point may roll on each other without slipping, their points of contact must all lie in one straight line: a property of plane, cylindrical, and conical surfaces only, the terms "cylindrical" and "conical" being used in the general sense, which comprehends cylinders and cones with bases of any figure, as well as those with circular bases.

In fig. 41, let the plane of the paper represent a plane of projection perpendicular to the straight line in which the fixed and the rolling surfaces touch each other; let T be the projection and trace of that straight line, which is the instantaneous axis of the rolling body. Let A be the projection at a given instant of a point in the rolling body; then at that instant A is moving with a velocity proportional to A T, and in a direction perpendicular to the plane traversing A and the instantaneous axis, of which plane A T is the trace.

It follows that the path traced by a point such as A in a rolling body is *a curve whose normal*, A T, *at any given point*, A, *passes through the corresponding position*, T, *of the instantaneous axis.* Curves of this class are called *rolled curves;* and some of them are useful in mechanism, as will be explained farther on.

73. **Composition of Rotation with Translation.**—From Article 52, page 30, it appears that the single rotation of a body about a fixed axis (such as O, fig. 19, page 26) may be regarded as compounded of a rotation with equal angular velocity about a moving axis parallel to the fixed axis (such as that whose trace is A, fig. 19), and a translation of that moving axis carrying the body along with it in a circle round the fixed axis of the radius O A. A similar resolution of motions may be applied to rotation about an instantaneous axis. For example, the rotation of the rolling body in fig. 41 about the instantaneous axis, T, may be conceived to be made up of a rotation about another axis, C, parallel to the instantaneous axis, and a translation of that axis.

The present Article relates to the converse process, in which there are given a rotation of a secondary piece about an axis occupying a fixed position in the piece, and a translation of that axis relatively to the frame in a direction perpendicular to itself—that is, parallel to the plane of rotation; and it is required to find, at any instant, the instantaneous axis so situated that a rotation about it with the same angular velocity shall express the resultant motion of the piece.

In fig. 41, let the plane of the paper be the plane of motion, and let C be the projection and trace of the moving axis—moving relatively to the frame, but fixed as to its position in the secondary piece. Let C U be the direction of the translation of that axis, carrying the moving piece with it; and let the velocity of translation be so related to the angular velocity of rotation as to be equal to the velocity of revolution about the axis C, of a particle whose distance from that axis is $\text{C T} = \dfrac{\text{velocity of translation}}{\text{angular velocity}}$. Draw C T of the length so determined, in a direction perpendicular to C U, and pointing towards the right or towards the left of C U, according as the rotation is right-handed or left-handed. Then T will be *the projection and trace of the required instantaneous axis;* so that if A is the projection of any point in the moving piece, the direction of motion of that point is perpendicular to the plane whose trace is A T; and the velocity-ratio of A and C is

$$\frac{A\,V}{C\,U} = \frac{T\,A}{T\,C}.$$

The proof that T is the instantaneous axis is, that any particle whose projection, at a given instant, coincides with T is carried backward relatively to C by the rotation with a speed equal and opposite to that with which C is carried forward by the translation; so that the resultant velocity of every particle at the instant when its projection coincides with T is nothing.

74. **Rolling of a Cylinder on a Plane—Trochoid—Cycloid.** (*A. M.*, 386.)—Every straight line parallel to the moving axis C, in a cylindrical surface described about C with the radius C T, becomes in turn the instantaneous axis. Hence the motion of the body is the same with that produced by the rolling of such a cylindrical surface on a plane, P T P, parallel to C and to $\overline{C\,U}$, at the distance C T.

The path described by any point in the body, such as A, which is not in the moving axis C, is a curve well known by the name of *trochoid.* The particular form of trochoid, called the *cycloid,* is described by each of the points in the rolling cylindrical surface.

75. **Rolling of a Plane on a Cylinder; Involute—Spirals.** (*A. M.*, 387.)—Another mode of representing the combination of rotation with translation in the same plane is as follows:—In fig. 42, let O be the projection and trace on the plane of motion of a fixed axis, about which let the plane whose trace and projection is O C (containing the axis O) rotate (righthandedly, in the figure) with a given angular velocity. Let a secondary piece have *relatively to the rotating plane,* and in a direction perpendicular to it, a translation with a given velocity. In the plane O C, and at right angles to the axis

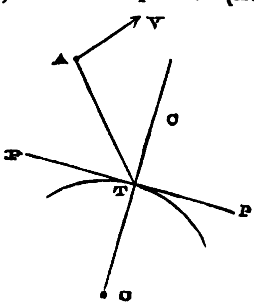

Fig. 42.

O, take $O\,T = \dfrac{\text{velocity of translation}}{\text{angular velocity}}$, in such a direction that the velocity which the point T *in the rotating plane* has at a given instant, shall be in the contrary direction to the equal velocity of translation which the secondary piece has relatively to the rotating plane. Then each point in the secondary piece which arrives at the position T, or at any position in a line traversing T parallel to the fixed axis O, is at rest *at the instant* of its occupying that position; therefore the line traversing T parallel to the fixed axis O is *the instantaneous axis;* the motion at a given instant of any point in the secondary piece, such as A, is at right angles to the plane whose trace is A T, perpendicular to the instantaneous axis; and the velocity of that motion bears to the velocity of the translation the ratio of T A to T O.

All the lines in the secondary piece which successively occupy the position of instantaneous axis are situated in a plane of that body, P T P, perpendicular to O C; and all the positions of the instantaneous axis are situated in a cylinder described about O with the radius O T; so that the motion of the secondary piece is such as is produced by the *rolling of the plane whose trace is* P P *on the cylinder whose radius is* O T. Each point in the secondary piece, such as A, describes a plane spiral about the fixed axis O. For each point in the *rolling plane,* P P, that spiral is the *involute of the circle* whose radius is O T, being identical with the curve traced by a pencil tied to a thread while the thread is being unwrapped from the cylinder. For each point whose path of motion traverses the fixed axis O—that is, for each point in a plane of the secondary piece traversing O and parallel to P P—the spiral is Archimedean, having a radius-vector increasing by a length equal to the circumference of the cylinder at each revolution.

76. **Composition of Rotations about Parallel Axes.** (*A. M.*, 388.) —In figs. 43, 44, and 45, let O be the projection and trace on the

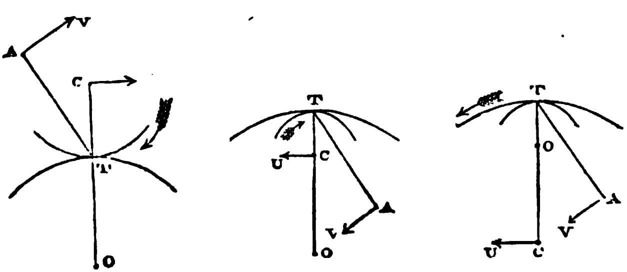

Fig. 43. Fig. 44. Fig. 45.

plane of rotation of a fixed axis, and O C the trace of a plane traversing that axis, and rotating about it with the angular velocity *a*. Let C be the projection and trace of an axis in that plane, parallel to the fixed axis O; and about the moving axis C let a secondary piece rotate with the angular velocity *b relatively to the* *plane* O C; and let the directions of the rotations a and b be distinguished by positive and negative signs. The body is said to have the rotations about the parallel axes O and C *combined* or *compounded*, and it is required to find the result of that combination of parallel rotations.

Fig. 43 represents the case in which a and b are similar in direction: fig. 44, that in which a and b are in opposite directions, and b is the greater; and fig. 45, that in which a and b are in opposite directions, and a is the greater.

Let the trace O C of the rotating plane be intersected in T by a straight line parallel to both axes, in such a manner that the distances of T (the trace of that line) from the fixed and moving axes respectively shall be inversely proportional to the angular velocities of the component rotations about them, as is expressed by the following proportion:—

$$a : b : : \mathrm{CT} : \mathrm{OT} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

When a and b are similar in direction, let T fall between O and C, as in fig. 43; when they are contrary, beyond, as in figs. 44 and 45. Then the velocity of the line T *of the plane* O C is $a \cdot \overline{\mathrm{OT}}$; and the velocity of the line T *of the secondary piece, relatively to the plane* O C, is $b \cdot \mathrm{CT}$, equal in amount and contrary in direction to the former; therefore each line of the secondary piece which arrives at the position T is at rest at the instant of its occupying that position, and is then *the instantaneous axis*. The *resultant angular velocity* is given by the equation

$$c = a + b; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

regard being had to the directions or signs of a and b; that is to say, if we now take a and b to represent *arithmetical* magnitudes, and affix explicit signs to denote their directions, the direction of c will be the same with that of the greater; the case of fig. 43 will be represented by the equation 2, already given; and those of figs. 44 and 45 respectively by

$$c = b - a; \; c = a - b \ldots\ldots\ldots\ldots\ldots (2\ \mathrm{A}.)$$

The relative proportions of a, b, and c, and of the distances between the fixed, moving, and instantaneous axes, are given by the equation

$$a : b : c : : \mathrm{CT} : \mathrm{OT} : \mathrm{OC} \ldots\ldots\ldots\ldots\ldots (3.)$$

The motion of any point, such as A, in the secondary piece, according to the principle of Article 72, is at each instant at right angles to the plane whose trace is A T, drawn from the point A perpendicular to the instantaneous axis; and the velocity of that motion is

$$v = c \cdot \mathrm{A\,T} \text{..........................} (4.)$$

77. **Rolling of a Cylinder on a Cylinder—Epitrochoids—Epicycloids.** (*A. M.*, 389.)—All the lines in the secondary piece which successively occupy the position of instantaneous axis are situated in a cylindrical surface described about C with the radius C T; and all the positions of the instantaneous axis are contained in a cylindrical surface described about O with the radius O T; therefore the resultant motion of the secondary piece is that which is produced by rolling the former cylinder on the latter cylinder.

In fig. 43 a convex cylinder rolls on a convex cylinder; in fig. 44 a smaller convex cylinder rolls in a larger concave cylinder; in fig. 45 a larger concave cylinder rolls on a smaller convex cylinder.

Each point in the rolling rigid body traces, relatively to the fixed axis, a curve of the kind called *epitrochoids.* The epitrochoid traced by a point in the surface of the rolling cylinder is an *epicycloid.*

In certain cases the epitrochoids become curves of a more simple class. For example, each point in the *moving axis* C traces a circle.

When a cylinder rolls, as in fig. 44, within a concave cylinder of *double its radius*, each point in the surface of the rolling cylinder moves backwards and forwards in a straight line, being a diameter of the fixed cylinder; each point in the axis of the rolling cylinder traces a circle of the same radius with that cylinder; and each other point in or attached to the rolling cylinder traces an ellipse of greater or less eccentricity, having its centre in the fixed axis O. This principle has been made available in instruments for drawing and turning ellipses.

There is one case of the composition of rotations about parallel axes in which there is *no instantaneous axis;* and that is when the two component rotations are of equal speed and in contrary directions; for then the resultant is simply a *translation* of the secondary piece along with the moving axis. This may be illustrated by referring to fig. 32, page 44, where the translation of the coupling-rod C D may be looked upon as the resultant of the combination of the rotation of the crank A C about A, with an equal and contrary rotation, *relatively to the crank*, of C D about C.

78. **Curvature of Involutes of Circles, Epicycloids, and Cycloids.**—It is often useful to determine the radius of curvature of a rolled curve at a given point, especially where the fixed curve and rolling curve are circles, and where the tracing point is in the circumference of the rolling curve.

In the case of the *involute of a circle*, the radius of curvature at a given point is simply the length of a normal to the curve at that point measured to the point where that normal touches the circle; that is to say, it is the length of the straight part of the thread used in drawing the involute.

In the case of a *cycloid* (traced by a point in the circumference of a cylinder which rolls on a plane) the radius of curvature at a given point is twice the length of the normal measured from that point to the corresponding instantaneous axis.

In the case of an epicycloid the construction for finding the radius of curvature is shown in fig. 46; the right-hand division of the figure giving the construction for an *external epicycloid*, I A, traced by a point, A, in the surface of a cylinder, the trace of

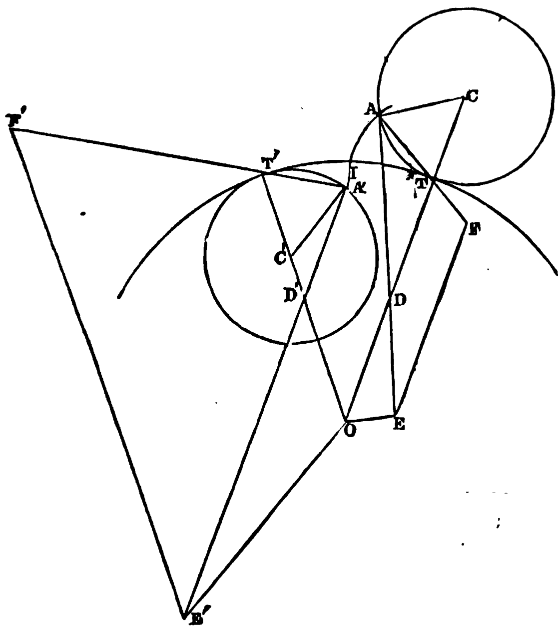

Fig. 46.

whose axis is C, rolling *outside* a fixed cylinder, the trace of whose axis is O; and the left-hand division giving the construction for an *internal epicycloid*, I A′, traced by a point, A′, in the surface of a cylinder, the trace of whose axis is C′, rolling *inside* the same fixed cylinder. The following description applies to both divisions of the figure: it being observed that at the left-hand side the letters are accented:—

Let T be the trace of the instantaneous axis, or line of contact of the cylinders, at the instant when the tracing point is at A; so that A T is the normal to the epicycloid at A, and O T and C T the radii of the fixed and rolling cylinders, being two parts of one straight line. Through O draw O E parallel to A C. Bisect O T in D, and draw the straight line A D E, cutting O E in E. Through E draw E F parallel to O T, and cutting A T (produced as far as required) in F. Then A F will be the radius of curvature of the epicycloid at the point A.

The following formula serves to find A F by calculation;

$$A\,F = \frac{A\,T \cdot O\,C}{C\,D}. \text{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

It is sometimes more convenient to calculate the distance, T F, of the *centre of curvature*, F, from the instantaneous axis, T, and that is done by the following formula:

$$T\,F = \frac{A\,T \cdot O\,D}{C\,D} = \frac{A\,T \cdot O\,T}{2\,C\,D}; \quad \ldots\ldots\ldots\ldots\ldots (2.)$$

the use of which, in designing the teeth of wheels by Mr. Willis's method, will appear farther on.

79. **To Draw Rolled Curves.**—A rolled curve may be drawn by actually rolling a disc of the form of the rolling curve, carrying a suitable tracing point, upon the edge of a disc of the form of the fixed curve. But it needs much care to perform that operation with accuracy, except with the aid of machinery specially contrived for the purpose, such as is to be found in certain kinds of turning lathes.

For ordinary purposes in designing machinery, approximate methods of drawing rolled curves are used, such as the following:—

I. *To draw approximately a rolled curve by the help of tangent circles.*—In fig. 47, let A B be the fixed curve, and A D the rolling curve, touching the fixed curve at A, which is also the position of the tracing point at starting. The curve A D rolls from A towards B; and it is required to draw approximately the curve traced by the point A. By Rule III. of Article 51, page 29, lay off on each of the two curves A B and A D a series of equal arcs, A 1, 12, 23, 34, &c. Measure the straight chord from 1 to A on the curve A D, and with 11 = 1A as a radius, and the point 1 on the curve A B as a centre, draw so much of a circle as lies near the probable position of the rolled curve; measure the straight chord from 2 to A on A D, and with 22 = 2 A as a radius, and the point 2 on the curve A B as a centre, draw in like manner part of a circle; and go on, in the same way, drawing a series of

* The proof of this is as follows:—Let the radius of the rolling cylinder, C A = C T = r; let that of the fixed cylinder, O T = R, which is to be

circular arcs with the points 1, 2, 3, 4, &c., in the fixed curve A B, for their centres, and for their radii the lines 11, 22, 33, 44, &c., respectively equal to the distances 1 A, 2 A, 3 A, 4 A, &c., as measured between points on the rolling curve. Then, with the

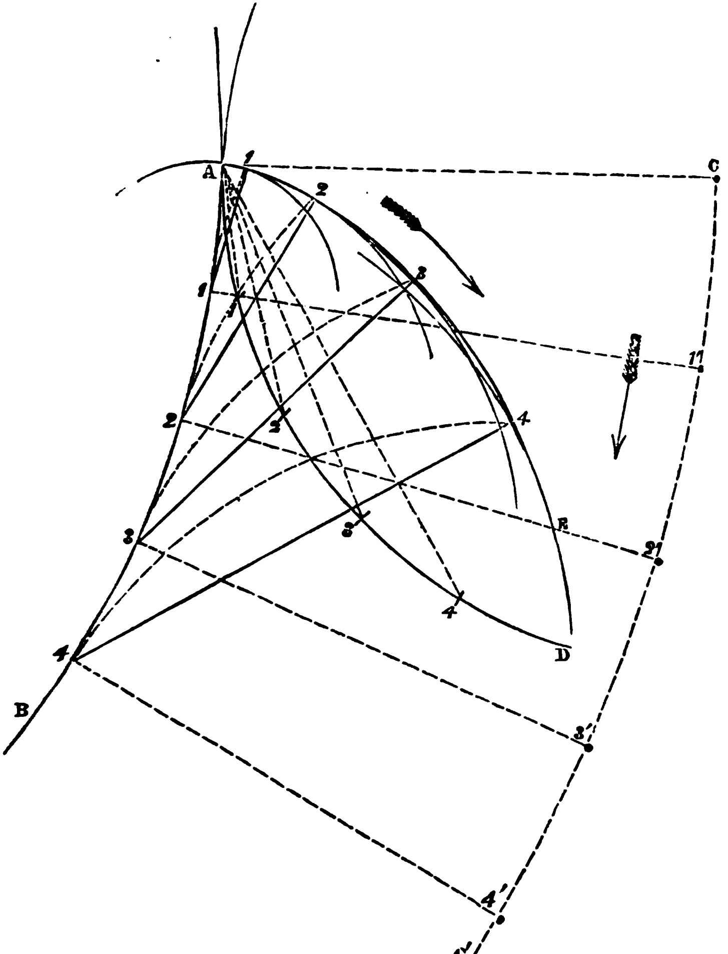

Fig. 47.

regarded as positive or negative according as the rolling cylinder rolls on the outer or inner surface of the fixed cylinder; let the instantaneous radius or free hand, or with the help of a bent spring, draw a curve, A E, so as to touch all those circular arcs; this will be very nearly the rolled curve required.

The curve A E is called the "Envelope" of the series of arcs that it touches.

II. *To find a series of points in a rolled curve.*—Draw a series of tangent circular arcs as in the preceding rule; then draw the several normals, 11, 22, 33, 44, &c., as radii of those arcs; the direction of each normal being determined by the principle, that at the point where it meets the fixed curve A B, it makes an angle with a tangent to that curve equal to the angle which the corre-

normal of the epicycloid, T A $= p$; and let the required radius of curvature, A F $= \rho$.

Let the angular velocity of the rolling cylinder, *relatively to the rotating plane* O C, be denoted by b, and that of the plane O C by a, so that the resultant angular velocity of the rolling cylinder is $a + b$. Then, because the angle C T A is the complement of one-half of the angle T C A, it is evident that the angular velocity of T A is $a + \frac{b}{2}$. But according to Article 76, $a \text{R} = b\, r$; therefore

$$a + b = b\left(1 + \frac{r}{\text{R}}\right);\ a + \frac{b}{2} = b\left(\frac{1}{2} + \frac{r}{\text{R}}\right).$$

In any indefinitely short time, $d\,t$, the normal sweeps through an angle whose value in circular measure is

$$d\,\theta = \left(a + \frac{b}{2}\right) d\,t = b\left(\frac{1}{2} + \frac{r}{\text{R}}\right) d\,t;$$

and the point A traces an arc of the length

$$d\,s = (a + b)\, p\, d\,t = b\left(1 + \frac{r}{\text{R}}\right) p\, d\,t;$$

therefore the radius of curvature of the epicycloid at the point A is

$$\rho = \frac{d\,s}{d\,\theta} = p \cdot \frac{1 + \frac{r}{\text{R}}}{\frac{1}{2} + \frac{r}{\text{R}}} = \frac{p\left(\text{R} + r\right)}{\frac{1}{2}\text{R} + r} = \frac{\text{A T} \cdot \text{O C}}{\text{C D}}.$$

This formula is made to comprehend the case of a cycloid by making $\text{R} = \infty$, when it becomes $\rho = 2\,p$; and that of the involute of a circle by making $r = \infty$, when we have $\rho = p$. When the epicycloid is internal, and R and r denote arithmetical values of those radii, the sign — is to be substituted for + both in the numerator and in the denominator of the formula. The symbolical expression for equation 2 of the text is

$$\rho - p = \frac{p\,\text{R}}{\text{R} + 2\,r},$$

with the same understanding as to the sign in the denominator. In the case already referred to at the end of Article 77, when a cylinder rolls inside a cylinder of twice its diameter, we have $\text{R} = -2\,r$, and the denominator of the expression for ρ becomes $= 0$; showing that the radius of curvature is infinite; or, in other words, that the epicycloid traced is a straight line, as stated in the text. When the rolling cylinder is concave, r is negative.

sponding chord on the rolling curve A D, makes with a tangent to that curve at the corresponding point. Thus are found a series of points, 1, 2, 3, 4, &c., on the rolled curve A E, at the ends of the normals from the corresponding points on the fixed curve A B.

The two preceding Rules are applicable to fixed and rolling curves of all figures whatsoever. When both curves are circles, the finding of a series of points is facilitated by drawing the circle C C′, which contains the successive positions of the centre of the rolling circle; then marking those successive positions, 1′, 2′, 3′, 4′, &c., on the circle C C′, by drawing radii through the corresponding points 1, 2, 3, 4, &c., on the circle A B; then drawing the rolling circle in its several successive positions (marked with dots in the figure), and laying off the chords 11, 22, 33, 44, &c., of their proper lengths upon those positions of the rolling circle, which chords will be a series of normals to the rolled curve A E.

III. *To approximate to the figure of an epicycloidal arc by means of one circular arc.* By the method of the preceding Rule draw the normal to the epicycloidal arc in question at a point near its middle. For example, if A 3 is the arc of the epicycloid A E, whose figure is to be approximated to by means of one circular arc, draw the normal 22 by Rule II. Then conceive that normal to be represented by A T in fig. 46, page 57; and by the method of Article 78 find the corresponding radius of curvature A F and centre of curvature F. A circular arc described about F, with the radius F A (fig. 46), will be an approximation to the epicycloidal arc.

This is the approximation used in Mr. Willis's method of designing teeth for wheels, to be described farther on. It ensures that the circular arc shall have, at or about the middle of its length, the same position, direction, and curvature with the epicycloidal arc for which it is substituted. Towards the ends of the arcs they gradually deviate from each other.

IV. *To approximate to the figure of an epicycloidal arc by means of two circular arcs.* This method of approximation is closer than the preceding, but more laborious. It substitutes for an epicycloidal arc a curve made up of two circular arcs; and the approximate curve coincides exactly with the true curve at the two ends and at one intermediate point, and has also the same tangents at its two ends

Suppose that A and B (fig. 48) are the two ends of the epicycloidal arc to which an approximation is required, and that A C and B C are normals to the arc at those points: the positions of the ends of the arc and directions of its normals having been determined by Rule II. of this Article. Let C be the point of intersection of the normals. Draw the tangents A D perpendicular to A C, and B D perpendicular to B C, meeting each other in D. Draw the straight line D C, and bisect it in E. About E, with the radius E D = E C, describe a circle, which will pass through the four points, A, D, B, C. Draw the diameter F E G, bisecting the arc A B in F and the arc B C A in G.

Draw the straight line G D, in which take G H = G A = G B. Through H, parallel to F E G, draw the straight line H K L, cutting A C in K and B C in L. Then about K, with the radius K A = K H, draw the circular arc A H; and about L, with the radius L H = L B, draw the circular arc H B: the curve made up of those two circular arcs will be a close approximation to the epicycloidal arc, having the same position and tangents at its two ends, and being very near to the true arc at all intermediate points.

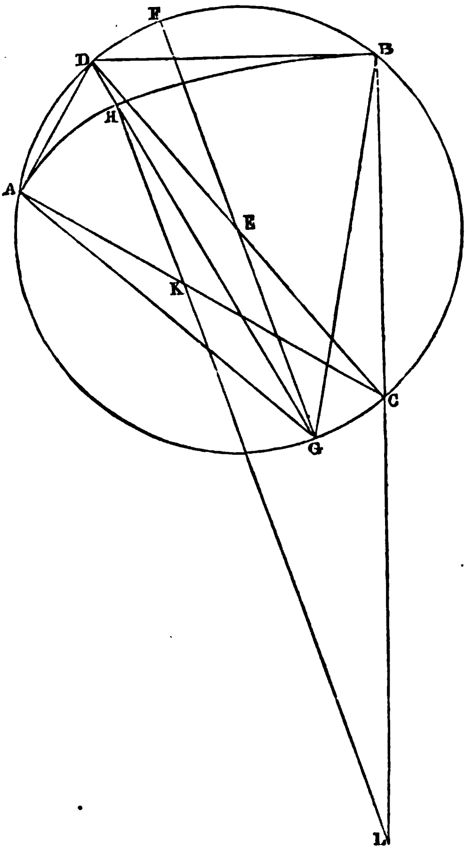

Fig. 48.

It may be remarked that $GH = GA = GB = \surd(HK \cdot HL)$ approximates very closely to the mean radius of curvature of the epicycloidal arc A B; also that the process described is applicable to the approximate drawing of many curves besides epicycloids; and that the ratio of the two radii, H L : H K, deviates less from equality than that of the radii of any other pair of circular arcs which can be drawn so as to touch A D in A and B D in B, and also to touch each other at an intermediate point.*

* This may be expressed symbolically by stating that $\left(\frac{HL^2 - HK^2}{HK \cdot HL}\right)^2$ is a minimum; or that $\left(\log. \frac{HL}{HK}\right)^2$ is a minimum.

80. **Resolution of Rotation in General.**—The following propositions show how the rotation of a rigid body about a given axis, fixed or instantaneous, may be resolved into two component rotations about any two axes in the same plane with the actual axis.

I. PARALLEL AXES.—*The rotation of a rigid body about a given axis is equivalent to the resultant of two component rotations about two axes parallel to the given axis and in the same plane, the angular velocity of each of the three rotations being proportional to the distance between the axes of the other two rotations.*

In fig. 49, let the plane of the paper be perpendicular to the three parallel axes, and let C be the trace of the axis of the resultant rotation, and A and B the traces of the axes of the component rotations; all three axes being in the same plane, whose trace is A C B. Let the angular velocities about

A, B, C,

be respectively proportional to

B C, C A, A B.

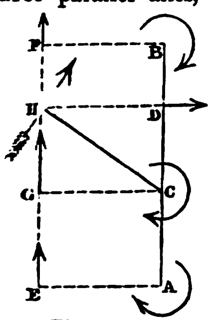

Fig. 49.

As the figure is drawn, all three angular velocities are of the same sign, because A B = B C + C A. If C lay beyond A and B, instead of between them, A B would be the difference of B C and C A, instead of their sum; and the lesser of these two distances and of the corresponding angular velocities would have to be considered as negative.

Let H be the projection of a particle in the rigid body, which particle is moving in a direction perpendicular to H C, with a velocity proportional to C H · A B. Then, *first*, from H let fall H D perpendicular to A B; then, by the principles of Article 55, page 33, the component velocity of H in the direction H D, whether due to rotation about A, B, or C, is the same with that of a particle at D. Now the velocities of a particle at D due to the rotations about

A, B, C

are proportional respectively to

$$+ AD \cdot BC; \ - BD \cdot CA; \ + CD \cdot AB;$$

and $CD \cdot AB = AD \cdot BC - BD \cdot CA$; therefore this component of the velocity of the particle H due to the rotation about C is the resultant of the corresponding components due to the rotations about A and B respectively.

Secondly. Through H draw E G H F parallel to A C B, and on it let fall the perpendiculars A E, B F, C G. Then, by the principles of Article 55, page 33, the component velocities of H along E F due to the rotations about the axes A, B, and C are respectively equal to the velocities of E due to rotation about A, of F due to rotation about B, and of G due to rotation about C; and because A E = B F = C G, these velocities are respectively proportional to

B C, C A, A B;

But A B = B C + C A; therefore the component along E F of the velocity of the particle H due to the rotation about C is the resultant of the corresponding component velocities due to the rotations about A and B respectively. Therefore the whole velocity of the particle H due to rotation about C, with an angular velocity proportional to A B, is the resultant of the velocities of the same particle due respectively to rotations about A, with an angular velocity proportional to B C, and about B, with an angular velocity proportional to C A. And this being true for every particle of the rotating body, is true for the whole body: Q. E. D.

II. Intersecting Axes.—*The rotation of a rigid body about a given axis is equivalent to the resultant of two component rotations about two axes in the same plane with the first axis, and cutting it in one point; the angular velocities of the component and resultant rotations being proportional respectively to the sides and diagonal of a parallelogram, which are parallel respectively to the three axes of rotation.*

In fig. 50 the upper right-hand part of the figure represents a plane perpendicular to the resultant axis of rotation, O″. F″ is the projection of any particle on that plane; and the direction of motion of any particle whose projection is F″ is perpendicular to O″ F″.

O″ Y″ and O″ Z″ are the traces of two planes perpendicular to the first plane of projection and to each other; and D″ and E″ are the projections of F″ on those planes respectively. According to the principle of Article 55, page 33, the component velocity parallel to O″ Y″ of the particle whose projection is F″ is the same with the velocity of a particle at D″; and its component velocity parallel to O″ Z″ is the same with that of a particle at E″.

The upper left-hand part of the figure represents the plane whose trace on the first plane of projection is O″ Z″; O′ X′, on this second plane, is the axis of rotation; O′ Z′ is the trace of the first plane of projection; and D′ is the projection of F″, and is the same point that is marked D″ on the first plane. The lower part of the figure represents the plane whose trace on the first plane of projection is O″ Y″, and on the second plane, O′ X′. On this third plane O X is the axis of rotation, and also the trace of the second plane; O Y is the trace of the first plane; E is the projection of

F″, and is the same point that is marked E″ on the first plane; O is the projection of D″.

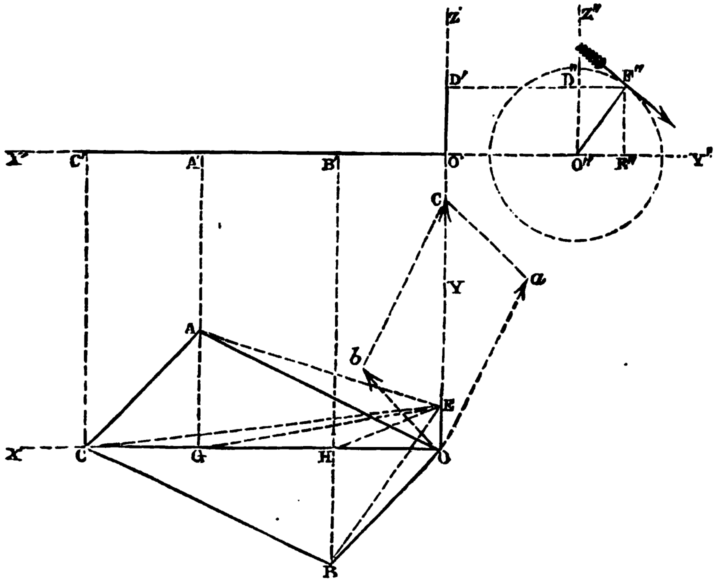

Fig. 50.

The positions of the second and third planes are arbitrary so long as they both traverse the axis of rotation, and are at right angles to each other.

In O X take O C proportional to the angular velocity; and make it point so that to an observer looking from O towards C the rotation shall seem right-handed.

From O draw any two lines, O A and O B, in the third plane; from C draw C B parallel to A O, and C A parallel to B O, so as to complete the parallelogram O B C A. Then the proposition states, that a right-handed rotation about O A, with an angular velocity proportional to O A, and a right-handed rotation about O B, with an angular velocity proportional to O B, being combined, are equivalent to the actual rotation.

To prove this for a particle at E, it is to be considered that the motions impressed on E by the three rotations separately are each of them perpendicular to the third plane; also, that the velocity of E due to any one of the three rotations is proportional to the angular velocity of that rotation multiplied by the perpendicular distance of E from the axis of that rotation, and is therefore proportional to the area of a triangle having for its base the length marked on that axis, to represent that angular velocity, and for its summit the point E; so that the velocities of a particle at E due respectively to the rotations about

O A, O B, O C

are proportional respectively to the areas of the triangles

O A E, O B E, O C E.

Through A and B draw A G and B H perpendicular to O C, and join E G and E H. Then, by plane geometry,

$$O A E = O G E; \text{ and } O B E = O H E = G C E;$$

therefore

$$O C E = O G E + G C E = O A E + O B E.$$

So that the velocity of E due to the actual rotation about O C is the resultant of the velocities due to the rotations about O A and O B; the angular velocities being proportional to the lengths laid off on the axes respectively.

To prove the same proposition for a particle at D″, whose projection on the third plane is O, it is to be considered that the perpendicular distance of this point from the three axes, O A, O B, and O C, is identical, being the line marked O″ D″ and O′ D′ on the first and second planes; so that the velocities of D due to the three rotations are simply proportional to the three angular velocities. To represent, then, those three velocities as projected on the third plane, draw O *a*, O *b*, and O *c* perpendicular and proportional respectively to O A, O B, and O C. It is evident that O *a*, O *b*, and O *c* are the sides and diagonal of a parallelogram similar to O B C A; and therefore that the velocity of D″ due to the actual rotation about O C is the resultant of the velocities due to the rotations about O A and O B, the angular velocities being proportional to the lengths laid off on the axes respectively.

The proposition, therefore, is proved for both components of the velocity of a particle at F″; and it holds for any particle whose projection on a plane perpendicular to the axis O C is F″; that is, for every particle of the body, and therefore for the whole body: Q. E. D.

It appears, then, that rotations, when represented by lengths laid off on their axes proportional to their angular velocities, can be compounded and resolved, like linear velocities, by constructing parallelograms.

81. **Rotations about Intersecting Axes Compounded.** (*A. M.*, 392).

—In fig. 51, let O A be a fixed axis, and about it let the plane A O C rotate with the angular velocity *a*. Let the plane of projection be that of those two axes at a given instant. Let O C be an axis in the rotating plane; and about that axis let a secondary piece rotate with the angular velocity *b* relatively to the rotating plane; and let it be required to find the instantaneous axis and the resultant angular velocity of the secondary piece. From the principles of Article 80, Proposition II., page 64, the following rule is deduced:—

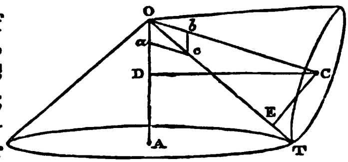

Fig. 51.

On O A take O *a* proportional to *a*; and on O C take O *b* proportional to *b*. Let those lines be taken in such directions that to an observer looking from O towards their extremities the component rotations shall seem both right-handed. Complete the parallelogram O *b c a*; the diagonal O *c* will be the instantaneous axis; and its length will represent *the resultant angular velocity.* Another mode of viewing the question is as follows:—

Because the point O in the secondary piece is fixed, the instantaneous axis must traverse that point. The direction of that axis is determined by considering that each point which arrives at that line must have, in virtue of the rotation about O C, a velocity relatively to the rotating plane, equal and directly opposed to that which the coincident point of the rotating plane has. Hence it follows that the ratio of the perpendicular distances of each point in the instantaneous axis from the fixed and moving axes respectively—that is, the ratio of the sines of the angles which the instantaneous axis makes with the fixed and moving axes—must be the reciprocal of the ratio of the component angular velocities about those axes; or if, in symbols, O T be the instantaneous axis,

$$\sin AOT : \sin COT :: b : a \text{..............} (1.)$$

The resultant angular velocity about this instantaneous axis is found by considering that if C be any point in the moving axis, the linear velocity of that point must be the same whether computed from the angular velocity, *a*, of the rotating plane about the fixed axis O A, or from the resultant angular velocity, *c*, of the rigid body about the instantaneous axis. That is to say, let C D, C E be perpendiculars from C upon O A, O T, respectively; then

$$a \cdot \overline{CD} = c \cdot \overline{CE};$$

but $\overline{CD} : \overline{CE} :: \sin AOC : \sin COT$; and therefore

$$\sin COT : \sin AOC :: a : c;$$

and, combining this proportion with that given in equation 1, we obtain the following proportional equation :—

$$\left.\begin{array}{lccccc} & \sin \mathrm{C\,O\,T} & : \sin \mathrm{A\,O\,T} & : \sin \mathrm{A\,O\,C} \\ :: & a & : & b & : & c \\ :: & \mathrm{O}\,a & : & \mathrm{O}\,b & : & \mathrm{O}\,c \end{array}\right\} \ldots\ldots(2.)$$

That is to say, *the angular velocities of the component and resultant rotations are each proportional to the sine of the angle between the axes of the other two;* and *the diagonal of the parallelogram* O b c a *represents both the direction of the instantaneous axis and the angular velocity about that axis.*

82. **Rolling Cones.** (*A. M.*, 393.)—All the lines which successively come into the position of instantaneous axis are situated in the surface of a cone described by the revolution of O T about O C; and all the positions of the instantaneous axis lie in the surface of a cone described by the revolution of O T about O A. Therefore the motion of the secondary piece is such as would be produced by the rolling of the former of those cones upon the latter. Circular sections of the two cones are sketched in perspective in fig. 51.

It is to be understood that either of the cones may become a flat disc, or may be hollow, and touched internally by the other. For example, should ∠ A O T become a right angle, the fixed cone would become a flat disc; and should ∠ A O T become obtuse, that cone would be hollow, and would be touched internally by the rolling cone; and similar changes may be made in the rolling cone.

The path described by a point in or attached to the rolling cone is a *spherical epitrochoid;* and if that point is in the surface of the rolling cone, that curve becomes a *spherical epicycloid.* It will be shown in the next chapter how to draw such curves—not exactly, but with a degree of accuracy sufficient for practical purposes.

83. **Resolution of Helical Motion.**—The resolution of helical or screw-like motion into rotation about an axis and translation along that axis has already been treated of in the last section of the preceding chapter. It remains to be shown how a helical motion may be regarded as compounded of two rotations about two axes which are in different planes.

In fig. 52, let the lower part of the figure represent a plane of projection, and O A and O B the projections upon that plane of two axes which are both parallel to it, but not in the same plane. Let the upper part of the figure represent a second plane of projection perpendicular to the first plane; and let F′ G′ be the projection on that second plane of the *common perpendicular* of those two axes (Article 36, page 14). Let a rigid body have a motion compounded of two rotations about the two axes respectively, with angular velocities represented by O A and O B, these lines being drawn, as before, so that to an observer at O each rotation shall appear right-handed.

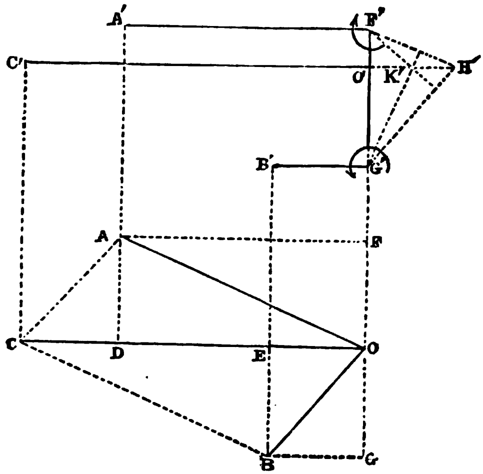

Fig. 52.

Complete the parallelogram O A C B, and draw its diagonal O C. Then, if the axes O A and O B were in the same plane, the rotations about them, being combined, would be equivalent simply to a rotation represented by O C, as in Articles 80 and 81, pages 64 to 66.

Let the second plane of projection be now supposed parallel to O C; and let F' A', O' C', and G' B' be the respective projections of O A, O C, and O B upon it. Draw A D, B E, and F O G perpendicular to O C, and A F and B G parallel to O C. It is obvious that O D = F' A', O E = G' B', and F O = O G.

According to Article 80, Proposition II., page 64, the rotation represented by O A may now be regarded as compounded of a rotation represented by O D, about an axis of which O D and F' A' are the projections, and a rotation represented by O F, about an axis of which O F and the point F' are the projections; also, the rotation represented by O B may be regarded as compounded of a rotation represented by O E, about an axis of which O E and G' B' are the projections, and a rotation represented by O G, about an axis of which O G and the point G' are the projections.

Then, according to Article 76, page 54, the rotations about the parallel axes F' A' and G' B', being combined, are equivalent to a rotation about an intermediate axis, O′ C′, in the same plane, with an angular velocity represented by

$$O\,C = O'\,C' = F'\,A' + G'\,B';$$

and that axis of resultant rotation divides the distance F′ G′ in the following proportion:—

$$O'\,C' : F'\,A' : G'\,B'$$
$$:: F'\,G' : O'\,G' : O'\,F'.$$

To find the point O′ by graphic construction, draw F′ H′, parallel to A O and G′ H′ parallel to B O, cutting each other in H′; then through H′ draw H′ O′ C′ parallel to O C.

Moreover, the component rotations represented by O F and O G, about the axes F′ and G′, are of equal and opposite angular velocities; and therefore, according to Article 76, page 54, they are equivalent to a translation in the direction O C, with a velocity represented by the product O F · F G.

That translation being compounded with the resultant rotation represented by O C, gives finally, for the resultant motion of the body, *a helical motion about the axis whose projections are* O C *and* O C′.

The *pitch* of that helical motion, or advance per turn, is found by multiplying the rate of advance, O F, F′ G′, by the time of one revolution, $\frac{6{\cdot}2832}{O\,C}$; and is therefore equal to the *circumference of a circle whose radius is* $\frac{O\,F \cdot F'\,G'}{O\,C}$. Draw F′ K′ perpendicular to O B, and G′ K′ perpendicular to O A, cutting each other in K′ (which will be in the straight line H′ O′ C′). Then it is evident that F′ K′ G′ and C A O are similar triangles; and because D A = O F, we have the following proportion:—

$$O\,C : O\,F :: F'\,G' : O'\,K' = \frac{O\,F \cdot F'\,G'}{O\,C};$$

Therefore *the pitch of the resultant helical motion is equal to the circumference of a circle whose radius is* O′ K′; and *the rate of advance may be represented by the product* O C · O′ K′.

84. **Rolling Hyperboloids.**—Conceive the straight line O C to represent an indefinitely long straight edge, rigidly fastened to the arm O′ F′, and sweeping along with that arm round the axis O A; then conceive the same straight line to be rigidly fastened to the arm O′ G′, and to sweep along with this arm round the axis O B. Thus are generated a pair of surfaces called *Rolling Hyperboloïds*,

which touch each other all along the straight line O C. Fig. 53 shows the general appearance of a pair of rollers of that form; and in fig. 54 the projections of their figures are given with greater precision. If one of those bodies is fixed, and the other made to roll upon it, they continue to touch each other in a straight line, which is the instantaneous axis of the rolling body; and the rotation about that instantaneous axis is accompanied by a sliding motion along the same axis, so as to give, as the resultant compound motion, a helical motion about the instantaneous axis, as described in the preceding Article. The following problem sometimes occurs in mechanism:—

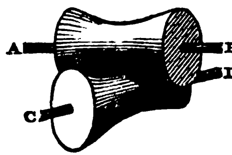

Fig. 53.

Given, the angle between the directions of two axes, and the length of their common perpendicular, to draw the projections of a pair of rolling hyperboloïds of which these shall be the axes, and of which one shall roll on the other, so as to have component angular velocities bearing to each other a given ratio.

Let the lower part of fig. 54 (see next page) represent a plane to which the two axes are parallel; and let O *a* and O *b* be their projections on that plane, with lengths laid off upon them proportional to the intended component angular velocities. Draw *b c* parallel to O *a*, and *a c* parallel to O *b*, cutting each other in *c*; O *c* will be the projection of the *line of contact*, or instantaneous axis; and the length O *c* will represent the resultant angular velocity (as in the preceding Article).

Through O, perpendicular to O *c*, draw O G′ F′, and lay off upon it G′ F′ equal to the given common perpendicular; and let the second plane of projection be perpendicular to the first plane, and parallel to O *c* and G′ F′. To find the projection of the line of contact upon this second plane, proceed as in the preceding Article; that is, draw F′ H′ and G′ H′ parallel respectively to O *a* and O *b*, and H′ O′ parallel to O *c*; H′ O′ will be the required projection. This projection may also be found, if convenient, by either of the following methods: Draw G′ K′ perpendicular to O *a*, and F′ K′ perpendicular to O *b*, cutting each other in K′; and then draw H′ K′ O′ parallel to A *c*; or otherwise:—Draw *g c f* perpendicular to O *c*, and divide F′ G′ in the following proportion:—

$$fg : cf : cg$$
$$:: F'G' : O'F' : O'G'.$$

Draw U O Y perpendicular to O *a*, making O U = O Y = F′ O′; also draw V O Z perpendicular to O *b*, making O V = O Z = G′ O′; then U O Y and V O Z will be the projections on the first plane of the *smallest transverse sections*, or what may be called the "*throats*" of the two hyperboloïds; which transverse sections are

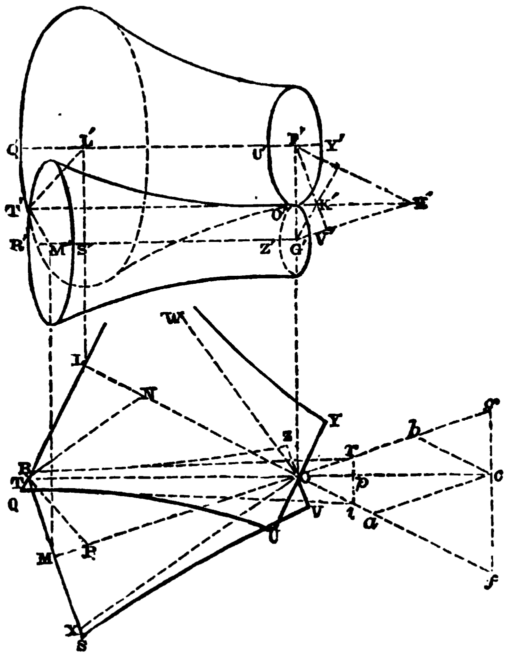

Fig. 54.

circles of the respective radii F′ O′ and G′ O′. The projections of those circles on the second plane of projection are the ellipses U′ O′ Y′ and V′ O′ Z′, drawn according to the principles of Article 37, page 15.

To find the projections of a pair of circular transverse sections of the two hyperboloïds, which shall cross each other in any given point of the line of contact, let T and T′ be the projections of that point. Then draw T L perpendicular to *a* O, and T M perpendicular to *b* O; L and M will be the projections of the centres of those circular sections on the first plane. Draw F′ L′ and G′ M′ parallel, and L L′ and M′ M′ perpendicular to O *c*; L′ and M′ will be the projections of those centres on the second plane. In L O take L N = F′ O′, and join N T; then in L T produced, take L Q = N T; this will be the radius of the required section of one hyperboloïd; and Q will be a point in the hyperbola

U Q, which is the longitudinal section or trace of that surface on a plane traversing the axis F′ L′, and parallel to the first plane of projection. Also, in M O take M P = G′ O′, and join P T; then in M T produced take M R = P T; this will be the radius of the required section of the other hyperboloïd; and R will be a point in the hyperbola Z R, which is the longitudinal section or trace of this surface on a plane traversing the axis G′ M′ and parallel to the first plane of projection.

The projection, O T, of the line of contact is an asymptote to both hyperbolas, U Q and Z R; and their other asymptotes are O W, making L O W = L O T, and O X, making M O X = M O T.

The projections on the second plane of projection of the two circular transverse sections which cross each other at the point whose projections are T and T′ are two ellipses, drawn according to the principles of Article 37, page 15.

By the same process may be found the projections of any required number of transverse sections of the two rolling hyperboloïds, and of any required number of points, such as Q and R, in their longitudinal sections.

Additional rules relating to the construction of such figures will be given in the next chapter, in the articles which treat of their application to skew-bevel wheels.

85. **Cylinder Rolling Obliquely.**—The same kind of resultant motion will take place, if for the rolling hyperboloïds there be substituted a pair of cylinders described about the axes whose projections are O A and O B, fig. 52, page 69, with the respective radii O′ F′ and O′ G′; provided the axis of the rolling cylinder is guided so that the point where it is met by the common perpendicular F′ G′ shall revolve in a circle of the radius F′ G′ round the axis of the fixed cylinder, and so that the inclination of those two axes to each other shall remain constant. The general appearance of such a pair of cylinders is shown in fig. 55. They touch each other in a point only, and not along a straight line, as the hyperboloïds do. The uniform transverse sections of such a pair of cylinders are identical with those at the *throats* of the corresponding pair of hyperboloïds. Further explanations as to obliquely-rolling cylinders will be given in the next chapter, under the head of screw-gearing.

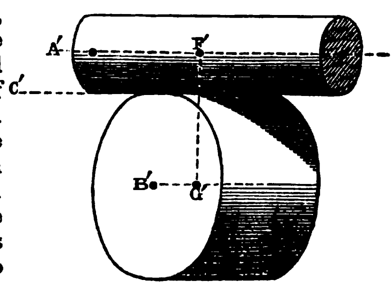

Fig. 55.

86. **Cones Rolling Obliquely.** — The same kind of resultant

motion may be effected also by substituting for the pair of hyperboloïds of fig. 54, page 72, the pair of cones which touch those hyperboloïds in the pair of circles that cross each other in any given point, T, of the instantaneous axis. To draw the projections of those tangent cones, let O′ T′ in fig. 56 (as in fig. 54) be the instantaneous axis, O′ the point where it cuts the common perpendicular, and T′ the intended point of contact of the cones. From O′, perpendicular to O′ T′, draw O′ h = O′ H′ in fig. 56; join T′ h; and perpendicular to T′ h draw $h\,p'$, cutting the instantaneous axis in p'. Then a plane normal to the instantaneous axis at p' passes through the summits of both the required tangent cones. Therefore, in fig. 54,

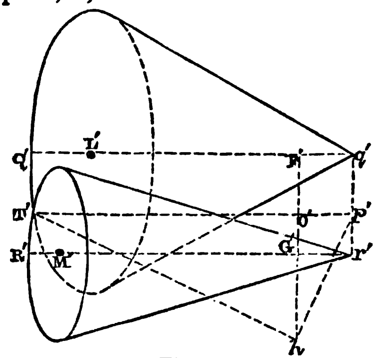

Fig. 56.

on O c lay off O p = O′ p' of fig. 56, and draw $q\,p\,r$ perpendicular to O c, cutting O a in q and O b in r; q and r will be *the projections of the summits of the tangent cones on the first plane of projection.* The projections on the same plane of the longitudinal sections or traces of these cones, upon planes traversing their axes parallel to the plane of projection, are q Q and r R. Also, let the plane of fig. 56 be the second plane of projection, and let F′ L′ and G′ M′, as in fig. 54, be the projections of the axes of the hyperboloïds, and F′ O′ G′ that of the common perpendicular. Draw $q'\,p'\,r'$, cutting those axes in q' and r'; these points will be *the projections of the summits of the tangent cones on the second plane of projection.* The projections of the bases of these cones on the same plane are the pair of ellipses, with L′ and M′ for their centres, which cross each other at the point T′, as in fig. 54. The cones touch each other in the point T′ only, and not along a straight line, as the hyperboloïds do. Further explanations as to obliquely-rolling cones will be given in the next chapter, under the head of skew-bevel wheels.

87. **Bands or Flexible Secondary Pieces—Cords—Belts—Chains.** (*A. M.*, 400, 401.)—The flexible pieces used in machinery may be classed under three heads:—*Cords*, which approximate to a round form in section; *Belts*, which are flat; and *Chains*, which consist of a series of rigid links so connected together that the chain as a whole is flexible. Mr. Willis gives them all the common name of *wrapping connectors;* and for the sake of brevity in stating principles that apply to them all, they may conveniently be called *bands*.

In treating of questions of pure mechanism, the *centre line* of a band is treated as being of invariable length; for although no substance is absolutely inextensible, and although when a band passes over a curved surface the concave side is shortened and the convex side lengthened, still the variations of length of the centre line of the band are, or ought to be, practically inappreciable.

In order that the figure and motion of a band may be determined from geometrical principles alone, independently of the magnitude and distribution of forces acting on it, its weight must be insensible compared with the tension on it, and it must everywhere be *tight;* and when that is the case, each part of the band which is not straight is maintained in a curved figure by passing over a *convex* surface. When a band is guided by a given actual surface, the centre line of that band may be regarded as guided by an imaginary surface parallel to the actual surface, and at a distance from it equal to half the thickness of the band. The line in which the centre line of a band lies on such guiding surface is the *shortest line* which it is possible to draw on that surface between each pair of points in the course of the band. (It is a well-known principle of the geometry of curved surfaces that the *osculating plane* at each point of such a line is perpendicular to the curved surface.)

Hence it appears that the motions of a tight flexible band, of invariable length along its centre line and insensible weight, are regulated by the following principles:—

I. *The length between each pair of points in the centre line of the band is constant.*

II. *That length is the shortest line which can be drawn between its extremities over the surface by which the centre line of the band is guided.*

The motions of a band are of two kinds—

I. Travelling of a band along a track of invariable form; in which case the velocities of all points of the centre line are equal.

II. Alteration of the figure of the track by the motion of the guiding surfaces.

Those two kinds of motion may be combined.

The most usual problems in practice respecting the motions of bands are those in which bands are the means of transmitting motion between two pieces in a train of mechanism. Such problems will be considered in the next chapter.

88. **Fluid Secondary Pieces.**—A mass of fluid may act as a secondary piece in a machine; and in order that the motion of such a mass may be a subject of pure mechanism, the volume occupied by the mass must be constant; and that not only for the whole mass, but for every part of it, how small soever. In other words, the fluid mass must in every part be of constant *bulkiness;*

this word being used to denote the volume filled by an unit of mass; for example, the number of cubic feet filled by a pound, or the number of cubic metres filled by a kilogramme. Every fluid, whether liquid or gaseous, undergoes variations of bulkiness through variations of pressure and of temperature; but in mechanism such variations of bulkiness may be either so small that they may be disregarded for the practical purpose under consideration (as in the case of most liquids), or, if the fluid employed be gaseous, they may be prevented by keeping the pressure and temperature constant.

Under such conditions the motions of the particles of a fluid mass are regulated by the following principle:—

At a given series of sections of a stream of fluid of constant bulkiness, the mean velocities at each instant of the particles in directions normal to those sections respectively, are inversely proportional to the areas of the sections.

CHAPTER IV.

OF ELEMENTARY COMBINATIONS IN MECHANISM.

SECTION I.—*Definitions, General Principles, and Classification.*

89. **Elementary Combinations Defined.** (*A. M.*, 431.)—An "Elementary Combination" in Mechanism (a term introduced by Mr. Willis) consists of a pair of primary moving pieces, so connected that one transmits motion to the other. In other words, (to quote the Article *Mechanics* (*Applied*), in the eighth edition of the *Encyc. Brit.*)—

"An *elementary combination* in mechanism consists of two pieces whose kinds of motion are determined by their connection with the frame, and their comparative motion by their connection with each other; that connection being effected either by direct contact of the pieces, or by a connecting" (secondary) "piece" (such as a band, or a link, or a mass of fluid), "which is not connected with the frame, and whose motion depends entirely on the motions of the pieces which it connects."

"The piece whose motion is the cause is called the *driver;* the piece whose motion is the effect, the *follower.*"

"The connection of each of those two pieces with the frame is in general such as to determine the path of every moving point. In the investigation, therefore, of the comparative motion of the driver and follower, in an elementary combination, it is unnecessary to consider relations of angular direction, which are already fixed by the connection of each piece with the frame; so that the inquiry is confined to the determination of the velocity-ratio, and of the directional-relation so far only as it expresses the connection between *forward* and *backward* movements of the driver and follower. When a continuous motion of the driver produces a continuous motion of the follower, forward or backward, and a reciprocating motion a motion reciprocating at the same instant, the directional-relation is said to be *constant.* When a continuous motion produces a reciprocating motion, or *vice versâ;* or when a reciprocating motion produces a motion not reciprocating at the same instant, the directional-relation is said to be *variable.*"

90. **Line of Connection.**—In every class of elementary combinations, except those in which the connection is made by reduplication of cords, or by an intervening fluid, there is at least one straight

line called the *line of connection* of the driver and follower; being a line traversing a pair of points in the driver and follower respectively, which points are so connected that the component of their velocity relatively to each other, resolved along the line of connection, is null.

91. **Comparative Motions of Connected Points and Pieces.**—From the definition of a line of connection it follows, that *the components of the velocities of a pair of connected points along their line of connection are equal.* And from this, and from the property of a rigid body already stated in Article 54, page 32, it follows, that *the components, along a line of connection, of all the points traversed by that line, whether in the driver or in the follower, are equal.*

The general principle which has just been stated serves to solve every problem in which—the mode of connection of a pair of pieces being given—it is required to find their comparative motion at a given instant, or *vice versâ.*

The following are the rules for applying that principle to the three classes of problems which most frequently occur with reference to elementary combinations:—

I. *Pair of Points; or Pair of Sliding Pieces.*—In fig. 57, let A B be a line of connection; and let it be taken as the axis of projection. Let A be a point in the driver, and B a point in the follower, both in the line of connection. Let A a', A a'' be the two projections of the direction of motion of A at a given instant; and let B b', B b'' be the two projections of the direction of motion of B at the same instant. Lay off, along the line of connection and in the same direction, the equal distances A a = B b; draw a'' a a', b'' b b' perpendicular to the line of connection; then A a' and A a'', B b' and B b'' will be the projections of a pair of lines proportional respectively to the velocities of A and B at that instant. The lengths of those lines may be found by the Rule of Article 19, page 7.

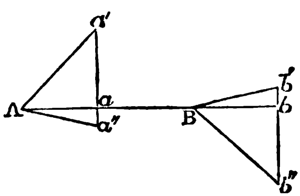

Fig. 57.

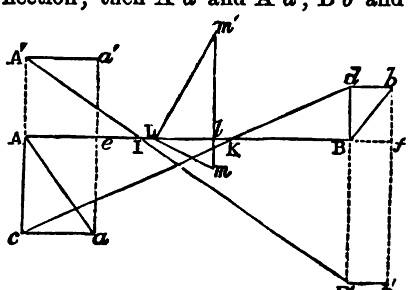

Fig. 58.

II. *Pair of Turning Pieces.*—In fig. 58, let A B be the line of connection of a pair of turning primary pieces. Let A and B be the points where that line is met by the common perpendiculars from the axes of rotation of the two pieces. (As to finding such common perpendiculars, see Article 36, page 15.) Let A A′ and B B′ be the *rabatments* of those two perpendiculars, drawn in opposite directions. Draw the straight line A′ B′ (called the *line of centres*), cutting the line of connection in I.

Then, because the component velocities of A and B along A B are equal, the angular velocities (or the component angular velocities) of the driver and follower about axes perpendicular to A B must be to each other *in the inverse ratio of the perpendiculars* A A′ *and* B B′; or, what is the same thing, *in the inverse ratio of the segments* I A′ *and* I B′ *into which the line of centres is cut by the line of connection.*

Hence the following construction:—In A B take A K = B I (or B K = A I); and through K draw an oblique straight line in any convenient direction, so as to cut A′ A produced in *c* and B′ B produced in *d*; then the component angular velocities of the pieces about two axes, A *c* and B *d*, perpendicular to the line of connection, will be to each other in the direct ratio of A *c* to B *d*. Also lay off, in opposite directions, the angles B A *a* and *f* B *b*, equal to the angles which the two axes of rotation respectively make with the line of connection, and draw *c a* and *d b* parallel to A B, cutting A *a* and B *b* in *a* and *b* respectively. Then *the ratio of* A *a* to B *b will be that of the resultant angular velocities of the two pieces.*

Through A′ and B′ draw A′ *a′* and B′ *b′* parallel to A B; and through *a* and *b* draw *a e a′* and *b f b′* perpendicular to A B. Then the proportion borne by *c a* = A *e* = A′ *a′* to *d b* = B *f* = B′ *b′* is that of the component angular velocities of the two pieces about axes parallel to the line of connection A B. Also A *a* and A′ *a′* represent the projections of the axis of rotation of the first piece upon a pair of planes which cut each other in A *e*, one perpendicular and the other parallel to the common perpendicular whose rabatment is A A′; and B *b* and B′ *b′* represent the projections of the axis of rotation of the second piece upon a pair of planes which cut each other in B *f*, one perpendicular and the other parallel to the common perpendicular whose rabatment is B B′

III. *Turning Piece and Sliding Piece.*—In fig. 58, let A L be the line of connection of a turning piece and a sliding piece, and let it be taken for the axis of projection; and let one of the planes of projection be parallel to the axis of the turning piece. Let A *a* and A *a′* be the projections of that axis; so that A A′ perpendicular to A L is the common perpendicular of the axis and the line of connection. Take A *a* to represent the angular velocity of the turning piece, and from *a* draw *a c* parallel to L A, cutting A′ A (produced if necessary) in *c*. Then A *c* will represent the component angular velocity of the turning piece about an axis, A *c*, perpendicular to A L; and the product A A′ · A *c* will represent *the component velocity of any point in* A L *along that line.*

Let L be a point in the line of connection and in the sliding piece; and let L *m* and L *m′* be the projections of the direction of motion of that piece. Lay off any convenient length, L *l*, to represent the component velocity of the sliding piece along the line of connection, and draw *m l m′* perpendicular to that line; then L *m* and L *m′* will represent *the two projections of the velocity of the sliding piece.*

Another construction is as follows:—Having determined the angle which the direction of motion of the sliding piece makes with the line of connection A L, draw A′ I, making the angle A A′ I equal to that angle; then the velocity of the sliding piece will be equal to that of *a point revolving at the end of the arm* A′ I, *with the angular velocity represented by* A *c*.

92. **Adjustments of Speed.**—The velocity-ratio of a driver and its follower is sometimes made capable of being changed at will, by means of apparatus for varying the position of their line of connection: as when a pair of rotating cones are embraced by a belt which can be shifted so as to connect portions of their surfaces of different diameters. Various such contrivances will be described in a later chapter.

93. A **Train of Mechanism** consists of a series of moving pieces, each of which is follower to that which drives it, and driver to that which follows it. In the case of a train of elementary combinations the comparative motion of the last follower and first driver is found by multiplying together all the velocity-ratios of the several elementary combinations of which the train consists, each ratio having the directional-relation with which it is connected denoted by means of the positive or negative algebraical sign, as the case may be. The product is the velocity-ratio of the last follower and first driver; and their directional-relation is indicated by the algebraical sign of that product, found by the rules, that any number of positive factors, and any even number of negative factors, give a positive product; and that any odd number of negative factors give a negative product.

94. **Elementary Combinations Classed.**—The only classification of elementary combinations that is founded, as it ought to be, on comparative motion, as expressed by velocity-ratio and directional-relation, is that first given by Mr. Willis in his Treatise on *Pure Mechanism.* Its general plan is as follows:—

Class A: Directional-relation constant; velocity-ratio constant.

Class B: Directional-relation constant; velocity-ratio varying.

Class C: Directional-relation changing periodically; velocity-ratio constant or varying.

Each of those classes is subdivided by Mr. Willis into five divisions, of which the characters are as follows:—

Division I.—Connection by *rolling contact* of surfaces, as in toothless wheels.

— II.—Connection by *sliding contact* of surfaces; as in toothed wheels, cams, &c.

— III.—Connection by *wrapping connectors* or *bands*, as in pulleys connected by belts, cords, or chains.

— IV.—Connection by *link-work*, as in levers and cranks connected by means of rods, &c.

— V.—Connection by *reduplication* of cords, as in blocks and tackle used on board ship;

and to those five divisions may be added—

Division VI.—Connection by an *intervening fluid*, as in the hydraulic press.

In the present treatise the principle of the classification of Mr. Willis is followed; but the arrangement (as in a *Manual of Applied Mechanics*, already referred to) is modified by taking the *mode of connection* as the basis of the primary classification.

With reference to classes B and C, in which the velocity-ratio is or may be varying, it is to be observed that two kinds of problems arise respecting velocity-ratio: the determination of the *instantaneous velocity-ratio* at the instant when the pieces are in one given position; and the determination of the *mean velocity-ratio* during the interval between two such instants: the latter quantity is the ratio of the entire motions of the pieces during the interval.

SECTION II.—*Of Rolling Contact and Pitch Surfaces.*

95. **Pitch Surfaces** are those surfaces of a pair of moving pieces which touch each other when motion is communicated by rolling contact. The LINE OF CONTACT is that line which at each instant traverses all the pairs of points of the pair of pitch surfaces which are in contact.

The motion, relatively to the line of contact of their surfaces, of a pair of primary pieces which move in rolling contact, is the same with that of a secondary piece and a fixed piece, of which the former rolls upon the latter, as already described in Article 72, page 51; Articles 74 and 75, pages 53, 54; Article 77, page 56; Article 82, page 68, and Articles 84, 85, 86, pages 70 to 74; and therefore the proper figures for the pitch surfaces of such primary pieces are the same; that is to say, cylinders, cones, and hyperboloïds.

96. **Toothless Wheels, Rollers, Toothless Racks.**—Of a pair of

primary moving pieces in rolling contact, both may rotate, or one may rotate and the other have a motion of straight sliding. A rotating piece, in rolling contact, is called a *toothless wheel*, and sometimes a *roller;* a sliding piece may be called a *toothless rack*.

97. **Ideal Pitch Surfaces.**—The designing of pitch surfaces is used not only with a view to the making of toothless wheels and toothless racks (which are seldom employed), but much oftener as a step towards determining the proper figures for wheels and racks provided with teeth.

The pitch surface of a toothed wheel or of a toothed rack is an ideal smooth surface, intermediate between the crests of the teeth and the bottoms of the spaces between them, which, by rolling contact with the pitch surface of another wheel, would communicate the same velocity-ratio that the teeth communicate by their sliding contact. In designing toothed wheels and racks the forms of the ideal pitch surfaces are first determined, and from them are deduced the forms of the teeth.

Wheels with cylindrical pitch surfaces are called *spur wheels;* those with conical pitch surfaces, *bevel wheels;* and those with hyperboloidal pitch surfaces, *skew-bevel wheels.*

98. The **Pitch Line** of a wheel, or, in circular wheels, the PITCH CIRCLE, is the trace of the pitch surface upon a surface perpendicular to it and to the axis; that is, in spur wheels, upon a plane perpendicular to the axis; in bevel wheels, upon a sphere described about the apex of the conical pitch surface; and in skew-bevel wheels, upon an oblate spheroid generated by the rotation of an ellipse whose foci are the same with those of the hyperbola that generates the pitch surface. The pitch line might be otherwise defined, in most cases which occur in practice, simply as the trace of the pitch surface upon a plane perpendicular to the axis of rotation.

The PITCH POINT of a pair of wheels is the point of contact of their pitch lines; that is, the trace of the line of contact upon the surface or surfaces on which the pitch lines are traced.

The *pitch line of a rack* is the trace of its pitch surface on a plane parallel to its direction of motion, and containing its line of connection with the wheel with which it works.

A SECTOR is a name given to a wheel whose pitch-line forms only part of a circumference: sectors are used where the motion required is reciprocating or "rocking," and does not extend to a complete revolution. Everything stated respecting the figures of complete wheels applies also to the figures of sectors.

99. **General Conditions of Perfect Rolling Contact.** (*A. M.*, 439.) —The whole of the principles which regulate the motions of a pair of primary pieces in perfect rolling contact follow from the single principle, *that each pair of points in the pitch surfaces which are in*

contact at a given instant must at that instant be moving in the same direction with the same velocity.

The direction of motion of a point in a rotating body being perpendicular to a plane passing through its axis, the condition, that each pair of points in contact with each other must move in the same direction, leads to the following consequences:—

I. That when both pieces rotate, their axes, and all their points of contact, lie in the same plane.

II. That when one piece rotates and the other slides, the axis of the rotating piece, and all the points of contact, lie in a plane perpendicular to the direction of motion of the sliding piece.

The condition, that the velocities of each pair of points of contact must be equal, leads to the following consequences:—

III. That the angular velocities of a pair of wheels, in rolling contact, must be inversely as the perpendicular distances of any pair of points of contact from the respective axes.

IV. That the linear velocity of a rack in rolling contact with a wheel is equal to the product of the angular velocity of the wheel by the perpendicular distance from its axis to a pair of points of contact.

Respecting the line of contact, the above principles III. and IV. lead to the following conclusions:—

V. That for a pair of wheels with parallel axes, and for a wheel and rack, the line of contact is straight, and parallel to the axes or axis; and hence that the pitch surfaces are either cylindrical or plane (the term "cylindrical" including all surfaces generated by the motion of a straight line parallel to itself).

VI. That for a pair of wheels with intersecting axes the line of contact is also straight, and traverses the point of intersection of the axes; and hence that the rolling surfaces are conical, with a common apex (the term "conical" including all surfaces generated by the motion of a straight line which traverses a fixed point).

There is a sort of *imperfect rolling contact* which takes place between hyperboloidal pitch surfaces; that is to say, there is a sliding motion, but along the line of contact of the surfaces only; so that the component motions of points in directions perpendicular to the line of contact are the same as in perfect rolling contact. This kind of motion will be considered in treating specially of skew-bevel wheels.

100. **Wheels with Parallel Axes.**—Given, the positions of the parallel axes of a pair of wheels, and their velocity-ratio at a given instant, to find the pitch-point. Fig. 59 represents the case in which the directions of the rotations are contrary; fig. 60 that in which they are the same. Let the plane of projection be perpendicular to the two axes, and let A and B be their traces; so that A B is the line of centres. Perpendicular to A B draw A *a* and B *b*

proportional to the intended angular velocities. Draw the straight line $a\ b$, cutting A B (produced if necessary) in K. Lay off B I =

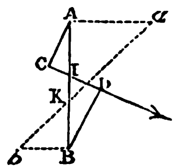

Fig. 59.

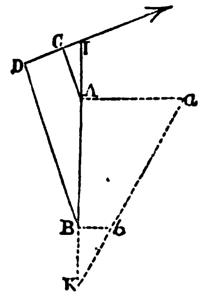

Fig. 60.

A K (or A I = B K); I will be the *required pitch-point* or trace of the line of contact.

The line of connection may be any straight line which traverses I, or whose projection traverses I; as C D. Let A C and B D be perpendicular to the line of connection; then the velocities of the points C and D are identical; and the perpendiculars A C and B D are inversely as the angular velocities of the pieces.

101. **Wheel and Rack.**—Given, at a given instant, the angular velocity of a wheel and the linear velocity of a rack, to find their pitch-point. In fig. 61, let the plane of projection be perpendicular to the axis of the wheel, and let A be the trace of that axis. Draw A I perpendicular to the direction of motion of the rack, and make its length such that a point in the wheel at I shall revolve with a velocity equal to that of the rack; that is to say, make

$$\text{A I} = \frac{\text{linear velocity of rack}}{\text{angular velocity of wheel}};$$

then I will be *the required pitch-point.*

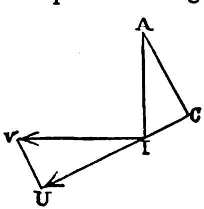

Fig. 61.

The line of connection may be any straight line which traverses I, or whose projection traverses I; as C U. Let fall the perpendicular A C; then the velocity of the point C in the wheel is equal to the component velocity of the rack along C U. Draw I V perpendicular to A I, to represent the whole velocity of the rack, and from V draw V U perpendicular to C U; it is evident that I U is the component velocity along the line of connection; and that A I : A C : : I V : I U.

102. **Circular Wheels in General.**—In order that, in an elementary combination of wheels, or of a wheel and rack, the velocity-

ratio may be constant (so that the combination shall belong to Mr. Willis's class A), it is obviously necessary that the pitch-point during the entire revolution of each wheel should occupy an invariable position in the line of centres; in other words, the pitch-line of each wheel must be a circle, and that of a rack a straight line. The corresponding forms of pitch-surface are:—for a spur-wheel, a circular cylinder; for a bevel-wheel, a cone with a circular base, and sometimes a plane circular disc; for a rack, a plane; for a skew-bevel wheel, a hyperboloïd of revolution. Circular wheels are by far the most common, the cases in which non-circular wheels are used being comparatively rare.

103. **Circular Spur-Wheels.**—Given, a pair of parallel axes and the constant velocity-ratio of a pair of wheels which are to turn about them, to draw the pitch-circles of those wheels. Fig. 62 represents the case in which the directions of rotation are contrary; fig. 63 that in which they are the same. Let A and B, as before, be the traces of the axes on a plane perpendicular to them. Find

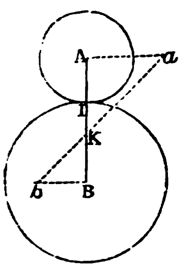

Fig. 62.

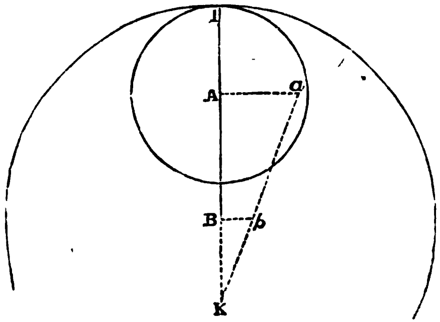

Fig. 63.

the pitch-point, I, as in Article 100, page 83. Then, about A and B, with the radii A I and B I respectively, draw two circles; these will be the pitch-circles required.

In fig. 62, where the rotations are contrary, and the pitch-point between the axes, the pitch-surfaces are both convex, and are said to be in "*outside gearing.*" In fig. 63, where the rotations are in the same direction, and the pitch-point beyond the axis of most rapid rotation, the smaller pitch-surface is convex and the larger concave; and these are said to be in "*inside gearing.*"

104. **Circular Wheel and Straight Rack.**—Given, the axis of a wheel, the direction of motion of a rack perpendicular to that axis, and the distance from the axis of a point in the wheel whose velocity is to be equal to that of the rack, to draw the pitch-lines of the wheel and rack. In fig. 64, let A be the trace

of the axis on a plane perpendicular to it. Draw A I perpendicular to the direction in which the rack is to move, and of a length equal to the given distance; then, about A, with the radius A I, draw a circle, and through I draw a straight line, M N, touching that circle; these will be the required pitch-lines.

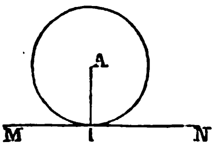

Fig. 64.

105. **Circular Bevel Wheels.**—Given, a pair of axes which intersect each other in a point, and the constant velocity-ratio of two wheels which are to turn about those axes, to draw projections of the pitch-surfaces of those wheels. Let the plane of fig. 65 represent the plane of the two axes; let O A and O B be their positions, and O their point of intersection. Lay off, on any convenient scale, along those axes, the distances O a and O b respectively proportional to the intended angular velocities (which, in the example shown, are contrary).

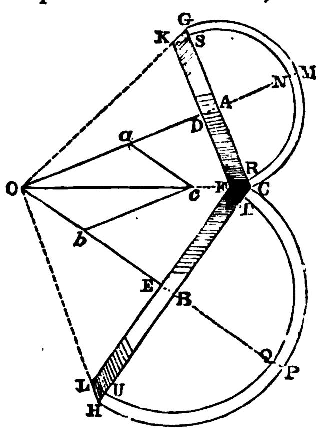

Fig. 65.

Draw $a\,c$ parallel to O b, and $b\,c$ parallel to O a, cutting each other in c; draw the diagonal O c C; this will be the line of contact; and the required pitch-surfaces will be parts of two cones described by making O C sweep round O A and O B respectively, and having their common summit at O. O C will be one of the traces of both these cones; and their other traces will be O G, making the angle A O G = A O C; and O H, making the angle B O H = B O C.

In any convenient position on the line of contact, mark a convenient breadth, C F, for the rims of both wheels, so that C F shall be their actual line of contact. Draw C A G and F D K perpendicular to O A, and C B H and F E L perpendicular to O B; then C G K F and C H L F will be the projections of the two wheels on the plane of their axes.

To draw the projection of one-half of each of those wheels on a plane perpendicular to its axis, about A, with the radius A C, draw the semicircle C M G, and with the radius A R = D F draw the semicircle R N S; these will be parts of the pitch-circles of which C A G and F D K are projections, and will form the required projection of one-half of the rim of the wheel whose axis is O A; then, about B, with the radius B C, draw the semicircle

C P H, and with the radius B T = E F draw the semicircle T Q U; these will be parts of the pitch-circles of which C B H and F E L are projections, and will form the required projection of one-half of the rim of the wheel whose axis is O B.

The proper widths for the rims of wheels will be considered farther on.

The perspective sketch, fig. 66, illustrates the case in which one of the pitch-surfaces becomes a flat disc or ring.

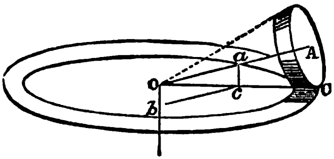

Fig. 66.

106. **Circular Skew-Bevel Wheels** are used when a constant velocity-ratio is to be communicated between two pieces which turn about axes that are neither parallel nor intersecting. Their pitch-surfaces are *rolling hyperboloïds;* and the figures and principal dimensions of such hyperboloïds are determined by the method already described in Article 84, page 70, and shown in fig. 54, page 72; it being understood that, in that figure, O *a* and O *b* represent the intended angular velocities *in contrary directions* of the two wheels.

For the actual wheels, narrow zones or frusta only of the hyperboloïds are used, as shown in fig. 67. Where approximate accuracy of form is sufficient, frusta of a pair of tangent cones (or of tangent cylinders, if the pitch-circles are the throats of the hyperboloïds) may be used; the figures of such cones and cylinders being determined as described in Articles 85 and 86, and shown in figs. 55 and 56, pages 73, 74.

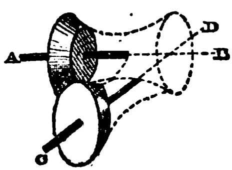

Fig. 67.

In all skew-bevel wheels the rolling motion is combined with *a relative sliding motion along the line of contact*, at a rate equal to the *rate of advance* described in Article 83, page 70.

The present Article contains some additional rules, which may have to be used in the designing and execution of skew-bevel wheels.

In fig. 68, let the vertical line through O represent the axis of a skew-bevel wheel, O A = O *a* the radius of its throat, and O C′ a generating line, or line of contact, in that position in which it is parallel to the plane of projection, which plane is supposed to pass through the axis.

Draw the semicircle A B *a*; this will be the projection of half the throat of the hyperboloïd on a second plane of projection, perpendicular to the axis of the wheel.

Let C′ G′ be the projection and trace of a plane perpendicular to the axis, and chosen as a convenient plane for the pitch-circle in the middle of the breadth of the rim of the intended wheel, and let that projection cut O C′ in C′.

I. *To find the radius of the middle pitch-circle, and to draw its projections.* Through B draw B C parallel to O A; through C′ draw C′ C parallel to the axis, cutting B C in C. Join O C; this will be the required radius, and the circle D C G will be the projection of the pitch-circle on the second plane; in G′ C′ produced take G′ D′ = O D = O G = O C; G′ D′ will be the projection of the pitch-circle on the first plane.

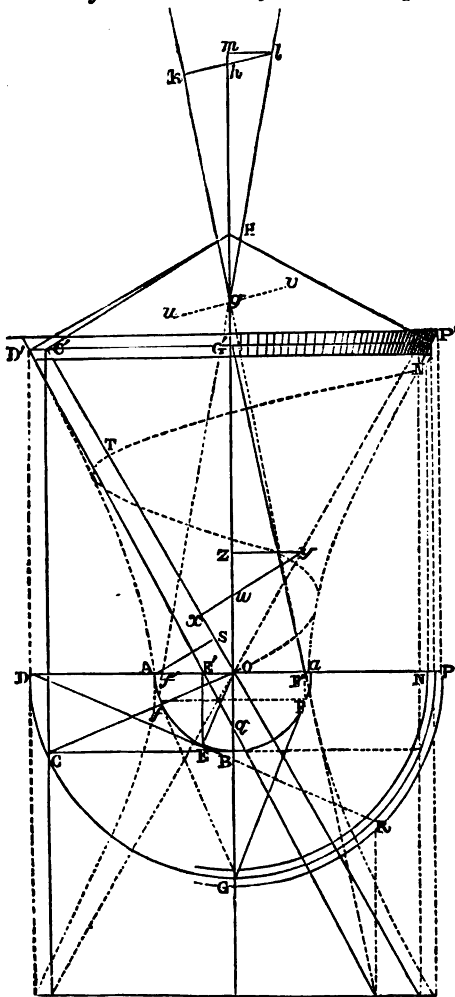

Fig. 68.

D′ is a point in the hyperbolic trace of the hyperboloïd on the first plane; and by the same process any number of points in that trace may be found.

II. *To draw a normal to the pitch-surface in the first plane of projection.* Perpendicular to O C′ draw C′ H, cutting the axis of the wheel in H. This line and O C will be the projections of a normal to the pitch-surface at the point whose projections are C′ and C. Join H D′; this line and O D will be the projections of a normal to the pitch-surface at the point whose projections are D′ and D.

III. *To draw the traces of a tangent plane to the pitch surface at the point* D′, D. The line D′ D is the trace on the second plane of projection; and the trace on the first plane is D′ q perpendicular to D′ H.

Another process for finding the trace D′ q is as follows:—About D, with a radius, D E, equal to C B, draw a circular arc, cutting the circle A B a in E. Through E draw E E′ parallel to B O, cutting O A in E′. The straight line D E′ q will be the trace required.

D E and D′ E′ are also the projections of a generating line of the hyperboloïd.

IV. *Tangent cone.*—The summit of the tangent cone at the pitch-circle D′ G′ is at the point q, and D′ q is the trace of that cone on the first plane of projection. When extreme accuracy of form is not required, a portion of that cone, having the pitch-circle D′ G′ at the middle of its breadth, may be used instead of the exact hyperboloidal surface (Article 86, p. 73).

V. *Normal spiral.*—The normal spiral is a curve on the hyperboloidal surface which cuts all the generating straight lines, such as those whose projections are E′ D′, O C′, &c., at right angles. Its general form is indicated by the winding dotted curve which traverses O and T in fig. 68. It has an *uniform normal pitch*, O T, which is found as follows:—From A let fall A S perpendicular to O C′; then the normal pitch of the normal spiral is equal to the circumference of a circle whose radius is O S; that is to say,

$$\text{O T} = \frac{710}{113}\text{O S}.$$

It is not necessary to draw precisely the projections of the normal spiral; but for purposes connected with the designing of teeth for skew-bevel wheels it is useful to know its radius of curvature at the pitch-circle chosen for the wheel. That is done as follows:—

About G, with the radius G F = G f = B C, describe a circle, cutting the circle A B a in F and f; from which two points draw F F′ and $f f'$ parallel to B O. (Or otherwise, lay off O F′ = $o f'$ = E E′. F′ G′ and F G will be the two projections of a generating line.) In O H take O g = E′ D′; join F′ g, f' g, and produce both these lines as far as may be necessary. O F′ g will be the rabatment of the triangle whose projection is O F′ G′. In O H produced, take g h = H D′; through h draw k h l perpendicular to F′ G′ k, and cutting f' g l in l: through l draw l m parallel to O A, cutting O H produced in m; then g m will be *the radius of curvature of every normal spiral at the point where it crosses the pitch-circle* G′ D′.

(The object of making this construction above instead of below the point g is merely to avoid confusion in the figure.)

Through g draw $u\,g\,v$ parallel to $k\,h\,l$; this will be the rabatment of a tangent to the normal spiral at the point G′.

To find the radius of curvature of a normal spiral at the throat of the hyperboloïd, in O H take O w = O A; draw $x\,w\,y$ perpendicular to O C′, and $y\,z$ parallel to O A; O z will be the required radius of curvature.

The lower part of the figure shows the projection on a plane through the axis, of a pitch-circle equal to the pitch-circle G′ D′, and at the same distance from the throat along the axis in the opposite direction. D E R and D′ E′ R′ are the two projections of one generating line extending from one of those pitch-circles to the other. G′ F′ R′ is the projection of another such generating line. The drawing of a pair of equal pitch-circles may sometimes be useful in the making of patterns for the wheel and for its teeth.

P, P′ and N, N′ are the projections of points in the two edges of the rim of the wheel. When the exact hyperboloidal pitch-surface is to be used, and not merely a tangent cone, those points are to be found by a process similar to that by which the projections D, D′ are found. When a tangent cone is used as an approximation, they are simply the intersections of two planes perpendicular to the axis, with a tangent in the plane of the axis.

VI. *Radius of curvature of hyperbolic trace.*—In constructing the pitch-surface of a skew-bevel wheel, it is sometimes useful to determine the radius of curvature of the hyperbolic trace of that surface on a plane traversing the axis, at the point where that trace cuts the pitch-circle, in order that a circular arc of that radius may, if required, be used as an approximation to a small arc of the hyperbolic curve.

In fig. 68 A, let O X be the axis of the hyperboloïd, O A the radius of its throat, O D an asymptote (being, as before, the projection of a line of contact that is parallel to the plane of projection), and X Y the trace of the plane of the intended pitch-circle. Part of the following process has already been described, but it will be described again here, to make the explanation complete:—Let D be the point where X Y cuts the asymptote. Lay off X E = O A; join D E; and make X Y = D E; then X Y will be the radius of the pitch-circle, and Y a point in the hyperbola. Perpendicular to O D draw D F, cutting the axis in F; join F Y; this will be a normal to the hyperbola at the point Y. Thus far the process has already been described.

Through A draw A B parallel to the axis, cutting the asymptote in B. From B, perpendicular to O B, draw B C, cutting O A produced in C. Then C will be *the centre of curvature*, and A C *the radius of curvature* of the hyperbola at A; that is, at the throat of the hyperboloïd.

In X Y, produced both ways as far as may be required, take

Y H = A O, Y L = A C, and Y G = Y F. In Y F take Y K = A O: join H F and K G. Through L, parallel to F H,

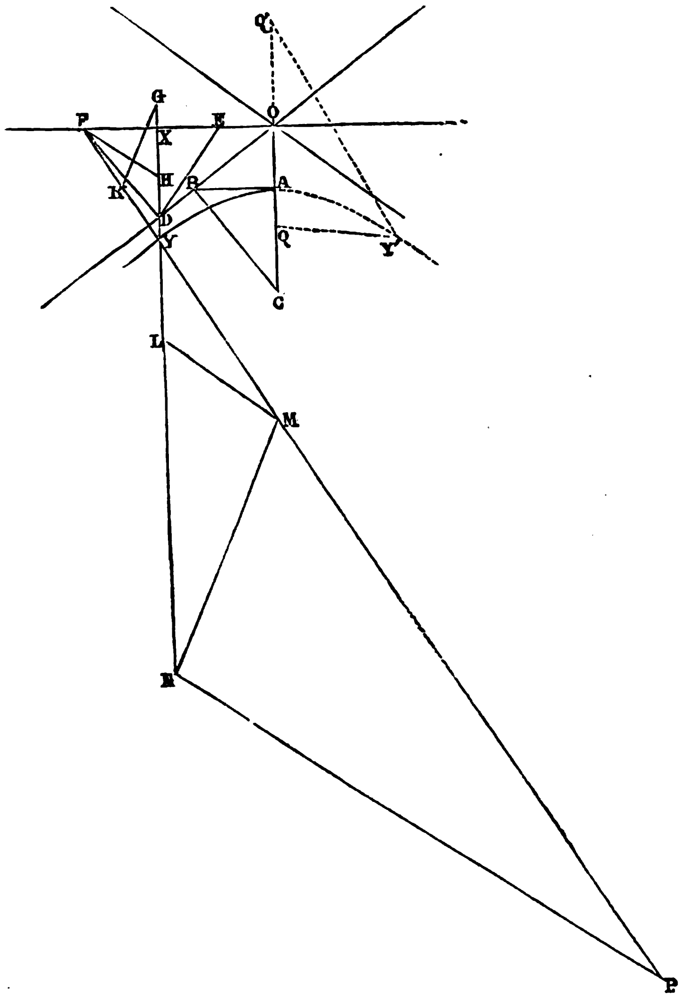

Fig. 68 A.

draw L M, cutting F Y produced in M; through M, parallel to G K, draw M N, cutting X Y L produced in N; and through N parallel to F H, draw N P, cutting F Y M produced in P; then P will be *the centre of curvature,* and Y P *the radius of curvature* of the hyperbola at Y.*

VII. *Foci of hyperbolic trace.*—To find, if required, the foci of the hyperbolic trace of the pitch-surface; produce, in fig. 68 A, the straight line O A, in both directions, as far as may be required, and lay off in it O Q = O Q′ = O B. Then Q and Q′ will be the two foci. The well-known property of a hyperbola, by means of which it can be drawn when one point in it and the two foci are given, is, that the difference of the distances from any point in the curve to the foci is a constant quantity; for example, Y′ Q′ — Y′ Q = A Q′ — A Q = 2 A O. Instruments founded on this principle are used for drawing hyperbolas.

107. **Non-Circular Wheels in General.** (*A. M.*, 443.)—Non-circular wheels are used to transmit a variable velocity-ratio between a pair of parallel axes. In fig. 69, let C_1, C_2 represent the axes of such a pair of wheels; T_1, T_2, a pair of points which at a given instant touch each other in the line of contact (which line is parallel to the axes and in the same plane with them); and U_1, U_2, another pair of points which touch each other at another instant of the motion; and let the four points, T_1, T_2, U_1, U_2, be in one plane perpendicular to the two axes and to the line of contact. Then, for every such set of four points, the two following equations must be fulfilled:—

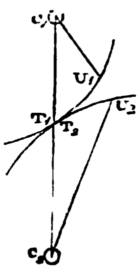

Fig. 69.

$$\left.\begin{array}{c} C_1 U_1 + C_2 U_2 = C_1 T_1 + C_2 T_2 = C_1 C_2; \\ \text{arc } T_1 U_1 = \text{arc } T_2 U_2; \end{array}\right\} (1.)$$

and those equations show the geometrical relations which must exist between a pair of rotating surfaces in order that they may move in rolling contact round fixed axes.

If one of the wheels be fixed and the other be rolled upon it, a point in the axis of the rolling wheel describes a circle of the radius $C_1 C_2$ round the axis of the fixed wheel.

The equations are made applicable to *inside gearing,* by putting − instead of + and + instead of −.

* The algebraical expressions of these operations are as follows:—

Let O A = b; A B = a; O X = x; X Y = y;

X F = m; Y F = n; Y P = ρ; A C = ρ_0; then

$$\rho_0 = \frac{a^2}{b}; \; y = \frac{b}{a}\sqrt{\left(a^2 + x^2\right)};$$

$$m = \frac{b^2 x}{a^2}; \; n = \sqrt{\left(y^2 + m^2\right)}; \; \rho = \rho_0 \frac{n^3}{b^3}.$$

The angular velocity-ratio at a given instant has the value

$$C_1 T_1 : C_2 T_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

Non-circular wheels, when without teeth, may be called **Rolling Cams**; and in order that motion may be communicated by means of a pair of rolling cams, and of a suitably shaped smooth rack or sliding bar, it is necessary that the forces exerted by the two pieces on each other should be such as to press their pitch-surfaces together.

The following are the general problems to be solved in designing non-circular wheels:—

I. *Given, the axis and pitch-line of a non-circular wheel; to find approximately the axis of another non-circular wheel which shall turn in rolling contact with the first wheel, and of which an arc of a given length on the pitch-line shall subtend a given angle.*

In fig. 70, let the plane of projection be a plane perpendicular to the axes of the wheels. Let A be the axis of the given wheel, B C its pitch-line, and B its pitch-point at a given instant; and let A B be part of the line of centres. Also, let B D be a straight tangent to B C at B; and let the length of B D be the length of the arc on the pitch-line of the second wheel which is to subtend a given angle.

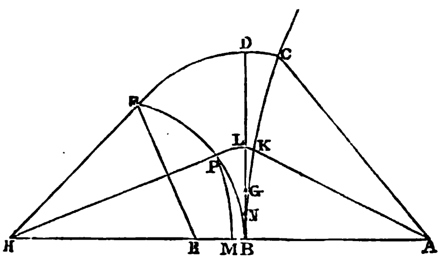

Fig. 70.

In B D take B G = $\frac{1}{4}$ B D, and about G, with the radius G D = $\frac{3}{4}$ B D, draw a circle, C D F, cutting the first pitch-line in C. Then, according to Rule IV., Article 51, page 29, the arc B C will be approximately equal in length to B D. Draw and measure the straight line A C; and in the line of centres take A E = A C. Then draw the straight line E F, making, with the line of centres, the angle H E F = the complement of half the angle that the arc of a length equal to B D is to subtend, and cutting the circle C D F in F. F will be approximately a point in the required pitch-line of the second wheel; and B and F will be the two ends of an arc approximately equal in length to B D and B C. To find the axis of that wheel, find, by plane geometry, in the line of centres, H B E, the centre, H, of a circle which shall traverse F and E; H will be approximately the trace of the required axis.

II. *To find a point in the second pitch-line whose distance from* B, *as measured on that pitch-line, shall be approximately equal to any given straight tangent,* B L. Take B N = $\frac{1}{4}$ B L; and about N, with a radius N L = $\frac{3}{4}$ B L, draw a circular arc, cutting the first pitch-line in K. Then B K will be approximately equal in length to B L. Join and measure A K, and in the line of centres take A M = A K. About H, with the radius H M, draw a circular arc, M P, cutting the arc K L P in P; P will be approximately the required point in the second pitch-line.

By repeating the same process, any number of points in the required pitch-line of the second wheel may be found approximately. The error of the two preceding rules, in what may be considered an extreme case—viz., where the pitch-line of the first wheel coincides with the straight tangent B D, and the angles B H F and B A C are each half a right angle (as in designing a roller to roll with a square roller)—is about 0·003 of the length B D to be laid off, and is in excess; the arc B F being too long by that fraction of its length; and the error, in fractions of the arc, varies nearly as the fourth power of the angle subtended by the arc. To ascertain whether the error is sensible, and to correct it by a second approximation, proceed as follows:—

III. *To obtain a closer approximation to the required axis and pitch-line.* Having drawn the pitch-line B F by Rule II., measure its length in subdivisions by Rule I. of Article 51, page 28, and compare that length with B D, so as to ascertain the error. Divide that error by B D, so as to express it in fractions of the required length. Multiply the half-sum of the greatest and least radii by the fraction expressing the ratio of the error to the required length; the product will be a *correction*, which is to be applied to the lengths of the line of centres, A H, and of each of the radii H B, H F, H P, &c., of the second pitch-line; that is to say, if the pitch-line, as at first drawn, is too long, each of those straight lines is to be shortened by having the correction subtracted from it.

For example, in the extreme case already cited, where the first pitch-line is a straight line, B D, perpendicular to A B, and subtending half a right angle at A, and the second pitch-line is to subtend also half a right angle at its axis H, let A B be taken as unity; then we have (to three places of decimals)

$$B\,C = B\,D = A\,B = 1{\cdot}000;$$

A C (coinciding with a straight line from A to D) = 1·414; and the application of Rule I. of this Article gives the following results as first approximations:—

$$A\,H = 2{\cdot}267;\ H\,B = 1{\cdot}267;\ H\,F = 0{\cdot}853.$$

Upon drawing the second pitch-line, B F, by Rule II. of this Article, and measuring it in subdivisions, it is found to be too long by 0·003 of its own length; which being multiplied by $\frac{\text{H B} + \text{H F}}{2} = \frac{2\cdot120}{2} = 1\cdot060$, gives 0·003 for the correction to be subtracted from the line of centres and from each of the radii of the second pitch-line. Thus are obtained the second approximations,

$$\text{A H} = 2\cdot264;\ \text{H B} = 1\cdot264;\ \text{H F} = 0\cdot850.$$

As examples of non-circular wheels, the following may be mentioned:—

I. An ellipse rotating about one focus rolls completely round in outside gearing with an equal and similar ellipse also rotating about one focus, the distance between the axes of rotation being equal to the major axis of the ellipses, and the velocity-ratio varying from

$$\frac{1 - \text{eccentricity}}{1 + \text{eccentricity}} \text{ to } \frac{1 + \text{eccentricity}}{1 - \text{eccentricity}}$$ (see Article 108).

II. Lobed wheels, of forms derived from the ellipse, roll completely round in outside gearing with each other (see Article 109).

III. A hyperbola rotating about its farther focus rolls in inside gearing, through a limited arc, with an equal and similar hyperbola rotating about its nearer focus, the distance between the axes of rotation being equal to the axis of the hyperbolas, and the velocity-ratio varying between

$$\frac{\text{eccentricity} + 1}{\text{eccentricity} - 1} \text{ and unity.}$$

IV. Two logarithmic spiral sectors of equal obliquity rotate in rolling contact with each other; or one logarithmic spiral sector rotates in rolling contact with the oblique plane surface of a sliding piece (see Article 110).

108. **Elliptic Wheels.**—The following rules are applicable to the drawing of the pitch-lines of elliptic wheels, and the determination of their comparative motions:—

I. *Given, the angle by which each wheel is alternately to overtake and to fall behind the other, and the length of the line of centres, to draw the ellipse which is the figure of both pitch-lines.*

From a point, B, draw two straight lines, B F = B F′ = half the line of centres, making with each other the given angle F B F′. Join F F′, bisect it in O, produce it both ways, and make O A = O A′ = half the line of centres. Draw B B′ perpendicular to A A′, making O B′ = O B. Then A A′ is the major axis, B B′ the minor axis, O the centre, and F, F′, the two foci of the required ellipse, which may be drawn by means of a suitable instrument or machine, or by the well-known process of putting an endless thread, of a length = 2 A F′ = 2 F A′, round two pins at the foci, and a pencil equal in diameter to those pins, and moving the pencil round so as to keep the thread tight. In the workshop ellipses of given dimensions can be drawn with great precision by means of the turning lathe, fitted with apparatus to be afterwards referred to.

The wheels are to be centred, as shown in fig. 72, each upon one of its foci. The *fixed foci*, which are thus placed in the axes of the wheels, are marked A, B, in this figure, and the *revolving foci*, C, D. The ellipses in fig. 72 are similar to that in fig. 71, but drawn on one-half of the scale.

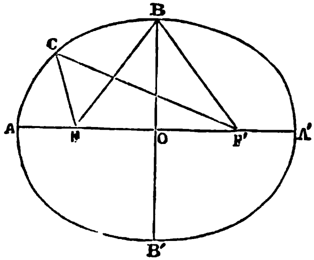

Fig. 71.

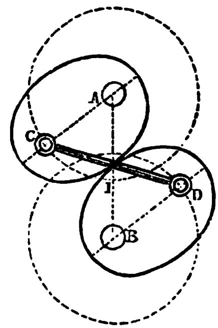

Fig. 72.

II. *To find the angular motions and the angular velocity-ratio corresponding to a given position of the pitch-point.* Suppose both wheels to have started from a position in which A, fig. 71, is the pitch-point, being at the distance A F from the axis of one wheel, and A F′ from that of the other, so that the angular velocity-ratio of the second wheel to the first is A F ÷ A F′. Let C be a new position of the pitch-point. Draw C F, C F′. Then the *angular motion* of the first wheel is A F C; that of the second wheel A F′ C; the first wheel has *overtaken* the second wheel to the extent represented by the angle F C F′ = A F C − A F′ C; and the *velocity-ratio* of the second wheel to the first is C F ÷ C F′.

The angular velocity-ratio ranges between the limits $\frac{\text{A F}}{\text{A F}'}$ and $\frac{\text{A F}'}{\text{A F}}$; and its mean value in each half-revolution is *unity;* because each half-revolution is made in the same time by both wheels. The instantaneous velocity-ratio is unity when the pitch-point is at B or B′; because B F = B F′.

III. *Given, at any instant, the position of one of the revolving foci, to find the position of the other revolving focus, and of the pitch-point.* In fig. 72, let A and B be the fixed foci. With a radius equal to the distance between the foci, or double eccentricity (F F′ in fig.

71), draw circles about A and B. Let C be the given position of one of the revolving foci. Then, with a radius C D = A B (the line of centres), draw a circular arc about C, cutting the circle round B in D; this will be the other revolving focus. Join C D, cutting A B in I; this will be the pitch-point.

If the wheels and their axles overhang the bearings, the revolving foci, being at a constant distance apart, may be connected by means of a link, C D, as shown in fig. 72. This is useful in elliptic toothed wheels of great eccentricity, because of the teeth in certain positions of the wheel being apt to lose hold of each other.

109. **Lobed Wheels** * are wheels such as those shown in figs. 74 and 75, having two, three, or any greater number of equal greatest radii (such as those marked F A″ in fig. 74, and F A‴ in fig. 75), and also of least radii (such as those marked F a″ in fig. 74, and F a‴ in fig. 75). Fig. 74 represents a two-lobed wheel, and fig. 75 a three-lobed wheel. An elliptic wheel may be regarded as a *one-lobed wheel*. Let the difference between the

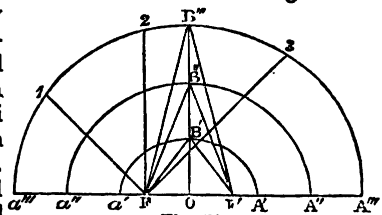

Fig. 73.

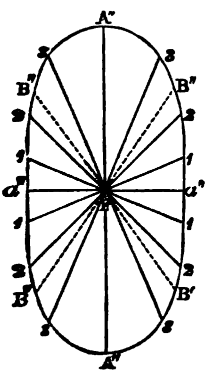
Fig. 74.

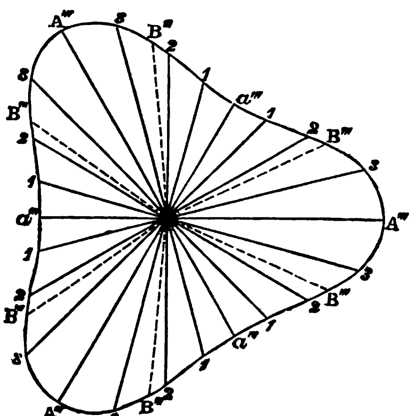
Fig. 75.

greatest and least radii of a lobed wheel be called the *inequality;* so that in an elliptic wheel (fig. 71) the inequality is the distance

* The properties of these wheels were discovered by the Reverend W. Holditch.

between the foci, F F′. Then any pair of lobed wheels in which the inequality is the same will, if properly shaped, work together in rolling contact, and that whether their numbers of lobes are many or few, the same or different; and this statement includes one-lobed or elliptic wheels.

The advantage of wheels with two or more lobes is their being self-balanced, which elliptic wheels are not.

The following are the rules for designing lobed wheels:—

I. *Given, in a pair of equal and similar lobed wheels, the angle by which each wheel is alternately to overtake and to fall behind the other wheel, the number of lobes, and the mean radius, to find the inequality, and thence the greatest and least radii.* Multiply the given angle by the number of lobes; then, from a point B‴, fig. 73, draw two lines, B‴ F, B‴ F′, making with each other an angle equal to the product, and make the length of each of them equal to the given mean radius. Draw the straight line F F′; this will be the required *inequality*. Bisect F F′ in O, and produce it both ways; then lay off O A‴ = O a‴ = B‴ F = B‴ F′, the mean radius; then F A‴ = F′ a‴ will be the greatest radius, and F′ A‴ = F a‴ the least radius.

II. *To find any number of points in the pitch-line.* In fig. 73, with the major axis A‴ a‴, and the foci F and F′, draw a semi-ellipse A‴ B‴ a‴. Then, in fig. 75, draw from the centre, F, the straight lines marked F A‴, dividing a complete revolution into as many equal parts as there are to be lobes (in the present case, three). Make each of these lines equal to the greatest radius (F A‴, fig. 73). Bisect the angles between them with the straight lines marked F a‴, fig. 75, and make each of the latter set of lines equal to the least radius (F a‴, fig. 73). Divide the half-revolution in fig. 73 into any convenient number of equal angles by the radiating lines F 1, F 2, &c., meeting the ellipse at 1, 2, &c. Divide each of the angles marked A‴ F a‴ in fig. 75 into the same number of equal parts by radiating lines, and lay off upon them lengths, F 1, F 2, &c., equal to those of the corresponding lines in fig. 73; the points 1, 2, &c., in fig. 75, thus found, will be *points in the required pitch-line.*

III. *To find the positions of the mean radii of the required pitch-line.* Divide the angle A‴ F B‴, in fig. 73 by the number of lobes, and lay off the quotient for each of the angles marked A‴ F B‴ in fig. 75; then make each of the radii F B‴ in fig. 75 equal to F B‴, in fig. 73; these will be the required mean radii.

REMARK.—The example in fig. 75 is a three-lobed wheel. The two-lobed wheel of fig. 74 is drawn by a similar process; the ellipse used for finding the radii being A″ B″ a″ in fig. 73; the inequality F F′; and the angle by which each wheel alternately overtakes and falls behind another equal and similar wheel being one-half of F B″ F′, fig. 73.

IV. *To draw the pitch-lines of a set of wheels of different numbers of lobes, all of which shall work with each other in rolling contact.* The inequality must be the same in each wheel. Let F F′, fig. 73, be that inequality; and let O be the centre, A‴ a‴ the major axis, and O B‴ the semi-minor axis of the ellipse which serves for finding the radii of one of the set of wheels, which one wheel is given. Divide O B‴ into as many equal parts as there are lobes in the given wheel; say, for example, three. To find the figure of a wheel having any other number of lobes (say two), take the point B″ at that number of divisions from O; join F B″, F′ B″; lay off O A″ = O a″ = F B″ = F′ B″; draw the ellipse A″ B″ a″ with A″ a″ for its major axis, and F and F′ for its foci; this will be the ellipse for determining the lengths of the radii of the new (two-lobed) wheel.

The ellipse A′ B′ a′ with the same foci, F F′, whose minor semi-axis, O B′, is *one* division of O B‴, is itself the pitch-line of the one-lobed wheel, which will work in rolling contact with any wheel of the set.

110. **Logarithmic Spiral Sectors or Rolling Cams.**—A pair of logarithmic spiral sectors may be used as rolling cams, to communicate by rolling contact an angular motion of limited extent, in the course of which it is desired that the velocity-ratio shall range between certain limits. The general nature of the figure and position of such a pair of sectors may be represented by fig. 69, page 92.

The only cases in which the dimensions and figures of such sectors can be determined by plane geometry alone, without the aid of calculation, are two, viz.: when the two sectors are equal and similar, so that the sum of the greatest and least radii of each of the two sectors is equal to the line of centres; and when the combination consists of one sector, working with a sliding bar or smooth rack. The following are the rules applicable to such cases:—

I. *Given, in fig.* 76, *the least and greatest radii,* O A *and* O B, *of a logarithmic spiral sector, and the angle* A O B *between them, to find intermediate points in the pitch-line of such a sector, and to draw that pitch-line approximately by means of one or more circular arcs.*

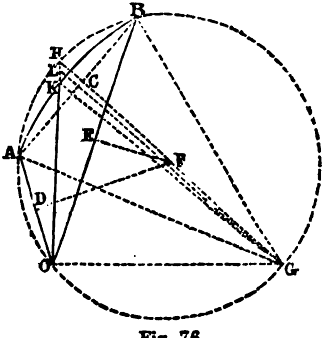

Fig. 76.

Describe a circle about the triangle O A B; that is to say, bisect any two of the sides of that triangle (C, E, and D being the three points of bisection), and from the points of bisection draw perpendiculars to the sides, meeting in F, which will be the centre of the circle through O, A, and B. Draw the diameter G F C H, bisecting the arc A H B in H and the arc A O G B in G. Join O H (which will be perpendicular to O G, and will bisect the angle A O B); and about G, with the radius G A = G B, draw the circular arc A K B, cutting O H in K. Then K will be a point in the required spiral; and A K B will be the nearest approximation to the spiral arc traversing the three points, A, K and B, that it is possible to make by means of one circular arc only.

To find two additional points, and a closer approximation to the curve, treat each of the triangles O A K and O K B as the triangle O A B was treated; the result will be the finding of two more points in the spiral, situated respectively in the radii which bisect the angles A O K and K O B; and the drawing of two circular arcs, one extending from A to K, and the other from K to B, which will make a closer approximation to the spiral arc than a single circular arc does.

The next repetition of the process will give four additional points and four circular arcs; the next, eight additional points and eight circular arcs; and so on to any degree of precision that may be required.

The radius O K is *a mean proportional* between O A and O B; and this property enables its length to be found by calculation, if required.

The *obliquity* of a logarithmic spiral, being the angle which a tangent to the spiral at a given point makes with a tangent to a circle described about the axis through that point, or the equal angle which a normal to the spiral at the same point makes with a radius drawn from that point to the axis, is uniform in a given spiral. In fig. 76 the equal angles, O A G, O H G, and O B G, are each of them less than the true obliquity of the spiral, and the angle O K G is greater than the true obliquity. To obtain the closest approximation to the true obliquity possible without further subdividing the angle A O B, proceed as follows:—

II. *To find the approximate obliquity.* In H K take H L = $\frac{1}{3}$ H K; join L G; then O L G will be the obliquity, very nearly. In other words, L G will be very nearly parallel to a normal, and perpendicular to a tangent, to the true spiral at the point K.

II A. *To find the approximate obliquity* (Another method). By Rule I. or Rule II. of Article 51, page 28, measure the approximate length of the arc A B in fig. 76. Then, in fig. 77, draw the straight line M N = O B − O A; draw M P perpendicular to M N; and about N, with a radius equal to the approximate length of the arc A B, draw a circular arc, cutting M P in P; join N P; then the angle M P N will be approximately the required obliquity.

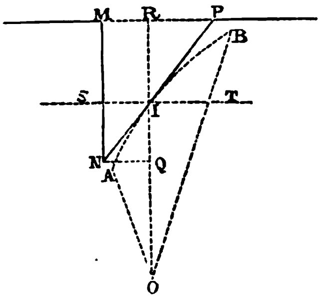

Fig. 77.

III. *Given* (in fig. 76), *one radius,* O K, *in a logarithmic spiral of a given obliquity, to draw approximately a short arc of that spiral through* K. Draw O G perpendicular to O K; draw K G, making O K G = the angle of obliquity, and cutting O K G in G; then, with the radius G K, draw a short circular arc through K.

IV. *To draw the pitch-line of a sliding bar which shall work in rolling contact with a given logarithmic spiral sector,* A O B (fig. 77). From the trace of the axis O draw O Q R perpendicular to the direction in which the bar is to slide, making O Q = O A, and O R = O B. Find the obliquity of the sector by means of one or other of the preceding rules. Let I be any given position of the pitch-point, and let T I S, traversing I perpendicularly to O Q R, be parallel to the direction in which the bar is to slide. Draw the straight line N I P, making the angle S I N = T I P = the obliquity; then draw Q N and R P parallel to T I S, and cutting N I P in N and P respectively. The straight line N I P will be the required pitch-line; and N and P will be the points in it corresponding to A and B respectively in the pitch-line of the sector.

At the instant when the pitch-point is at I, the velocity of the sliding bar is equal to that of the point I in the sector; that is to say, angular velocity × O I; agreeably to the general principle of Article 101, page 84.

The following rules relate to the solution of questions respecting logarithmic spiral sectors by calculation.

V. *Given, two radii of a logarithmic spiral sector* (as O A and O B, fig. 76), *and the angle between them* (A O B), *to find the obliquity of the spiral.* Take the hyperbolic logarithm * of the ratio $\frac{OB}{OA}$; divide it by the angle A O B in

* Hyperbolic logarithm of a ratio = common logarithm × 2·3026 nearly.

circular measure;* the quotient will be the tangent of the obliquity.

VI. *Given, the least and greatest radii of a logarithmic spiral sector, and the angle between them, to find the lengths of a series of intermediate radii, which shall divide that angle into a given number of equal smaller angles.* Take the difference between the logarithms of the greatest and least radii; divide it by the given number; then, commencing with the logarithm of the least radius, compute by successive additions of the quotient a series of logarithms, increasing by uniform differences up to the logarithm of the greatest radius; these will be the logarithms of the required intermediate radii.

VII. *Given, one radius and the obliquity of a logarithmic spiral, to find the length of a radius making a given angle with the given radius.* Multiply the given angle in circular measure (see first footnote below) by the tangent of the obliquity; to the product add the hyperbolic logarithm of the given radius; the sum will be the hyperbolic logarithm of the required radius;—or otherwise, multiply the product by 0·4343, and to the new product add the common logarithm of the given radius; the sum will be the common logarithm of the required radius.

VIII. *Given, the difference between the greatest and least radii of a logarithmic spiral sector, and the obliquity of its pitch-line, to find the length of its pitch-line.* Multiply the difference of the radii by the cosecant of the obliquity.†

111. **Frictional Gearing.**—To increase that friction or adhesion between a pair of wheels which is the means of transmitting force and motion from one to the other, their surfaces of contact are sometimes formed into alternate ridges and grooves parallel to the

* Reduction of angles to circular measure—

1 degree	= 0·0174533	radius	length,	nearly.
30 degrees	= 0·5236	,,	,,	,,
60 degrees	= 1·0472	,,	,,	,,
90 degrees	= 1·5708	,,	,,	,,

† In symbols, the equations of a logarithmic spiral are as follows:—Let a be the radius from whose directions angles are reckoned; r, any other radius; θ, the angle which r makes with a, in circular measure; ϕ, the obliquity of the spiral; s, the length of the arc from a to r; ρ, the radius of curvature at the end of the radius r. Then

$$r = a e^{\theta \tan \phi}; \ \tan \phi = \frac{1}{\theta} \text{hyp. log.} \frac{r}{a}$$

$$\theta = \text{cotan } \phi \cdot \text{hyp. log.} \frac{r}{a};$$

$$s = (r - a) \text{ cosec } \phi = a \text{ cosec } \phi \left(e^{\theta \tan \phi} - 1 \right);$$

$$\rho = r \tan \phi.$$

plane of rotation, constituting what is called *frictional gearing*. Fig. 78 is a cross-section of the rim of a wheel, illustrating the kind of frictional gearing invented by Mr. Robertson. The comparative motion of a pair of wheels thus ridged and grooved is nearly the same with that of a pair of smooth wheels in rolling contact, having cylindrical or conical pitch-surfaces lying midway between the tops of the ridges and bottoms of the grooves.

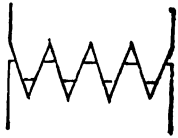

Fig. 78.

The relative motion of the surfaces of contact of the ridges and grooves is a rotatory sliding or grinding motion about the line of contact of the ideal pitch-surfaces as an instantaneous axis; and the angular velocity of that relative grinding motion is equal to the angular velocity of one wheel considered as rolling upon the other as a fixed wheel; which may be found by the principles of Article 77, page 56, and Article 82, page 68.

The angle between the sides of each groove is about 40°; and it is stated that the mutual friction of the wheels is about once and a-half the force with which their axes are pressed towards each other.

Section III.—*Of the Pitch and Number of the Teeth of Wheels.*

112. **Relation between Teeth and Pitch-Surfaces—Nature of the Subject.** (*A. M.*, 446.)—The most usual method of communicating motion between a pair of wheels, or a wheel and a rack, and the only method which, by preventing the possibility of the rotation of one wheel unless accompanied by the other, insures the preservation of a given velocity-ratio exactly, is by means of a series of alternate ridges and hollows parallel, or nearly parallel, to the successive lines of contact of the ideal toothless wheels or *pitch-surfaces*, whose velocity-ratio would be the same with that of the toothed wheels. The ridges are called *teeth;* the hollows, *spaces.* The teeth of the driver push those of the follower before them, and in so doing sliding takes place between them in a direction across their lines of contact.

The properties of pitch-surfaces and pitch-lines, and the art of designing them, have been explained in the preceding section. The figures of teeth depend on the principles of sliding contact, which belong to the ensuing section. The present section relates to questions connected with the manner in which the pitch-line of a wheel is divided by the acting surfaces of its teeth, without reference to the figures of those surfaces; for such questions do not require the principles of sliding contact for their solution.

113. **Pitch Defined.** (*A. M.*, 447.)—The distance, measured

along the pitch-line, from the front of one tooth to the front of the next, is called the PITCH.

114. **General Principles.** (*A. M.*, 447.)—The pitch, and the number of teeth in wheels, are regulated by the following principles:—

I. In wheels which rotate continuously for one revolution or more, it is obviously necessary *that the pitch should be an aliquot part of the circumference of the pitch-line.*

In racks, and in wheels which reciprocate without performing a complete revolution, this condition is not necessary. Such wheels are called *sectors*, as has been already stated.

II. In order that a pair of wheels, or a wheel and a rack, may work correctly together, it is in all cases essential *that the pitch should be the same in each.*

III. Hence, in any pair of wheels which work together, *the numbers of teeth in a complete circumference are inversely as the numbers of whole revolutions in a given time;* or, in other words, *directly as the times of revolution.*

IV. Hence, also, in any pair of wheels which rotate continuously for one revolution or more, the ratio of the times of revolution (being equal to that of the numbers of teeth), and its reciprocal, the ratio of the numbers of revolutions in a given time, *must both be expressible in whole numbers.*

V. In circular wheels, everything stated in the preceding principles regarding the ratio of the numbers of revolutions in a given time (in other words, of the *mean angular velocity-ratio*) is true also of the angular velocity-ratio at every instant.

115. **Frequency of Contact—Hunting-Cog.**—Let n, N, be the respective numbers of teeth in a pair of wheels, N being the greater. Let t, T, be a pair of teeth in the smaller and larger wheel respectively, which at a particular instant work together. It is required to find, first, how many pairs of teeth must pass the pitch-point before t and T work together again (let this number be called a); secondly, with how many different teeth of the larger wheel the tooth t will work at different times (let this number be called b); and thirdly, with how many different teeth of the smaller wheel the tooth T will work at different times (let this be called c).

CASE 1. If n is a divisor of N,

$$a = \mathrm{N};\ b = \frac{\mathrm{N}}{n};\ c = 1 \ldots\ldots\ldots\ldots (1.)$$

CASE 2. If the greatest common divisor of N and n be d, a number less than n, so that $n = m\,d$, $\mathrm{N} = \mathrm{M}\,d$, then

$$a = m\,\mathrm{N} = \mathrm{M}\,n = \mathrm{M}\,m\,d;\ b = \mathrm{M};\ c = m \ldots\ldots (2.)$$

CASE 3. If N and n be prime to each other,

$$a = \mathrm{N}\,n;\ b = \mathrm{N};\ c = n \ldots\ldots\ldots\ldots (3.)$$

It is considered desirable by millwrights, with a view to the preservation of the uniformity of shape of the teeth of a pair of wheels, that each given tooth in one wheel should work with as many different teeth in the other wheel as possible. They therefore study to make the numbers of teeth in each pair of wheels which work together such as to be either prime to each other, or to have their greatest common divisor as small as is possible consistently with the purposes of the machine.

When the ratio of the angular velocities of two wheels, being reduced to its least terms, is expressed by numbers less than those which can be given to wheels in practice, and it becomes necessary to employ multiples of those numbers by a common multiplier, which becomes a common divisor of the numbers of teeth in the wheels, millwrights and engine-makers avoid the evil of frequent contact between the same pairs of teeth, by giving one additional tooth, called a *hunting-cog*, to the larger of the two wheels. This expedient causes the velocity-ratio to be not exactly but only approximately equal to that which was at first contemplated; and therefore it cannot be used where the exactness of certain velocity-ratios amongst the wheels is of importance, as in clockwork.

116. **Smallest Pinion.**—The *smallest* number of teeth which it is practicable to give to a pinion (that is, a small wheel), is regulated by principles which will appear when the forms of teeth are considered. The following are the least numbers of teeth which can be *usually* employed in pinions having teeth of the three classes of figures named below, whose properties will be explained in the sequel:—

I.	Involute teeth,	25
II.	Epicycloidal teeth,	12
III.	Round teeth, or *staves*,	6

117. **Arithmetical Rules.**—For convenience sake the following arithmetical rules are here given, as being useful in the designing of toothed gearing.

I. *To find the prime factors of a given number.* Try the prime numbers, 2, 3, 5, 7, 11, &c., as divisors in succession, until a prime number has been found to divide the given number without a remainder; then try whether and how many times over the quotient is again divisible by the same prime number, so as to obtain a quotient not divisible again by the same prime number; then try the division of that quotient by the next greater prime number; and so on until a quotient is obtained which is itself a prime number; that is, a number not divisible by any other number except 1. This final quotient and the series of divisors will be the prime factors of the given number. To test the accuracy of the process, multiply all the prime factors together; the product should be the given number.

II. *To find the greatest common measure* (otherwise called the *greatest common divisor*) *of two numbers.* Divide the greater number by the less, so as to obtain a quotient, and a remainder less than the divisor; divide the divisor by the remainder as a new divisor; that new divisor by the new remainder; and so on, until a remainder is obtained which divides the previous divisor without a remainder. That last remainder will be the required greatest common measure.

If the last remainder is 1, the two numbers are said to be "prime to each other."

Example.—Required, the greatest common measure of 1420 and 1808.

Divisor, 1420) 1808 (1, Quotient.
1420
Remainder, 388) 1420 (3, Quotient.
1164
Remainder, 256) 388 (1, Quotient.
256
Remainder, 132) 256 (1, Quotient.
132
Remainder, 124) 132 (1, Quotient.
124
Remainder, 8) 124 (15, Quotient.
120
Remainder, 4) 8 (2, Quotient.

The last remainder, 4, is the required greatest common measure.

III. To reduce the ratio of two numbers to its least terms, divide both numbers by their greatest common measure.

$$\text{For example, } \frac{1808 \div 4}{1420 \div 4} = \frac{452}{355}.$$

IV. *To express the ratio of two numbers in the form of a continued fraction.* Let A be the lesser of the two numbers, and B the greater; and let *a*, *b*, *c*, *d*, &c., be the quotients obtained during the process of finding the greatest common measure of A and B. Then, in the equation

$$\frac{B}{A} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \&c.,}}}$$

the right-hand side is the continued fraction required.

To save space in printing, a continued fraction is often arranged as follows:—

$$a + \frac{1}{b+} \frac{1}{c+} \frac{1}{d+} \&c.$$

The ratio of two incommensurable quantities is expressed by an endless continued fraction. For example, the ratio of the diagonal to the side of a square is expressed by $1 + \frac{1}{2+} \frac{1}{2+} \frac{1}{2+} \frac{1}{2+}$ &c., without end.

V. *To form a series of approximations to a given ratio.* Express the ratio in the form of a continued fraction. Then write the quotients in their order; and in a line below them write $\frac{0}{1}$ to the left of the first quotient, and $\frac{1}{0}$ directly under the first quotient. Then calculate a series of fractions by the following rule:—Multiply the first quotient by the numerator of the fraction that is below it, and add the numerator of the fraction next to the left; the sum will be the numerator of a new fraction: multiply the first quotient by the denominator of the fraction that is below it, and add the denominator of the fraction that is next to the left; the sum will be the denominator of the new fraction; then write that new fraction under the second quotient, and treat the second quotient, the fraction below it, and the fraction next to the left, as before, to find a fraction which is to be written under the third quotient, and so on. For example:

Quotients,....a, b, c, d, &c.

Fractions, $\frac{0}{1}, \frac{1}{0}, \frac{n}{m}, \frac{n'}{m'}, \frac{n''}{m''}$;

$$\frac{n}{m} = \frac{0+a}{1+0} = \frac{a}{1};\ \frac{n'}{m'} = \frac{1+b\,n}{0+b\,m};\ \frac{n''}{m''} = \frac{n+c\,n'}{m+c\,m'};\ \&c.$$

To take a particular case; let the given ratio be as before, $\frac{452}{355}$. then we have the following series:—

Quotients,....................		1	3	1	1	1	15	2	
Fractions,...............	$\frac{0}{1}$	$\frac{1}{0}$	$\frac{1}{1}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{9}{7}$	$\frac{14}{11}$	$\frac{219}{172}$	$\frac{452}{355}$
Less or greater than given ratio,.........	L	G	L	G	L	G	L	G	

The fractions in a series formed in the manner just described are called *converging fractions*, and they have the following properties:—*First*, each of them is in its least terms; *secondly*, the difference between any pair of consecutive converging fractions is equal to unity divided by the product of their denominators; for example, $\frac{9}{7}-\frac{5}{4}=\frac{36-35}{7\times 4}=\frac{1}{28}$; $\frac{9}{7}-\frac{14}{11}=\frac{99-98}{7\times 11}=\frac{1}{77}$; *thirdly*, they are alternately less and greater than the given ratio towards which they approximate, as indicated by the letters L and G in the example; and, *fourthly*, the difference between any one of them and the given ratio is less than the difference between that one and the next fraction of the series.

Fractions intermediate between the converging fractions may be found by means of the formula $\frac{hn+kn'}{hm+km'}$; where $\frac{n}{m}$ and $\frac{n'}{m'}$ are any two of the converging fractions, and h and k are any two whole numbers, positive or negative, that are prime to each other.

118. A **Train of Wheelwork** (*A. M.*, 449,) consists of a series of axes, each having upon it two wheels, one of which is driven by a wheel on the preceding axis, while the other drives a wheel on the following axis. If the wheels are all in outside gearing, the direction of rotation of each axis is contrary to that of the adjoining axes. In some cases a single wheel upon one axis answers the purpose both of receiving motion from a wheel on the preceding axis and giving motion to a wheel on the following axis. Such a wheel is called an *idle wheel:* it affects the direction of rotation only, and not the velocity-ratio.

Let the series of axes be distinguished by numbers 1, 2, 3, &c. . . . . m; let the numbers of teeth in the *driving wheels* be denoted by N's, each with the number of its axis affixed; thus, N_1, N_2, &c. . . . . N_{m-1}; and let the numbers of teeth in the *driven* or *following* wheels be denoted by n's, each with the number of its axis affixed; thus, n_2, n_3, &c. . . . . n_m. Then the ratio of the angular velocity a_m of the m^{th} axis to the angular velocity a_1 of the first axis is the product of the $m-1$ velocity-ratios of the successive elementary combinations, viz.:—

$$\frac{a_m}{a_1}=\frac{N_1 . N_2 . \&c. \ldots . N_{m-1}}{n_2 \cdot n_3 \cdot \&c. \ldots . n_m};$$

that is to say, the velocity-ratio of the last and first axes is the ratio of the product of the numbers of teeth in the drivers to the product of the numbers of teeth in the followers; and it is obvious, that so long as the same drivers and followers constitute the train, the *order* in which they succeed each other does not affect the resultant velocity-ratio.

Supposing all the wheels to be in outside gearing, then, as each elementary combination reverses the direction of rotation, and as the number of elementary combinations, $m - 1$, is one less than the number of axes, m, it is evident that if m is odd, the direction of rotation is preserved, and if even, reversed.

It is often a question of importance to determine the numbers of teeth in a train of wheels best suited for giving a determinate velocity-ratio to two axes. It was shown by Young, that to do this with the *least total number of teeth*, the velocity-ratio of each elementary combination should approximate as nearly as possible to 3·59. This would in some cases give too many axes; and as a convenient practical rule it may be laid down, that from 3 to 6 ought to be the range of the velocity-ratio of an elementary combination in wheelwork. *

Let $\frac{B}{C}$ be the velocity-ratio required, reduced to its least terms, and let B be greater than C.

If $\frac{B}{C}$ is not greater than 6, and C lies between the prescribed minimum number of teeth (which may be called t), and its double $2t$, then one pair of wheels will answer the purpose, and B and C will themselves be the numbers required. Should B and C be inconveniently large, they are if possible to be resolved into factors, and those factors, or, if they are too small, multiples of them, used for the numbers of teeth. Should B or C, or both, be at once inconveniently large, and prime, or should they contain inconveniently large prime factors, then, instead of the exact ratio $\frac{B}{C}$,

* The following are some examples of the results of Young's rule, the first line containing velocity-ratios, and the second, the numbers of elementary combinations of wheels suited to give velocity-ratios intermediate between the numbers in the first line:—

1	7	24	88	315	1132	4064	14596
1	2	3	4	5	6	7	

The following are examples of the results of the modified rule, that the lowest of the velocity-ratios for each elementary combination should range from 3 to 6:—

1	6	36	216	1296	7776
1	2	3	4	5	

some ratio approximating to that ratio, and capable of resolution into convenient factors, is to be found by the method of continued fractions (see Article 117, page 106); also Willis *On Mechanism*, pages 223 to 238).

Should $\frac{B}{C}$ be greater than 6, the best number of elementary combinations is found by dividing by 6 again and again till a quotient is obtained less than unity, when the number of divisions will be the required number of combinations, $m - 1$.

Then, if possible, B and C themselves are to be resolved each into $m - 1$ factors, which factors, or multiples of them, shall be not less than t, nor greater than $6t$; or if B and C contain inconveniently large prime factors, an approximate velocity-ratio, found by the method of continued fractions, is to be substituted for $\frac{B}{C}$, as before. When the prime factors of either B or C are fewer in number than $m - 1$, the required number of factors is to be made up by inserting 1 as often as may be necessary. In multiplying factors that are too small to serve for numbers of teeth, prime numbers differing from those already amongst the factors are to be preferred as multipliers; and in general, where two or more factors require to be multiplied, different prime numbers should be used for the different factors.

So far as the resultant velocity-ratio is concerned, the *order* of the drivers N, and of the followers n, is immaterial; but to secure equable wear of the teeth, as explained in Article 115, page 104, the wheels ought to be so arranged that for each elementary combination the greatest common divisor of N and n shall be either 1, or as small as possible; and if the preceding rules have been observed in the choice of multipliers, this will be ensured by so placing each driving wheel that it shall work with a following wheel whose number of teeth does not contain any of the same multipliers; for the original numbers B and C contain no common factor except 1.

The following is an example of a case requiring the use of additional multipliers:—Let the required velocity-ratio, in its least terms, be

$$\frac{B}{C} = \frac{360}{7}.$$

To get a quotient less than 1, this ratio must be divided by 6 three times, therefore $m - 1 = 3$. The prime factors of 360 are $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$; these may be combined so as to make three factors in various different ways; and the preference is to be given to that which makes these factors least unequal, viz., $5 \cdot 8 \cdot 9$. Hence, resolving numerator and denominator into three factors each, we have

$$\frac{B}{C} = \frac{5 \cdot 8 \cdot 9}{1 \cdot 1 \cdot 7}.$$

It is next necessary to multiply the factors of the numerator and denominator by a set of three multipliers. Suppose that the wheels to be used are of such a class that the smallest pinion has 12 teeth, then those multipliers must be such that none of their products by the existing factors shall be less than 12; and for reasons already given, it is advisable that they should be different prime numbers. Take the prime numbers, 2, 13, 17 (2 being taken to multiply 7); then the numbers of teeth in the followers will be

$$13 \times 1 = 13;\ 17 \times 1 = 17;\ 2 \times 7 = 14.$$

In distributing the multipliers amongst the factors of the numerator, let the smallest multiplier be combined with the largest factor, and so on; then we have

$$17 \times 5 = 85;\ 13 \times 8 = 104;\ 2 \times 9 = 18.$$

Finally, in combining the drivers with the followers, those numbers are to be combined which have no common factor; the result being the following train of wheels:—

$$\frac{85}{14} \cdot \frac{18}{13} \cdot \frac{104}{17} = \frac{360}{7}.$$

119. **Diametral and Radial Pitch.**—The *diametral pitch* of a circular wheel is a length bearing the same proportion to the pitch proper, or *circular pitch*, that the diameter of a circle bears to its circumference; and the *radial pitch* is half the diametral pitch. In other words, the diametral pitch is to be found by dividing the diameter of the pitch-circle by the number of teeth in the whole circumference, and the radial pitch by dividing the radius by the same number. In symbols, let p be the pitch, properly so called, or circular pitch, as measured on the pitch circle, r the radius of the pitch circle, or *geometrical radius*, and n the number of teeth; q the diametral pitch, and $\frac{q}{2}$ the radial pitch; then

$$q = \frac{113}{355} p = \frac{2r}{n};\ 2r = nq;\ p = \frac{355}{113} q;$$

$$\frac{q}{2} = \frac{113}{710} p = \frac{r}{n};\ r = \frac{nq}{2};\ p = \frac{710}{113} \cdot \frac{q}{2}.$$

Wheels are sometimes described by stating how many teeth they have for each inch of diameter; that is to say, by stating *the reciprocal of the diametral pitch in inches* $\left(\frac{1}{q} = \frac{n}{2\ r}\right)$; and the phrases used in so describing them are such as the following:—*A three-pitch wheel* is a wheel having three teeth for each inch of diameter; so that $q = \frac{1}{3}$ inch, and $p = \frac{355}{113 \times 3}$ inch $= 1{\cdot}0472$ inch; a *ten-pitch wheel* is a wheel having ten teeth for each inch of diameter; so that $q = 0{\cdot}1$ inch, and $p = \frac{35{\cdot}5}{113}$ inch $= 0{\cdot}31416$ inch; and so on.

The following are rules for solving questions regarding radial and circular pitch by graphic construction.

I. *Given, the circular pitch of a wheel, to find the radial pitch.* Draw a straight line equal to one-sixth part of the given circular pitch, and then, by Rule IV. of Article 51, page 29, find the two ends of a circular arc approximately equal in length to that straight line, and subtending 60°. The chord of that arc will be the required radial pitch very nearly, being too long by about one-900th part only.

This may be expressed in other words, as follows (see fig. 79):— Let A B be a straight line equal to one-sixth of the given circular pitch. Draw the equilateral triangle A B C, bisect B C in D, and join A D; in A B, take A E = $\frac{1}{4}$ A B, and about E, with the radius E B = $\frac{3}{4}$ A B, draw the circular arc B F, cutting A D produced in F; A F will be the required approximate radial pitch.

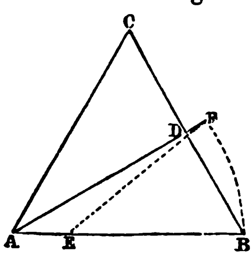

Fig. 79.

If greater accuracy is required, make the straight line equal to one-twelfth of the circular pitch, and let the angle subtended by the arc be 30°; the *radius* of that arc will be the required radial pitch, correct to one-14,400th part.

II. *Given, the radial pitch of a wheel, to find the circular pitch.* With a radius equal to six times the given radial pitch describe a circle: mark upon the circumference of that circle a chord equal to the radius, so as to lay off an arc equal to one-sixth part of the circumference; then, by Rule I. or II. of Article 51, page 28, draw a straight line approximately equal in length to that arc; the length of that straight line will be the required circular pitch, very nearly.

If Rule I. is used, the straight line will be too short by about one-900th part; if Rule II. is used, it will be too long by about one-3,600th part. If a closer approximation is required, measure the circular pitch by both rules; then to the length, as measured by Rule I., add *four times* the length as measured by Rule II., and divide the sum by *five;* the quotient will be the required circular pitch, correct to about one-40,000th part.

120. **Relative Positions of Parallel Axes in Wheelwork.**—I. Given, the radial pitch and the numbers of teeth of a pair of wheels with parallel axes, to find the length of the line of centres, or distance between the axes. Multiply the radial pitch by the sum or by the difference of the numbers of teeth, according as the wheels are in outside or inside gearing.

II. Given, the length of the line of centres, and the numbers of teeth, to find the radial pitch. Divide the given length by the sum or the difference of the numbers, according as the wheels are in outside or inside gearing.

III. Given, in fig 80, the perpendicular distance A A‴ between the first and last axes of a train of wheels, which are to turn about parallel axes all in one plane, and the numbers of teeth of the wheels; required, the positions of the several pitch-points and intermediate axes. From one end, A, of the straight line A A‴, draw, in any convenient different direction, another straight line A *a*‴, on which lay off, on any convenient scale, a series of lengths proportional to the numbers of teeth, viz.: A *i* for the first driver, *i a*′ for the first follower; *a*′ *i*′ for the second driver, *i*′ *a*″ for the second follower; and so on. Let *a*‴ be the end of that series of lengths. Draw the straight line *a*‴ A‴, and parallel to that line draw a series of straight lines, *i* I, *a*′ A′, &c., through the points of division of A *a*‴, cutting A A‴ in a corresponding series of points of division. Then A′, A″, &c., will represent the intermediate axes, and I, I′, &c., the pitch-points.

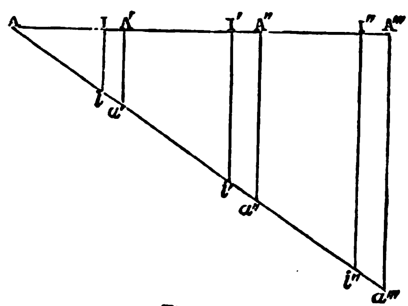

Fig. 80.

121. **Laying-off Pitch, and Subdivision of Pitch-Lines.**—The laying-off of the pitch, or of any multiple of the pitch, on the pitch-line of a wheel, is to be performed by means of Rule III. of Article 51, page 29. The laying-off of the same length upon several different pitch-lines, so as to find *corresponding pitch-points* upon them, may be performed at one operation, as follows:—Let the straight line A G represent the given length. In A G take A C = ¼ AG; and about C, with the radius C G = ¾ A G, draw a circular arc,

D G D.''''' Let A D, A D′, &c., be arcs of different pitch-lines, touching A G in A, and cut off by the dotted circular arc; each of these arcs will be approximately equal in length to A G.

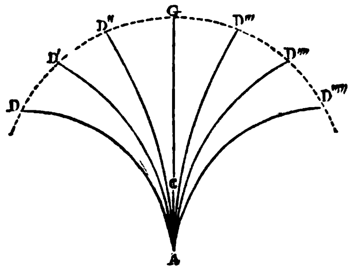

Fig. 81.

As to the operation of "*pitching*"—that is, division of a pitch-circle or other pitch-line, or of any part of a pitch-line, into any required number of parts, each equal to the pitch—see Article 51, Rules V. and VI., pages 29 and 30.

Circular and straight pitch-lines may be subdivided by means of "Dividing Engines." In a dividing engine the piece upon which divisions are to be marked is fixed upon a suitable support, capable of turning about an axis or of sliding in a straight line, as the case may be, and moved by means of a screw. By turning the screw a motion of any required extent can be given to the piece, and repeated as often as may be necessary; and after each such movement, a mark is made on the surface to be divided by means of a sharp point or edge, having a movement transverse to that of the piece to be divided.*

Machines are used by mechanical engineers, with movements on the principle of dividing engines, which serve both to *pitch* wheels or divide their pitch-circles, and to cut their teeth to the proper shape. Such machines will be again mentioned further on.

SECTION IV.—*Sliding Contact—Teeth, Screw-Gearing, and Cams.*

122. **General Principle of Sliding Contact.**—The *line of connection*, in the case of sliding contact of two moving pieces, is the common normal to their surfaces at the point where they touch; and the principle of their comparative motion is, that *the components, along that normal, of the velocities of any two points traversed by it, are equal.* This being borne in mind, all questions of the comparative motion of a pair of primary pieces in sliding contact may be solved by means of the Rules of Article 91, pages 78 to 80.

The acting surfaces of a pair of pieces in sliding contact may be both plane or both convex, or one convex and one plane; but one

* For descriptions of dividing engines for purposes of great precision, see Ramsden's *Description of an Engine for Dividing Mathematical Instruments*, 1777; Ramsden's *Description of an Engine for Dividing Straight Lines*, 1779; Holtzapffel *On Turning and Mechanical Manipulation*, vol. ii., pages 639 to 654.

of them only can be concave; and in that case the other must be convex, and of a curvature not flatter than that of the concave surface.

123. **Teeth of Wheels and Racks. General Principle.** (*A. M.*, 451.)—The figures of the teeth of wheels and racks are regulated by the principle, *that the teeth shall give the same velocity-ratio by their sliding contact which the ideal toothless pitch-surfaces would give by their rolling contact.*

Let B_1, B_2, in fig. 82, be parts of the pitch-lines of a pair of wheels, I the pitch-point, and C_1, C_2 the traces of the axes. According to Article 91, pages 78 to 80, the comparative velocity of two connected pieces depends on the position of the point where the line of connection cuts the line of centres. For a pair of smooth pitch-surfaces, that point is the pitch-point I; and for a pair of surfaces in sliding contact, it is the point where the line of connection of these surfaces (being, as stated in the preceding Article, their common normal at the point where they touch) cuts the plane of the axes. Hence the condition of the correct working of the teeth of wheels and racks is the following:—

The line of connection of the teeth should always traverse the pitch-point.

For example, in fig. 82, A_1 T_1 and A_2 T_2 may represent the traces of parts of the acting surfaces of a pair of teeth belonging to the driver and follower respectively, T_1 and T_2 a pair of particles in these surfaces, which at a given instant touch each other in one point, and P_1 T_1 T_2 P_2 the common normal at that point; then that normal ought always to traverse the pitch-point I.

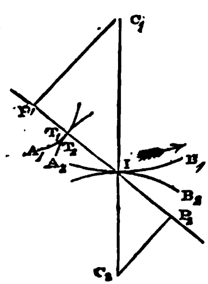

Fig. 82.

At the instant of passing the line of centres the point of contact of a pair of teeth coincides with the pitch-point.

124. **Teeth—Definitions of their Parts.**—That part of the FRONT or acting surface of a tooth which projects beyond the pitch-surface is called the FACE; that part which lies within the pitch-surface, the FLANK. The flanks of the teeth of the driver drive the faces of the teeth of the follower, and the faces of the teeth of the driver drive the flanks of the teeth of the follower. The corresponding divisions of the BACK of a tooth may be called the BACK-FACE and BACK-FLANK. The face of a tooth in outside gearing is always convex; the flank may be convex, plane, or concave.

When the motion of a pair of wheels is reversed, the backs of the teeth become the acting surfaces.

By the PITCH-POINT OF A TOOTH is meant the point where the pitch-line of the wheel cuts the front of the tooth. At the instant of passing the line of centres, the pitch-points of a pair of teeth coincide with each other, with the point of contact, and with the pitch-point of the pitch-lines.

The DEPTH of a tooth is the distance in the direction of a radius from root to crest; the extent to which the crest of a tooth projects beyond the pitch-surface is called the ADDENDUM; and a line parallel to the pitch-line, and touching the crests of all the teeth of a wheel or rack, is called the ADDENDUM-LINE, or, in a circular wheel, the ADDENDUM-CIRCLE. The radius of the addendum-circle of a circular wheel is called the REAL RADIUS, to distinguish it from the radius of the pitch-circle, which is called the GEOMETRICAL RADIUS.

CLEARANCE or FREEDOM is the excess of the total depth above the working depth, or, in other words, the least distance between the crest of a tooth of one wheel and the bottom of the hollow between two teeth of another wheel, with which the first wheel gears.

The pitch of a pitch-line is divided by the fronts and backs of the teeth into THICKNESS and SPACE. The excess of the space between the teeth of one wheel above the thickness of the teeth of another wheel with which the first wheel gears is called PLAY or BACK-LASH; because it is the distance through which the pitch-line of the driver moves after having its motion reversed before the backs of the teeth begin to act.

125. **Customary Dimensions of Teeth.**—The following are *customary dimensions* for teeth, taken from a table which Mr. Fairbairn gives in his treatise *On Millwork* (see fig. 83).

It is to be understood that these customary dimensions may be departed from when there is any sufficient reason for doing so. Examples of this will appear in the sequel.

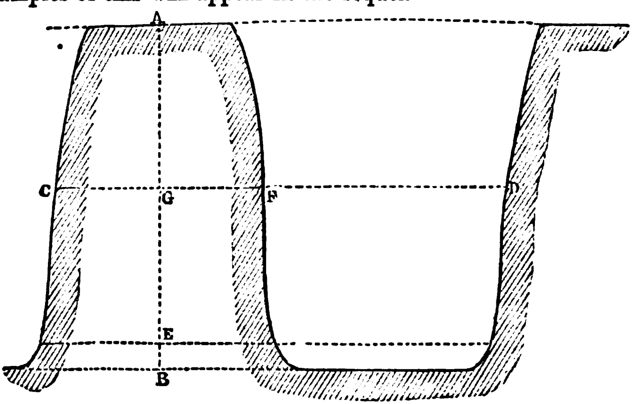

Fig. 83.

Let the pitch C D = p; then

Depth, total, A B = $0{\cdot}75\,p$;

$$\left.\begin{array}{l}\text{Clearance or freedom, E B,}\ldots\ldots\ldots\ldots \\ \text{Also, play or back-lash, F D} - \text{C F}\end{array}\right\} = f = 0{\cdot}06\,p + 0{\cdot}04 \text{ inch;*}$$

Depth, working, A E = $0{\cdot}75\,p - f$;

Addendum, A G = $\frac{1}{2}$ A E;

Thickness, C F = $\frac{p - f}{2}$;

Space, F D = $\frac{p + f}{2}$.

Thickness of *ring* which carries the teeth (in a cast-iron wheel) = thickness of tooth at root.

The *least thickness sufficient* for the teeth of a given pair of wheels is a question of strength, depending on the force to be exerted; and although such questions properly belong to a later division of this treatise, it may be convenient to state here the rule generally relied on:—*Divide the greatest pressure to be exerted between a pair of teeth in pounds by* 1,500; *the square root of the quotient will be the least proper thickness in inches.*

For pressures expressed in kilogrammes, and thicknesses in millimètres, the divisor becomes 1·055; the rule being in other respects the same.

The *least breadth sufficient* for the fronts of teeth is a quantity depending on dynamical principles, and belonging properly to the next division; but for convenience it may here be stated that an ordinary rule is as follows:—*Divide the greatest pressure to be exerted in pounds by the pitch in inches, and by* 160; *the quotient will be the breadth in inches.*

For pressures in kilogrammes and dimensions in millimètres, instead of dividing by 160, multiply by 9.

126. **Teeth for Inside Gearing.**—The figures of the acting surfaces of teeth for a pitch-circle in inside gearing are exactly the same with those suited for the same pitch-circle in outside gearing; but the relative positions of teeth and spaces, and those of faces and flanks, are reversed; and the addendum-circle is of less radius than the pitch-circle. All the rules in the ensuing Articles, with these modifications, may be applied to inside gearing.

127. **Common Velocity and Relative Velocity of Teeth—Approach and Recess—Path of Contact.**—The *common velocity* of a pair of

* 0·04 inch = 1 millimètre, nearly.

teeth is that component velocity along the line of connection which is common to the pair of particles that touch each other at a given instant. In fig. 82, page 115, let $C_1 P_1$ and $C_2 P_2$ be the two common perpendiculars of the line of connection and the two axes respectively, and let a_1 and a_2 denote the angular velocities about those axes; then the common component in question has the value

$$a_1 \cdot C_1 P_1 = a_2 \cdot C_2 P_2 \text{.....................(1.)}$$

The *relative velocity* of a pair of teeth is the velocity with which their acting surfaces slide over each other; and it is found as follows:—Conceive one of the pitch-surfaces to be fixed, and the other to roll upon it, so that the line of contact (I, fig. 82, page 115) becomes an instantaneous axis; find the resultant angular velocity (see Articles 73 to 77, pages 52 to 56, and Articles 81 and 82, pages 66 to 68), and multiply it by the perpendicular distance of the point of contact of the teeth (T, fig. 82) from the instantaneous axis; the product will be the relative velocity required. That is to say, let c denote the resultant angular velocity about the instantaneous axis of the pitch-surface which is supposed to roll; and in fig. 82 let I T be the perpendicular distance of the point of contact from the instantaneous axis; then the relative velocity of sliding is

$$c \cdot \mathrm{I\,T} \text{..............................(2.)}$$

The values of the resultant angular velocity c (as has been shown in the previous Articles, already referred to) are, for parallel axes in outside gearing, $c = a_1 + a_2$; for parallel axes in inside gearing, $c = a_1 - a_2$; and for intersecting axes, the diagonal of a parallelogram, of which a_1 and a_2 are the sides.

While the point of contact, T, is advancing towards the pitch-point I, the roots of the teeth are sliding towards each other; and this relative motion is called the APPROACH.

The relative velocity gradually diminishes as the approach goes on, and vanishes at the instant when I T = 0; that is, when the point of contact coincides with the pitch-point; so that at that precise instant the pair of teeth are in rolling contact.

After the point of contact has passed the pitch-point, the roots of the teeth are sliding away from each other with a gradually increasing relative velocity; and this relative motion is called the RECESS.

During the approach the flank of the driver drives the face of the follower; during the recess the face of the driver drives the flank of the follower.

The *extent of the sliding motion* of a pair of teeth is equal, during the approach, to the excess of the length of the face of the driven tooth above the length of the flank of the driving tooth; and during the recess, to the excess of the length of the face of the driving tooth above the length of the flank of the driven tooth.

The PATH OF CONTACT is the line traversing the various positions of the point of contact, T (fig. 82, page 115). If the line of connection preserves always the same position, the path of contact coincides with it, and is straight; in other cases the path of contact is curved.

It is divided by the pitch-point I into two parts: the *path of approach*, described by T in approaching the pitch-point; and the *path of recess*, described by T after having passed the pitch-point.

The path of contact is bounded where the approach commences by the addendum-line of the follower; and where the recess terminates, by the addendum-line of the driver. The length of the path of contact must be such that there shall always be at least one pair of teeth in contact; and it is better still, when practicable, to make it so long that there shall always be at least two pairs of teeth in contact; but this is not always possible.

128. **Arc of Contact.** (*A. M.*, 454.)—The arc of contact on a pitch-line is that part of the pitch-line which passes the pitch-point during the action of one given tooth with the corresponding tooth of the other wheel.

In order that one pair of teeth at least may be in action at each instant, the length of the arc of contact must be *greater than the pitch;* and when practicable, it should be *double the pitch*, in order that two pairs of teeth, at least, may be in action at each instant; but this is not always practicable; and the most common values are from 1·4 to 1·8 times the pitch. It is divided by the front of the tooth to which it belongs into two parts: the *arc of approach*, lying in advance of the front of the tooth; and the *arc of recess*, lying behind the front of the tooth. It is usual to make the arcs of approach and of recess of equal length; and in that case each of them must be *greater than half the pitch*, and should, if practicable, be made *equal to the pitch*. For a given pitch-line, and a given pitch and figure of tooth, the length of those arcs depends on the addendum, in a manner to be afterwards described.

129. **Obliquity of Action.**—The obliquity of action of a pair of teeth is the angle which the line of connection makes at any instant with a tangent plane to the two pitch-surfaces; for example, in fig. 82, page 115, the *complement* of the angle at I. When the path of contact is a straight line, coinciding at every instant with the line of connection, the obliquity is constant; in other cases it is variable; and its mode of variation is usually such that it diminishes during the approach, and increases again during recess. In a dynamical point of view, it is advantageous to make

the obliquity as small as possible; and, on the other hand, there is a connection between the obliquity of action and the number of teeth which makes it impracticable to use pinions of fewer than a certain number of teeth with less than a certain maximum obliquity of action. Mr. Willis, from an examination of the results of ordinary practice, concludes that the best value on the whole for the *mean obliquity* of action in toothed gearing is between 14° and 15°. Such an angle may be easily constructed by drawing a right-angled triangle whose three sides bear to each other the proportion of the numbers

$$65 : 63 : 16;$$

when the required angle will lie opposite to the shortest side of the triangle. The values of its chief trigonometrical functions are—

sine,................	$16 \div 65$	=	0·2461538, nearly.
cosine,.............	$63 \div 65$	=	0·9692308, nearly.
tangent,............	$16 \div 63$	=	0·2539683, nearly.
cosecant,...........	$65 \div 16$	=	4·0625.
cotangent,.........	$63 \div 16$	=	3·9375.

The corresponding angle is 14° 15′; being a little less than one-25th part of a revolution.

130. The **Teeth of Spur-Wheels and Racks** have acting surfaces of the class called *cylindrical surfaces*, in the comprehensive sense of that term; and their figures are designed by drawing the traces of their surfaces on a plane perpendicular to the axes of the wheels (or, in the case of a rack, to the axis of the wheel that is to gear with the rack); which plane contains the pitch-lines and the line of connection, and may be represented by the plane of the paper in fig. 82, page 115. The path of contact, also, is situated in the same plane; and the angle of obliquity of action is at each instant equal to the angle I C P, which the common perpendicular, C P, of the line of connection and one of the axes makes with the line of centres, C_1 I C_2. Because of the comparative simplicity of the rules for drawing the figures of the teeth of spur-wheels, those rules are used, with the aid of certain devices to be afterwards described, for drawing the figures of the teeth of bevel wheels and skew-bevel wheels also.

131. **Involute Teeth for Circular Wheels.** (*A. M.*, 457.)—The simplest of all forms for the teeth of circular wheels is that in which the path of contact is a straight line always coinciding with the line of connection, which makes a constant angle with the line of centres, and is inclined at a constant angle of obliquity to the common tangent of the pitch-lines.

In fig. 84, let C_1, C_2, be the centres of two circular wheels, whose pitch-circles are marked B_1, B_2. Through the pitch-point I draw the intended *line of connection*, P_1 P_2, making, with the line of centres, the angle C I P = the complement of the intended obliquity.

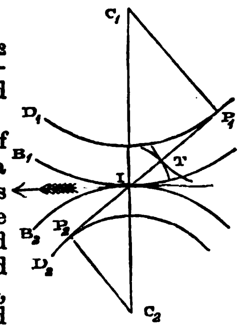

Fig. 84.

From C_1 and C_2 draw C_1 P_1 and C_2 P_2 perpendicular to P_1 P_2, with which two perpendiculars as radii describe circles (called *base-circles*) marked D_1, D_2.

Suppose the base-circles to be a pair of circular pulleys, connected by means of a cord whose course from pulley to pulley is P_1 I P_2. As the line of connection of those pulleys is the same with that of the proposed teeth, they will rotate with the required velocity-ratio. Now, suppose a tracing point, T, to be fixed to the cord, so as to be carried along the path of contact, P_1 I P_2. That point will trace, on a plane rotating along with the wheel 1, part of the involute of the base-circle D_1, and on a plane rotating along with the wheel 2, part of the involute of the base-circle D_2, and the two curves so traced will always cut the line of connection at right angles, and touch each other in the required point of contact T, and will therefore fulfil the condition required by Article 122, page 114. The teeth thus traced are called *Involute Teeth*.

All involute teeth of the same pitch work smoothly together.

The following is the process by which the figures of involute teeth are to be drawn in practice:—

In fig. 85, let C represent the centre of the wheel, I the pitch-point, C I the geometrical radius, B I B the pitch-circle, and let the intended angle of obliquity of action be given, and also the pitch. (In the example represented by the figure, the obliquity is supposed to be $14\frac{1}{4}°$, as stated in Article 129, page 120; and the wheel has 30 teeth.) Then proceed by the following rules:—

I. *To draw the base-circle and the line of connection.* About C, with the radius C P = C I × cosine of obliquity (that is to say, in the present example, $\frac{63}{65}$ C I), draw a circle, D P D; this is the *base-circle*. Then about I, with a radius I P = C I × sine of obliquity (that is to say, in the present example, $\frac{16}{65}$ C I), draw a short circular arc, cutting the base-circle in P. Draw the straight line P F I E; this will be the line of connection; and it will touch the base-circle at P.

II. *To find the normal pitch, the addendum, and the real* *radius, and to draw the addendum-circle and the flank-circle.* At the pitch-point, I, draw the straight line I A, touching the pitch-

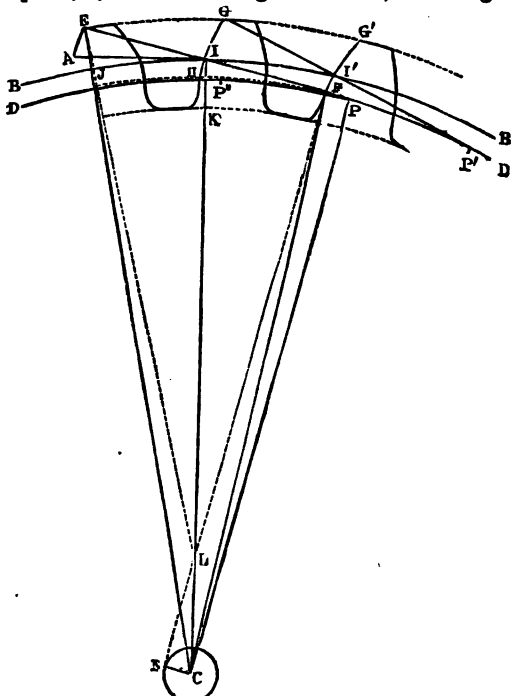

Fig. 85.

circle, and lay off upon it the length I A equal to the pitch. From A let fall A E perpendicular to I E. Then I E will be what may be called the NORMAL PITCH, being the distance, as measured along the line of connection, from the front of one tooth to the front of the next tooth.

The normal pitch is also the *pitch on the base-circle;* that is, the distance, as measured on the base-circle, between the front of one tooth and the front of the next.

The ratio of the normal pitch of involute teeth to the circular pitch is equal to the ratio of the radius of the base-circle to that of the pitch-circle; that is to say,

$$\frac{IE}{IA} = \frac{CP}{CI} = \text{cosine of obliquity} \left(= \frac{63}{65} \right.$$

in the present example).

In order that two pairs of teeth at least may always be in action, the *arc of contact* is to consist of two halves, each equal to the pitch (see Article 128, page 119). Lay off on the line of connection, E P, the distance I F = I E. Then E F will be the *path of contact* (Article 127, page 119), consisting of two halves, each equal to the normal pitch.

Draw the straight line C E; this will be the *real radius*, and the circle E G G′, drawn with that radius, will be the *addendum-circle*, which all the crests of the teeth are to touch. Then, with the radius C F, draw the circle F H (marked with dots in the figure); this may be called the FLANK-CIRCLE, for it marks the inner ends of the flanks of all the teeth.

The *addendum* is C E − C I.

III. *To draw the* ROOT CIRCLE; that is, the circle which the bottoms of all the hollows between the teeth (or CLEARING CURVES, as they are called) are to touch. First find, by drawing or by calculation, the *greatest addendum* of any wheel with which the given wheel may have to gear; that is, the addendum of the smallest practicable pinion of the same pitch and obliquity; that is, the addendum of a pinion in which the pitch subtends at the centre an angle approximately equal to the obliquity. With the obliquity already stated, such a pinion has 25 teeth. To find the addendum of such a pinion by drawing:—Through F, parallel to P C, draw F L, as cutting I C in L. Join L E; then L E − L I will be the required *greatest addendum*. To find the greatest addendum by calculation, let ϕ denote the obliquity, and p the pitch; then

$$\text{L E} - \text{L I} = p \operatorname{cotan} \phi \left\{ \sqrt{(3 \sin^2 \phi + 1)} - 1 \right\}.$$

With the angle of obliquity already stated, this gives

$$\text{L E} - \text{L I} = 0{\cdot}343\, p, \text{ very nearly};$$

and this is the origin of the value $0{\cdot}35\, p$, which is very commonly used for the addendum of teeth.

To the greatest addendum, thus found, add a suitable allowance for clearance (Article 125, page 116), and lay off the sum I K inwards from the pitch-circle along the radius. Then C K will be the radius of the required root-circle.

IV. *To draw the traces of the teeth.* Mark the pitch-points of the fronts of the teeth (I, I′, &c.), according to the principles of Article 121, page 113, and those of their backs, by laying off a suitable thickness on the pitch-circle (see Article 125, page 116). Obtain a "*templet*," or thin flat disc of wood or metal, having its edge accurately shaped to the figure of the base-circle. Such a templet is represented in plan by C D D, fig. 86, and in elevation by D′ D′. A piece of watch-spring, marked P M in plan, and P′ M′ in elevation, is to have its edges filed so as to leave a pair of sharp

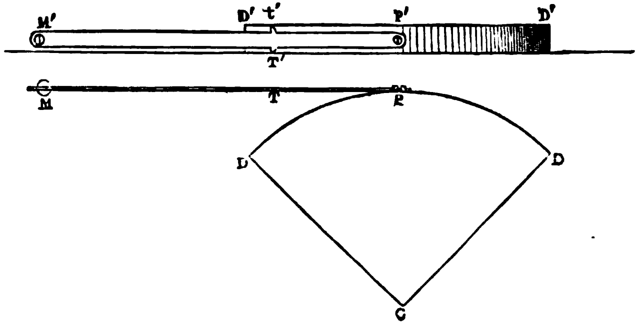

Fig. 86.

projecting tracing-points, marked T′, t′, in elevation, and T in plan. One end of that spring, P, P′, is to have a round hole drilled in it, and to be fixed to the middle of the edge of the templet by means of a screw, about which the spring is to be free to turn; and the other end, M, M′, is to be fitted with a knob to hold it by. Place the templet on the drawing (or pattern, as the case may be), so that C shall coincide with the centre of the wheel, and D D with the base-circle; and also so that the lower of the two tracing-points, when the spring is moved to and fro, shall pass through the pitch-point of a tooth; then that tracing-point will draw the trace of the front of the tooth; and by turning the templet about C, and repeating the process, the traces of the fronts of any required number of teeth may be drawn.

To draw the traces of the backs of the teeth, the position of the spring relatively to the templet is to be reversed, by turning it about the screw at P, so as to use the tracing-point that was previously uppermost.

The distance, P T, from the screw to the tracing-points should not be less than *twice the normal pitch.*

V. The *Clearing Curves* are the traces of the hollows which lie inside the flank-circle, F H, fig. 85. Their side parts ought to be tangents to the inner ends of the flanks of the teeth (at F and H, for example), and their bottom parts ought to coincide with the root-circle through K. Those different parts may be joined to each other by means of small circular arcs. In connection with the figures of the side parts of those clearing curves, it may be observed, that F L is a tangent to the inner end of the flank I′ F, and therefore to the clearing curve at that point; and that tangents to the inner ends of other flanks may be drawn by repeating the process by which F L is drawn, or by the following process:—About C, with the radius C N = P F, draw a circle; F L N will be a straight tangent to that circle; and so also will all the tangents to the flanks at their inner ends. Therefore, from the inner ends of all the flanks, both front and back, draw straight lines touching the circle C N, and so placed that the straight lines from the front and back flanks of the same tooth shall not cross each other; these lines will show the proper positions for the side parts of the clearing curves. When the flank-circle coincides with the base-circle (as in the smallest pinion of a given pitch), the side parts of the clearing curves coincide with the radii drawn from the centre C to the inner ends of the flanks.

132. **Involute Teeth for Racks.**—The following is the process of designing the teeth of a straight rack which is to gear with an involute-toothed wheel of a given pitch and a given obliquity:—In fig. 87, let A B be the pitch-line of the rack, and let A I = I I′ be the pitch. Lay off A I E = the given angle of obliquity, and from A let fall A E perpendicular to I E; then I E will be the *normal pitch*; further, if the path of contact is to consist of two halves, each equal to half the normal pitch, I E will be one of those halves; then in E I produced make I F = I E, and I F will be the other half of the path of contact. Through E, parallel to A B, draw E G G′; this will be the *addendum line;* through F, parallel to B A, draw F H; this will be the *flank-line,* marking the inner ends of the acting surfaces of the teeth. Perpendicular to A B draw I K, equal to the greatest addendum in the set of wheels of the given pitch and obliquity with an allowance for clearance added, as in Rule III. of Article 131, page 123; through K, parallel to A B, draw a straight line; this will be the *root-line,* with which the bottoms of all the hollows between the teeth are to coincide.

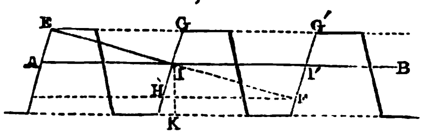

Fig. 87.

The traces of the fronts of the teeth are straight lines perpendicular to E F, and the fronts themselves are planes perpendicular to E F. The backs of the teeth are planes inclined at the same angle to A B in the contrary direction.

133. **Peculiar Properties of Involute Teeth.**—Involute teeth have some peculiar properties not possessed by teeth of other figures.

I. Sets of involute teeth have *a definite and constant normal pitch;* being, as already explained, the distance between the fronts of successive teeth, measured on the path of contact, or on the circumference of the base-circle; and *all wheels and racks with involute teeth of the same normal pitch gear correctly with each other.*

II. The length of the line of centres, or perpendicular distance between the axes, of a pair of wheels with involute teeth of the same normal pitch, or the perpendicular distance from the axis of a wheel with involute teeth to the addendum-line of a rack with which it gears, *may be altered;* and so long as the wheels, or wheel and rack, are sufficiently near together to make the path of contact longer than the normal pitch, and sufficiently far asunder for the crests of each set of teeth to clear the hollows between the teeth of the other set, the wheels, or the wheel and rack, will continue to work correctly together, and to preserve their velocity-ratio; although, in the case of a pair of wheels, the pitch-lines, the pitch as measured on the pitch-lines, and the obliquity, will all be altered when the length of the line of centres is altered. In other words, the velocity-ratio of a pair of wheels with involute teeth of the same normal pitch is the reciprocal of the ratio of the radii of their

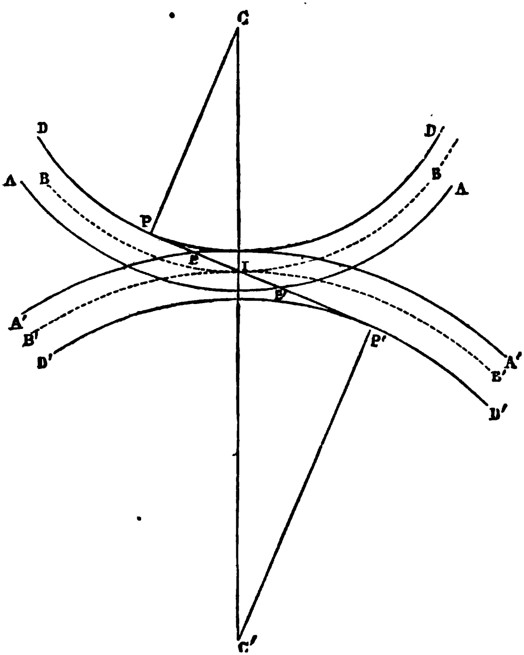

Fig. 88.

base-circles, and depends on this ratio alone; and the velocity-ratio of a wheel and rack with involute teeth of the same normal pitch depends solely on the radius of the base-circle of the wheel and on the angle of obliquity of the line of connection.

Another way of stating this property of involute teeth is, that the pitch-lines of wheels and racks with such teeth are arbitrary to an extent limited only by the necessity of having a path of contact of a certain length.

One practical result of this is (as Mr. Willis first pointed out), that the *back-lash* of involute teeth is variable at will, being capable of being increased or diminished by moving the wheels, or the wheel and rack, farther from or nearer to each other, and may thus be adjusted so as to be no greater than is absolutely necessary in order to prevent jamming of the teeth—a property not possessed by teeth of any other figure.

III. *Given* (in fig. 88), *the centres*, C, C′, *the base-circles*, D D, D′ D′, *and the addendum-circles*, A A, A′ A′, *of a pair of spur-wheels with involute teeth of a given normal pitch, to find the line of connection, the pitch-point, the pitch-circles, the pitch on the pitch-circles, and the path of contact.*

Draw a common tangent, P P′, to the two base-circles in such a position as to run from the driver to the follower in the direction of motion. That common tangent will be the line of connection: the point I, where it cuts the line of centres, will be the pitch-point: two circles, B B and B′ B′, described about C and C′ respectively, and touching each other in I, will be the pitch-circles: the pitch on the pitch-circles will be greater than the normal pitch in the ratio $\frac{C\,I}{C\,P} = \frac{C'\,I}{C'\,P'}$; and the part E E′ of the line of connection which lies between the two addendum-circles will be the path of contact.

IV. *Given* (in fig. 89), *the centre*, C, *the base-circle*, D D, *and the addendum-circle*, A A, *of a spur-wheel with involute teeth of a given*

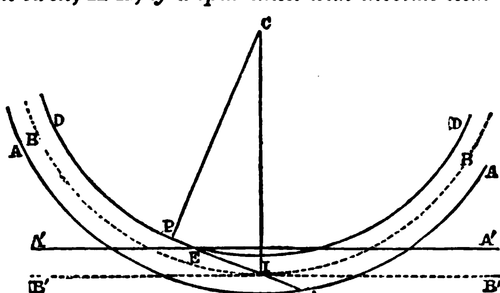

Fig. 89.

normal pitch; also the pitch-line, B′ B′, *and the addendum-line*, A′ A′, *of a rack which is to have involute teeth of the same normal pitch; to* *find the pitch-point, the pitch-circle of the wheel, the line of connection, the pitch, as measured on the pitch-lines, the path of contact, and the position of the fronts of the teeth of the rack.*

From C let fall C I perpendicular to B′ B′; then I will be the pitch-point; and a circle, B B, of the radius C I, will be the pitch-circle of the wheel. From I draw I P, touching the base-circle D D; I P will be the line of connection. The pitch, as measured on the pitch-lines, will be greater than the normal pitch, in the ratio $\frac{C\,I}{C\,P}$ of the radius of the pitch-circle to that of the base-circle. The path of contact will be the part E E′ of the line of connection, which is contained between the addendum-line of the rack, A′ A′, and the addendum-circle of the wheel, A A. The fronts of the teeth of the rack are to be planes perpendicular to I P, or, in other words, parallel to P C.

V. By the application of the preceding principles, two or more wheels of different numbers of teeth, turning about one axis, can be made to gear correctly with one wheel or with one rack; or two or more parallel racks, with different obliquities of action, may be made to gear correctly with one wheel, the normal pitches in each case being the same; and thus *differential movements* of various sorts may be obtained. This is not possible with teeth of any other form.

The obliquity of the action of involute teeth is by many considered an objection to their use; and that is the reason why, notwithstanding their simplicity and their other advantages, they are not so often used as other forms. In anticipation of the subject of the dynamics of machinery it may be stated, that the principal effect of the obliquity of the action of involute teeth is to increase the pressure exerted between the acting surfaces of the teeth, and also the pressure exerted between the axles of the wheels and their bearings, nearly in the ratio in which the radius of the pitch-circle of each wheel is greater than the radius of the base-circle, and that a corresponding increase of friction is produced by that increase of pressure. In the example of Article 131, that ratio is 65 : 63.

134. **Teeth for a Given Path of Contact.**—In the three preceding Articles the forms of the teeth are found by assuming a figure for the path of contact—viz., the straight line. Any other convenient figure may be assumed for the path of contact, and the corresponding forms of the teeth found, by determining what curves a point moving along the assumed path of contact will trace on two discs, rotating round the centres of the wheels with angular velocities, which bear that relation to the component velocity of the tracing-point along the line of connection which is given by the principles of Article 127, page 118. This method of finding the forms of the teeth of wheels is the subject of an interesting treatise by Mr. Edward Sang.

All wheels having teeth of the same pitch, traced from the same path of contact, work correctly together, and are said to belong to the *same set*.

135. **Teeth Traced by Rolling Curves.** (*A. M.*, 452.)—From the principles of Articles 122 and 123, pages 114, 115, it appears that at every instant the position of the point of contact, T, of the acting surfaces of a pair of teeth (fig. 82, page 115), and the corresponding position of the pitch-point I in the pitch-lines of the wheels to which those teeth belong, are so related, that the line, I T, which joins them, is normal to the surface of each of the teeth at the point T. Now this is the relation which exists between the *tracing-point* T, and the *instantaneous axis or line of contact* I, in a rolling curve of such a figure, that, being rolled upon the pitch-line, its tracing-point T traces the outline of a tooth. (As to rolling curves and rolled curves, see Articles 72, 74, 75, 77, 78, 79, pages 51 to 62.)

In order that a pair of teeth may work correctly together, it is necessary and sufficient that the *instantaneous normals* from the pitch-point to the acting surfaces of the two teeth should coincide at each instant; and this condition is fulfilled *if the outlines of the two teeth be traced by the motion of the same tracing-point, in rolling the same rolling curve on the same side of the pitch-lines of the respective wheels.*

The *flank* of a tooth is traced while the rolling curve rolls *inside* of the pitch-line; the *face*, while it rolls *outside*.

To illustrate this more fully, the following explanation is quoted from the Article "*Mechanics* (*Applied*)," in the *Encyclopædia Britannica* (see fig. 90):—"If any curve, R, be rolled on the inside of the pitch-line, B B, of a wheel, the instantaneous axis of the rolling curve at any instant will be at the point I, where it touches the pitch-line for the moment; and consequently the line A T, traced by a tracing-point T, fixed to the rolling curve, will be everywhere perpendicular to the straight line T I; so that the traced curve A T will be suitable for the flank of a tooth, in which T is the point of contact corresponding to the position I of the pitch-point. If the same rolling curve R, with the same tracing-point T, be rolled on the *outside* of any other pitch-line, it will trace the *face* of a tooth suitable to work with the *flank* A T.

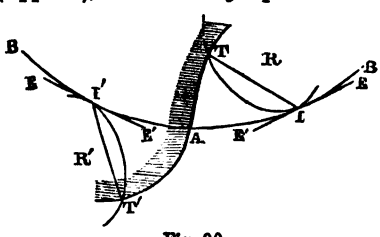

Fig. 90.

"In like manner, if either the same or any other rolling curve

R′ be rolled the opposite way, on the *outside* of the pitch-line B B, so that the tracing-point T′ shall start from A, it will trace the *face* A T′ of a tooth suitable to work with a *flank* traced by rolling the same curve R′ with the same tracing-point T′ *inside* any other pitch-line.

"The figure of the *path of contact* is that traced on a fixed plane by the tracing-point, when the rolling curve is rotated in such a manner as always to touch a fixed straight line E I E (or E′ I′ E′, as the case may be) at a fixed point I (or I′).

"If the same rolling curve and tracing-point be used to trace both the faces and the flanks of the teeth of a number of wheels of different sizes, but of the same pitch, all those wheels will work correctly together, and will form a *set*. The teeth of a *rack* of the same set are traced by rolling the rolling curve on both sides of a straight line.

"The teeth of wheels of any figure, as well as of circular wheels, may be traced by rolling curves on their pitch-lines; and all teeth of the same pitch, traced by the same rolling curve with the same tracing-point, will work together correctly if the pitch-surfaces are in rolling contact."

Involute teeth themselves might be traced by rolling a logarithmic spiral on the pitch-circle; but it is unnecessary to explain this in detail, as the ordinary method of tracing them is much more simple.

136. **Epicycloidal Teeth in General.**—For tracing the figures of teeth, the most convenient rolling curve is the circle. The path of contact which a point in its circumference traces is identical with the circle itself; the flanks of the teeth for circular wheels are internal epicycloids, and their faces external epicycloids, and both flanks and faces are cycloids for a straight rack. (See Article 74, page 53, and Article 77, page 56.)

Wheels of the same pitch, with epicycloidal teeth traced by the same rolling circle, all work correctly with each other, whatsoever may be the numbers of their teeth; and they are said to belong to *the same set*.

For a pitch-circle of twice the radius of the rolling or *describing* circle (as it is called), the internal epicycloid is a straight line, being a diameter of the pitch-circle; so that the flanks of the teeth for such a pitch-circle are planes radiating from the axis. For a smaller pitch-circle, the flanks would be convex, and *incurved* or *under-cut*, which would be inconvenient; therefore the smallest wheel of a set should have its pitch-circle of twice the radius of the describing circle, so that the flanks may be either straight or concave.

In fig. 91, let B B be the pitch-circle of a wheel, C C the line of centres, I the pitch-point, R the internal describing circle, and R′

the external describing circle, so placed as to touch the pitch-circle and each other at I; let E E be a straight tangent to the pitch-circle at the pitch-point; and let T I T′ be the path of contact, consisting of the path of approach, T I, and the path of recess, I T′. Each of those arcs should be equal to the pitch when practicable, in order that there may be always at least two pairs of teeth in action; but this is not always possible; and the length of each of them in many cases is only from 0·7 to 0·9 of the pitch, being regulated by the customary practice of making the addendum from 0·3 to 0·35 of the pitch.

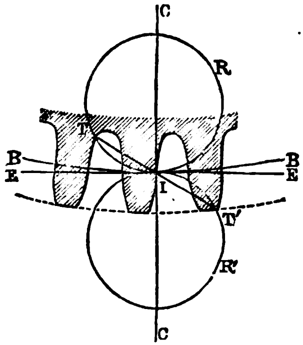

Fig. 91.

The *real radius* of the wheel is the distance from its centre to the point T′, at the outer end of the face of a tooth; the dotted circle traversing T′ is the *addendum-circle*, and the perpendicular distance from T′ to the pitch-circle B B is the *addendum*.

The *flank-circle* is a circle described about the centre of the wheel, and traversing the point T; and the *clearing curves* (as in the case of involute teeth, Article 131, Rule V., page 124) must have a depth sufficient to clear the greatest addendum given to the teeth of any one of the set of wheels that are capable of gearing with the wheel under consideration.

In passing the line of centres, the line of connection coincides with the tangent E E, and the *obliquity* is nothing. The greatest angle of obliquity of action is, during the approach, E I T, and during the recess E I T′; and the mean angles of obliquity during the approach and recess are the halves of those greatest angles respectively. From the results of practical experience, Mr. Willis deduces the rule that the mean obliquity should not exceed 15°, or one-twenty-fourth of a revolution; therefore the maximum obliquity should not exceed 30°, or one-twelfth of a revolution; therefore the arcs I T and I T′ should neither of them in any case exceed one-sixth of the circumference of the describing circles to which they respectively belong; from which it follows, that if either of those arcs is to be equal to the pitch, the circumference of the describing circle ought *not to be less than six times the pitch;* therefore the smallest pinion of a set should have *twelve teeth.*

137. **Tracing Epicycloidal Teeth by Templets.**—The face of an epicycloidal tooth may be traced by rolling a templet of the form of the describing circle upon a convex templet of the form of the pitch-circle; and the flank, by rolling a templet of the form of the

describing circle upon a concave templet of the form of the pitch-circle.

When the fixed templet is either convex (as when the face of the tooth of a wheel is to be traced) or straight (as when either the face or the flank of the tooth of a rack is to be traced), the rolling templet may be prevented from slipping on the fixed templet by connecting them together by means of a slender piece of watch-spring, as follows:—In fig. 92, C B B represents the fixed templet

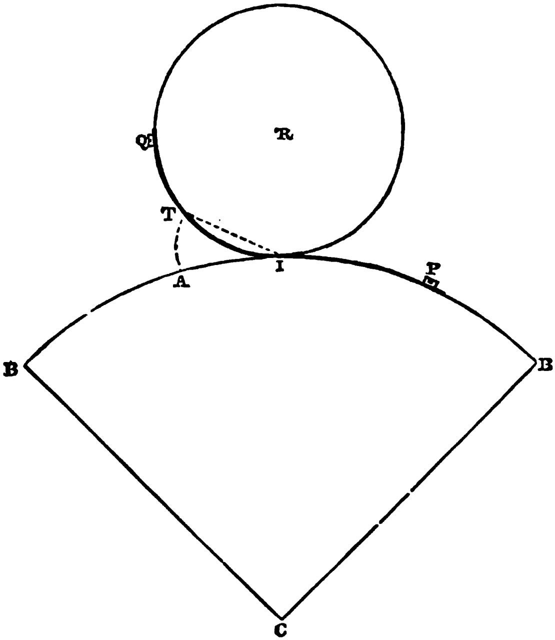

Fig. 92.

of the form of the pitch-circle, and R the rolling templet of the form of the describing circle. Q I P is a slender piece of watch-spring, fastened by a screw at P to the edge of the fixed templet, and by a screw at Q to the edge of the rolling templet. The spring may have a sharp tracing-point formed at T on one of its edges, as already described in Article 131, Rule IV., and shown in fig. 86, page 124. A T, in fig. 92, represents part of the epicycloid traced by the point T, and I the point of contact of the pitch-circle and describing circle. The radius of each of the templets ought to be made less than the radius of the circle which it represents, by *half the thickness* of the spring P Q.

When the fixed templet is concave (for tracing the flanks of teeth) this method of preventing the rolling templet from slipping is not available.

138. **Straight-Flanked Epicycloidal Teeth.**—In the oldest form of epicycloidal teeth, the traces of the flanks are straight lines radiating from the centre of the wheel, being the lines which would be traced by a describing circle, of half the radius of the pitch-circle, rolling inside the pitch-circle. Hence, in order that a pair of wheels with teeth described according to this principle may gear correctly together, the faces of the teeth of each wheel must be traced by rolling upon the outside of its pitch-circle a describing circle of *half the radius of the other pitch-circle.*

For example, in fig. 93, let C and C′ be the centres of a pair of

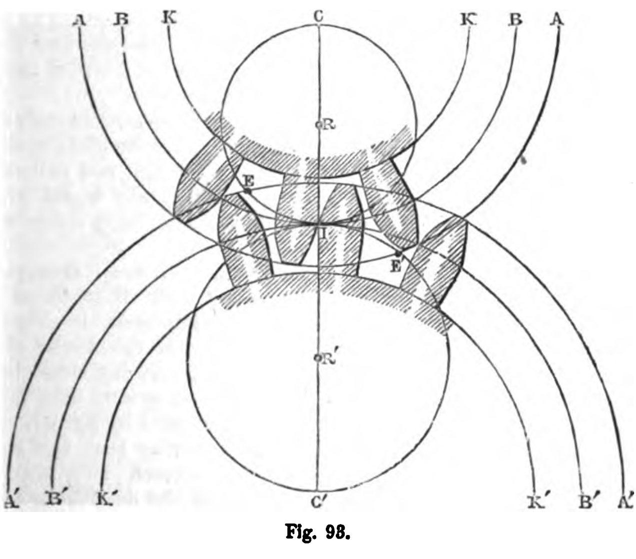

Fig. 93.

spur-wheels, and B B and B′ B′ their pitch-circles, touching each other in the pitch-point I. Lay off the pitch-points of the fronts and backs of the teeth on each of the pitch-circles, and draw straight lines from the centres of the wheels to the points of division of their respective pitch-circles; these lines will be the traces of the flanks of the teeth. Bisect C I in R, and C′ I in R′, and about R and R′ respectively describe circles traversing I; these will be the two describing circles for the faces of the teeth. Lay off, on those two circles, sufficient lengths, I E and I E′, for the two divisions of the path of contact; that is to say, each of these lengths must be greater than half the pitch, and should be made as nearly equal to the pitch as practicable. Then a circle, A A, described about C through E′, will be the addendum-circle of the first wheel, and a circle, A′ A′, described about C′, through E, will be the addendum-circle of the second wheel. The two root-circles, K K and K′ K′, are to be drawn so as to leave a sufficient clearance between each of them and the opposite addendum-circle.

To trace the front and back faces of the teeth of the first wheel, roll the describing circle R′ on the pitch-circle B B; to trace the front and back faces of the teeth of the second wheel, roll the describing circle R on the pitch-circle B′ B′. This may be done with the aid of templets connected together by means of a spring, as described in Article 137, page 131.

The traces of the flanks of the teeth of a rack, according to this system, are straight lines perpendicular to the pitch-line, and those of the faces are cycloids. The traces of the faces of a wheel that is to gear with a rack are involutes of the pitch-circle.

Epicycloidal teeth described by this method are very smooth and accurate in their action; but they labour under the disadvantage that the faces of the teeth of any given wheel are not suited to work accurately with the flanks of the teeth of any wheel whose radius differs from double the radius of the describing circle with which they were traced.

139. **Epicycloidal Teeth Traced by an Uniform Describing Circle.**—The property of working accurately with all teeth of the same pitch, whatsoever the radius of the pitch-circle, is given to epicycloidal teeth by tracing both the faces and the flanks of all teeth of the same pitch, by rolling the same describing circle upon the outside and the inside of the pitch-circle—a system first introduced by Mr. Willis. This method is illustrated by fig. 91, page 131, already described in Article 136. In order that the mean obliquity of action may not in any case exceed 15°, nor the maximum obliquity 30°, the circumference of the describing circle employed is *six times the pitch;* so that its radius is *six times the radial pitch* (see Article 119, page 111). According to this system, the traces of both the flanks and the faces of the teeth of a rack are cycloids.

140. **Approximate Drawing of Epicycloidal Teeth.**—Various approximate methods of drawing epicycloids have already been described in Article 79, pages 59 to 62. The following are the additional explanations required in order to show the application of those methods to epicycloidal teeth:—

I. *By two pairs of Circular Arcs.* In fig. 94, let I A be part of the pitch-circle of a wheel. Draw the describing circle touching the pitch-circle at any convenient point, I, and outside or inside,

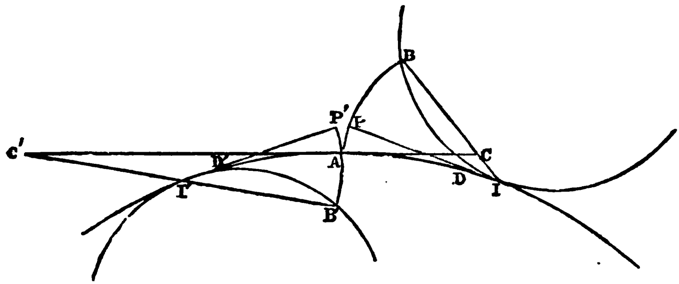

Fig. 94.

according as the face or the flank of a tooth is to be traced. (Letters without an accent refer to the face; letters with an accent, to the flank.)

Draw the straight tangent I P, equal in length to one of the two divisions of the arc of contact, and in it take I D = $\frac{1}{4}$ I P. Then, with the radius D P = $\frac{3}{4}$ I P, draw the circular arc A B; A and B will be the two ends of the required epicycloidal arc. Join B I; and from A draw the straight tangent A C, cutting B I in C. Then A C and B C will be the normals at the two ends of the epicycloidal arc. Then proceed, according to Rule IV. of Article 79, pages 61 and 62, fig. 48, to draw two circular arcs approximating to the required curve; and perform the same operation both for the face and for the flank of the tooth.

According to this method, the traces of the face and flank of a tooth consist each of a pair of circular arcs, and the two arcs which join each other at the pitch-point, A, of a tooth have a common tangent there; because their centres are in the straight line C′ A C.

II. *By one pair of Circular Arcs—Mr. Willis's Method.* Mr. Willis first showed how to approximate to the figures of epicycloidal teeth by means of two circular arcs—one concave, for the flank, the other convex, for the face; and each having for its radius the mean radius of curvature of the epicycloidal arc. Mr. Willis's rules may be deduced from the formula for finding the centre of curvature of an epicycloid, which is given in Article 78, equation 2, page 59; that formula being applied to the point in the epicycloid whose normal meets the pitch-circle at a distance from the pitch-point of the tooth to be traced equal to one-half of the pitch, and the obliquity of that normal to the pitch-circle being 15°.

In fig. 95, let B C be part of the pitch-circle, and A the pitch-point of a tooth whose front is to be traced. Lay off, in opposite directions from the point A, the arcs A B and A C, each equal to one-half of the pitch. Draw the radii of the pitch-circle, B D and C E, and through the points B and C draw the straight lines B F and C G, making angles of 75° with the radii respectively; these lines are normals to the face and to the flank of the tooth respectively. Let n denote the number of teeth in the wheel. Lay off along the two normals the distances B F and C G, as calculated by the following formulæ:—

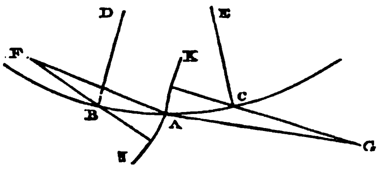

Fig. 95.

$$B\,F = \frac{\text{pitch}}{2} \cdot \frac{n}{n + 12};\ C\,G = \frac{\text{pitch}}{2} \cdot \frac{n}{n - 12};$$

then F will be the centre of curvature for the face, and G the centre of curvature for the flank.

About F, with the radius F A, draw the circular arc A H; this will be the trace of the face of the tooth. About G, with the radius G A, draw the circular arc A K; this will be the trace of the flank of the tooth.

To facilitate the application of this rule, Mr. Willis has published tables of the values of B F and C G, and invented an instrument called the "*Odontograph.*" That instrument is an oblong piece of card-board, F G K H, fig. 96, measuring about 13 inches by 7½ inches. The oblique edge, L H, makes an angle of 75° with the edge G F; so that when the edge L H is laid along a radius, O I, of a pitch-circle, B B, the edge G I F shows the positions of normals to acting surfaces of teeth whose pitch-points are at a distance from I equal to half the pitch. Along the edge G I F two scales of equal parts are laid off in opposite directions from the point I, where the straight line coinciding with H L meets G F; the scale I F serving to mark the centres for faces, and the scale I G the centres for flanks, at distances

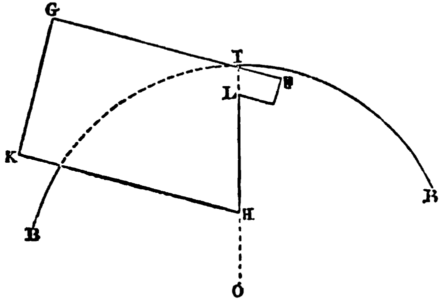

Fig. 96.

from I computed by the formulæ. Values of those distances for different pitches and numbers of teeth, and other useful dimensions, are given in tables which are printed on the sides of the card-board.

141. **Teeth Gearing with Round Staves—Trundles and Pin-wheels.** —When two wheels gear together, and one of them has cylindrical pins (called *staves*) for teeth, that one is called, if it is the larger of the two, a *pin-wheel*, and if the smaller, a *trundle*. The traces of the teeth of the other wheel are drawn in the following manner:— In fig. 97, let B_2 be the pitch-circle and C_2 the centre of the trundle or pin-wheel, and let B_1 B_1 be the pitch-circle of the other wheel. Divide the pitch-circle, B_1 B_1, into arcs equal to the pitch, and through the points of division trace a set of external epicycloids by rolling the pitch-circle B_2 on the pitch-circle B_1, with the centre of a stave for a tracing-point, as shown by the dotted lines; then draw curves parallel to and within the epicycloids, at a distance from them equal to the radius of a stave. These will be the fronts and backs of the required teeth. The clearing curves are circular arcs of a radius equal to that of the staves.

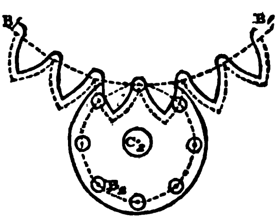

Fig. 97.

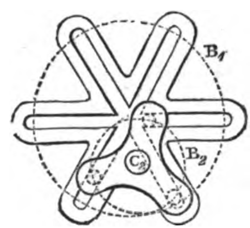

Fig. 98.

When the teeth drive the staves, the whole path of contact consists of *recess*, and there is no approach; for the teeth begin to act on the staves at the instant of passing the line of centres. When the staves drive the teeth, the whole path of contact consists of *approach*, and there is no recess; for the staves cease to act on the teeth at the instant of passing the line of centres. The latter mode of action is avoided where economy of power is studied, because it tends to produce increased friction, for reasons to be stated under the head of the Dynamics of Machines.

To drive a trundle in *inside gearing*, the outlines of the teeth of the wheel should be curves parallel to internal epicycloids. A peculiar case of this is represented in fig. 98, where the radius of the pitch-circle of the trundle is exactly one-half of that of the pitch-circle of the wheel; the trundle has three equidistant staves; and the internal epicycloids described by their centres, while the pitch-circle of the trundle is rolling within that of the wheel, are three straight lines, diameters of the wheel, making angles of 60° with each other. Hence the surfaces of the teeth of the wheel form three straight grooves intersecting each other at the centre, each being of a width equal to the diameter of a stave of the trundle, with a sufficient addition for back-lash.

The following is the construction given by Mr. Willis for finding in pin-wheels and trundles *what is the greatest radius of stave consistent with having an arc of contact not less than the pitch* (see fig. 99):—

Let C be the centre of the wheel with teeth, and C′ that of the wheel with staves. On their two respective pitch-circles lay off the arcs I D and I B, each equal to the pitch. Draw the straight line I B; draw also the straight line C E, bisecting the angle I C D, and cutting I B in E; then B E will be the greatest radius that can be given to the staves consistently with having an arc of contact not less than the pitch.

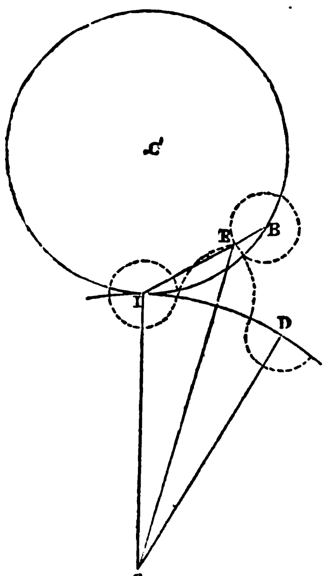

Fig. 99.

The proof is as follows:—Because the fronts and backs of the teeth are similar, the crest of the tooth that acts on a stave at B must be in the straight line C E, that bisects the angle I C D. When the centre of a stave is at B, the point of contact of the stave and tooth must be in the line of connection I B. When the staves have the greatest radius consistent with the continuance of action, while the centre of a stave moves from I to B, the point of contact and the crest of the tooth coincide, and are therefore at the point E, where I B and C E intersect.

Should C E pass beyond B, the proposed pair of wheels will not work, and the design must be altered; and such is also the case when C E either traverses the point B or cuts I B so near to B as to give a radius too small for strength.

In practice, the radius B E ought to be made a little less than that given by the Rule, in order that there may be no risk of imperfect working through the effects of tear and wear.

The smallest number of staves commonly met with in a trundle is five.

A straight rack may have staves instead of teeth; it is then called a *pin-rack;* and it is evident that the fronts and backs of the teeth of a wheel to gear with it should be parallel to involutes of the pitch-circle of that wheel. On the other hand, a toothed straight rack may gear with a trundle, and then the teeth of the rack are to be traced by first rolling the pitch-circle of the trundle on the pitch-line of the rack, so as to draw cycloids, and then drawing curves parallel to and inside those cycloids, at a distance equal to the radius of the staves.

142. **Intermittent Gearing.**—The action of a pair of wheels is said to be *intermittent* when there are certain parts of the revolution of the driver during which the follower stands still. This is effected by having a *dead arc*, or portion without teeth, such as A E, fig. 100, in the circumference of the driver, to which there corresponds a suitable *gap* in the series of teeth of the follower, as between C and D; and in most cases there are also required a *guide-plate*, G H, fixed to one side of the follower, which, when the connection of the wheels is renewed, is acted upon by a *pin*, F, in the driver.

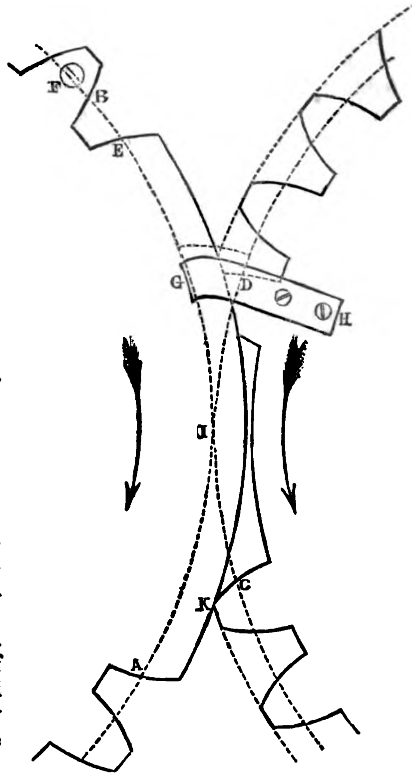

Fig. 100.

Supposing the radii and the pitch of a pair of wheels to be given, and also the *arc of repose*—by which term is meant the length upon the pitch-circle of the driver of that part which is to pass during the pause in the movement of the follower—the method of designing those wheels so as to work with smoothness and precision is as follows:—

Draw the pitch-circles, and divide them as usual; draw also the addendum-circles and root-circles according to the ordinary rules. Mark the point, K, where the addendum-circles cut each other at the *receding* side; this will be the point at which the action of the teeth will terminate, at the instant when the pause begins. Through K draw a curve suited for the front of a tooth of the follower, and let C be the pitch-point of that tooth. Then, starting from C, lay off the pitch-points of the fronts and backs of the teeth of the follower, and draw those fronts and backs. Then, looking towards the *approaching* side, mark the furthest tooth from the line of centres, which is cut by the addendum-circle of the driver; let D be the pitch-point of the face of that tooth. The crest of the tooth D is to be cut away so as exactly to fit the addendum-circle of the driver, and the teeth between it and the tooth C are to be omitted, leaving a smooth part of the root-circle between the front of C and the back of D; this is the required *gap*.

Measure the arc C D on the pitch-circle of the follower between the fronts of the teeth C and D, and to its length add the length of the intended *arc of repose;* from the sum subtract the *space*, B E, that is to be left between each pair of teeth on the pitch-circle of the driver; the remainder will be the *dead arc*, A E, which is to be laid off on the pitch-circle of the driver. The two ends of that arc are to be bounded by curves like the front and back of a tooth of the driver respectively: a front at A, a back at E; and the intervening part of the rim of the driver is to have a smooth edge coinciding with its addendum-circle.

For the purpose of renewing the connection between the driver and follower, the cylindrical pin F is to be fixed with its centre in the pitch-circle of the driver, and the guide-plate G H is to be fixed to the corresponding side of the follower. The acting edge of the guide-plate is to be shaped like the front of a tooth for working with the pin F (as in Article 141, page 137); and the distance on the pitch-circle of the follower from the front of that edge to the front of the tooth D is to be equal to the distance on the pitch-circle of the driver from the front of the pin F to the front of the tooth B; so that when B is driving D, F shall at the same time be driving G H. The end G of the guide-plate in the position of repose should project just far enough inside the pitch-circle of the driver to insure that the pin F shall meet it.

The action in working is as follows:—Just before the pause, the front, A, of the dead-arc drives the front of the tooth, C, in the usual way throughout the ordinary path of contact; and then, as there is a gap following C, the crest of the front A continues to drive C until the crest of C reaches the position K, and clears the addendum-circle of the driver. At that instant the driver loses hold of the follower, and at the same instant the top of the tooth D comes in contact with the rim of the dead arc, which it is shaped to fit; and this prevents the follower from moving until the dead arc has passed clear of the tooth D. At this instant the pin F begins to drive the guide-plate G H, and continues to do so until the tooth B has begun to drive the tooth D, and the connection is renewed.

If the pressure to be exerted is considerable, there may be a pair of pins at F, one at each side of the driver, and a pair of guide-plates at G H, one at each side of the follower.

The shortest arc through which the follower can be driven in the interval between two pauses is C D; and such is the case when B is the front of a second dead arc, and the tooth D is immediately followed by a second gap. In this case it may be necessary to cut away part of the outer side of the pin F, in order to insure its clearing the tip, G, of the guide-plate when the next pause begins.

It is easy to see how the same principles may be applied to the designing of a *wheel and rack* with intermittent action. When the rack is the follower, a pair of similar and parallel racks, rigidly framed together, may be made to gear with opposite edges of a spur-wheel, having a toothed arc and a dead arc so arranged as to drive the two racks alternately in opposite directions, and thus produce a reciprocating motion of the piece of which they are parts. This combination belongs to Class C of Mr. Willis's arrangement. As to the form which it takes when one tooth only acts at a time, see Article 164, further on. (See also Addendum, page 286.)

143. **The Teeth of Non-Circular Wheels** may be traced by rolling circles or other curves on the pitch-lines; and when those teeth are small, compared with the wheels to which they belong, each tooth is nearly similar to the tooth of a circular wheel whose pitch-circle has a radius equal to the radius of curvature of the pitch-line of the actual wheel at the point where the tooth is situated; the tooth being traced by means of the same describing circle which is used for the circular wheel.

It is obvious that the use of an uniform describing circle for teeth of a given pitch (as explained in Article 139, page 134) is the most easily practicable method of tracing teeth for a non-circular wheel. It may be carried out by means of templets, as in Article 137, page 131.

The operation is necessarily much more laborious than the corresponding operation for a circular wheel; because in a non-circular wheel the teeth have figures varying with the curvature of the pitch-line.

If the pitch-line of a non-circular wheel is one whose radii of curvature at a series of points can be easily found, a series of figures may be used for the teeth similar to the figures suited for circular wheels of those radii; and in drawing those figures the approximate methods of Article 140, pages 134 to 136, may be employed.*

* The following relation between the radii of curvature at a pair of corresponding points of a pair of pitch-lines that roll together, may be useful to determine one of those radii of curvature when the other is known. Let r and r' be the two segments into which the pitch-point divides the line of centres at the instant when the pair of corresponding points in question are in contact; let ρ and ρ' be the two radii of curvature at these points, and let θ be the angle which those radii make with the line of centres at the instant before mentioned; then

$$\frac{1}{\rho} + \frac{1}{\rho'} = \left(\frac{1}{r} + \frac{1}{r'}\right) \cos \theta \quad \text{(1.)}$$

When the pitch-lines are in inside gearing, the greater of the two segments, r, r', is to be made negative, and each radius of curvature is to be considered as positive for a convex and negative for a concave pitch-line.

For *a pair of equal elliptic pitch-lines*, as in Article 108, page 93, the radii of curvature at a pair of corresponding points are equal, and are therefore both given by the following formulæ:—

$$\frac{1}{\rho} = \frac{1}{\rho'} = \frac{\cos \theta}{2} \left(\frac{1}{r} + \frac{1}{r'}\right); \quad \text{(2.)}$$

or,

$$\rho = \rho' = \frac{2 r r'}{(r + r') \cos \theta}; \quad \text{(2 A.)}$$

and the same formulæ apply to any pair of *equal and similar lobed pitch-lines* of the class described in Article 109, page 97.

For a *logarithmic spiral pitch-line* (Article 110, page 99) the radius of curvature at any point is given by the formula

$$\rho = \frac{r}{\cos \theta}; \quad \text{(3.)}$$

and may be found approximately by construction, as already described in the article referred to.

If *one of the pitch-lines is straight* (a case already used as an example in Article 107, page 92), the reciprocal of the radius of curvature of that line is at every point equal to nothing; so that equation 1 of this note becomes (for the other pitch-line)

$$\frac{1}{\rho} = \left(\frac{1}{r} + \frac{1}{r'}\right) \cos \theta \quad \text{(4.)}$$

Let c denote the length of the line of centres, and a the shortest distance of the straight pitch-line from its own axis of motion; then $r' = \frac{a}{\cos \theta}$; and $r = c - r' = c - \frac{a}{\cos \theta}$; consequently equation 4 becomes

$$\frac{1}{\rho} = \frac{c \cos^2 \theta}{a c \cos \theta - a^2}; \quad \text{(4 A.)}$$

or

$$\rho = \frac{a}{\cos^2 \theta} - \frac{a^2}{c \cos^3 \theta}. \quad \text{(4 B.)}$$

When a pair of non-circular wheels are connected by means of teeth alone, care must be taken that the obliquity of the action of the teeth does not become too great in certain positions of the wheels. That obliquity is greatest at the instant when the obliquity of the common tangent of the two pitch-lines at their pitch-point to a perpendicular to the line of centres at that point is greatest, such obliquity being in the direction of rotation of the follower; for, as that is also the direction of the obliquity of the line of connection of the teeth to the pitch-lines, those two obliquities are added together at the instant in question; their sum being the total obliquity of the line of connection to a perpendicular to the line of centres. Excessive obliquity of action tends to produce great friction, and involves also the risk of the teeth either getting jammed or losing hold of each other. In practice, the total obliquity of action of the teeth of non-circular wheels is seldom allowed to exceed about 50°; or say, about 15° for the obliquity of the line of connection of the teeth to the pitch-lines, and 35° for the greatest obliquity of the pitch-lines to a line perpendicular to the line of centres.

There is one case, however, in which it is not necessary to confine the obliquity within such narrow limits; and that is when the wheels have a pair of equal and similar elliptic pitch-lines centred on two of their foci, and it is practicable to link the revolving foci together, as shown in Article 108, fig. 72, page 96; for the link preserves the connection accurately at the time when the obliquity of the pitch-lines is greatest. In this case, indeed, the teeth may be omitted throughout a pair of arcs at the two sides of each elliptic pitch-line, each such toothless arc having a smooth rim of the form of the pitch-line, and extending both ways from the end of the minor axis to a pair of points perpendicularly opposite the foci, or nearly so. (See page 292.)

144. **Teeth of Bevel-Wheels.** (*A. M.*, 467.)—The teeth of a bevel-wheel have acting surfaces of the conical kind, generated by the motion of a line traversing the apex of the conical pitch-surface, while a point in it is carried round the traces of the teeth upon a spherical surface described about that apex.

The operations of drawing the traces of the teeth of bevel-wheels exactly, whether by involutes or by rolling curves, are in every respect analogous to those for drawing the traces of the teeth of spur-wheels; except that in the case of bevel-wheels all those operations are to be performed on the surface of a sphere described about the apex, instead of on a plane, substituting *poles* for *centres*, and *great circles* for *straight lines*.

In consideration of the practical difficulty, especially in the case of large wheels, of obtaining an accurate spherical surface, and of drawing upon it when obtained, the following *approximate* method, proposed originally by Tredgold, is generally used:—

I. *Development of Teeth.*—Let O, fig. 101, be the common apex of the pitch-cones, O B I, O B′ I, of a pair of bevel-wheels; O C, O C′, the axes of those cones; O I their line of contact. Perpendicular to O I draw A I A′, cutting the axes in A, A′; make the outer rims of the patterns and of the wheels portions of the cones A B I, A′ B′ I, of which the narrow zones occupied by the teeth will be sufficiently near for practical purposes to a spherical surface described about O. As the cones, A B I, A′ B′ I, cut the pitch-cones at right angles in the outer pitch-circles, I B, I B′, they may be called the *normal cones.* To find the traces of the teeth upon the normal cones, draw on a flat surface circular arcs, I D, I D′, with the radii A I, A′ I; those arcs will be the *developments* of arcs of the pitch-circles, I B, I B′, when the conical surfaces, A B I, A′ B′ I, are spread out flat. Describe the traces of teeth for the developed arcs as for a pair of spur-wheels, then wrap the developed arcs on the normal cones, so as to make them coincide with the pitch-circles, and trace the teeth on the conical surfaces.

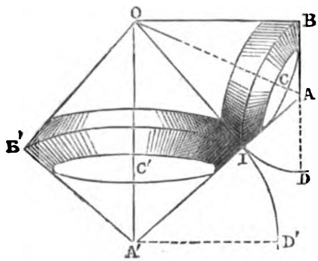

Fig. 101.

II. *Traces and Projections of Teeth.*—Fig. 102 illustrates the process of drawing the *projection of a tooth of a bevel-wheel on a plane perpendicular to the axis.* In the first place, let A C represent the common axis of the pitch-cone and normal cone; A being the apex of the normal cone. Let A I be the trace of the normal cone on a plane traversing the axis; and let I I′, perpendicular to I A, be part of the trace of the pitch-cone on the same plane, of a length equal to the intended breadth of the toothed rim of the wheel. C I perpendicular to A C is the radius of the pitch-circle in which the pitch-cone and normal cone intersect each other. About A, with the radius A I, draw the circular arc D I D, making D I = I D = half the pitch; D I D will be the development of an arc of the pitch-circle of a length equal to the pitch. On the arc D I D lay off I G = I G = half the thickness of a tooth on the outer pitch-circle. Then, by the rules for spur-wheels, draw the trace, H G E G H, of one tooth and a pair of half-spaces, with a suitable addendum-circle through E, and a suitable root-circle, H F H.

The straight line F I E will be the trace, upon a plane traversing the axis, of the outer side of a tooth; and E and F will be the traces, on that plane, of the outer addendum-circle and root-circle respectively. From E and F draw straight lines, E E′ and

F F′, converging towards the apex of the pitch-cone; these will be the traces of the *addendum-cone* and *root-cone* respectively. (For want of space, the apex of the pitch-cone is not shown in fig. 102.) Through I′, parallel to F I E, draw F′ I′ E′; this will be the trace,

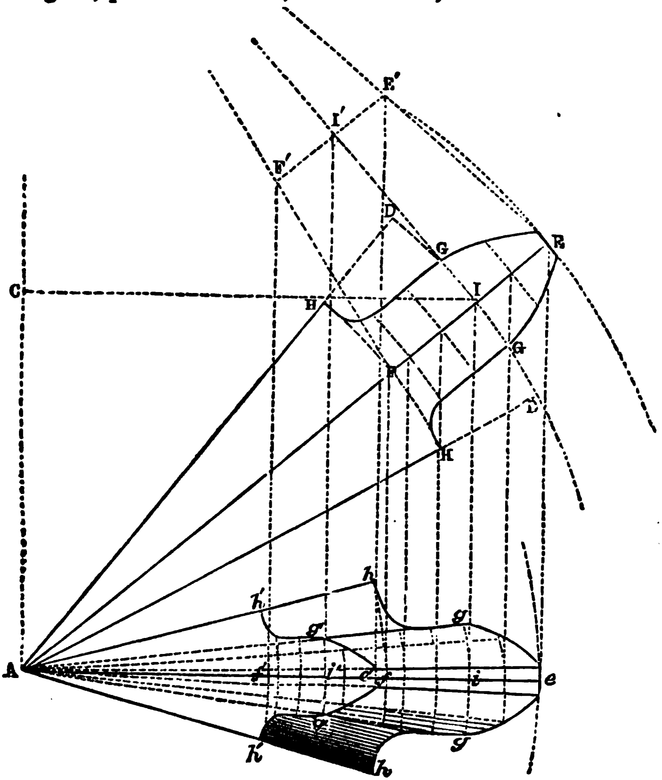

Fig. 102.

on a plane traversing the axis, of the inner side of a tooth; and the points E′, I′, and F′ will be respectively the traces of the inner addendum-circle, inner pitch-circle, and inner root-circle.

Through A, parallel to C I, draw the straight line A *i e*, and conceive this line to be traversed by a plane perpendicular to the axis, as a new plane of projection. Through the points F, I, E, F′, I′, E′, draw straight lines parallel to C A, cutting A *e* in *f*, *i*, *e*, *f*′, *i*′, *e*′; these points, marked with small letters, will be the projections, on the new plane, of the points marked with the corresponding capital letters.

Divide the depth, F E, of the tooth at its outer side into any convenient number of intervals. Through the points of division draw straight lines parallel to C A; these will cut *f e* in a series of points, which will be the projections of the points of division of F E. Through the points of division of F E, and also through the projections of those points, draw circular arcs about A as a centre. Measure a series of thicknesses of the tooth on the arcs which cross F E, and lay off the same series of thicknesses on the corresponding arcs which cross *f e;* a curve, *h g e g h,* drawn through the points thus found, will be the required projection, on a plane parallel to the axis, of the outer side of a tooth.

The projection, *h' g' e' g' h'*, of the inner side of a tooth is found by a similar process, except that the measuring and laying-off the thicknesses is rendered unnecessary by the fact that each pair of corresponding points in the projections of the outer and inner sides lie in one straight line with A. For example, having drawn about A a circular arc through *e'*, draw the two straight lines A *g*, A *g*; these will cut that arc in the points *g'*, *g'*, being the points in the projection of the inner side corresponding to *g*, *g* in the projection of the outer side; and thus it is unnecessary to lay off the thickness *g' g'*.

145. **Teeth of Skew-bevel Wheels—General Conditions.**—The surfaces of the teeth of a skew-bevel wheel belong, like its pitch-surface, to the hyperboloïdal class, and may be conceived to be generated by the motion of a straight line which, in each of its successive positions, coincides with the line of contact of a tooth with the corresponding tooth of another wheel. Those surfaces may also be conceived to be traced by the rolling of a hyperboloïdal roller upon the hyperboloïdal pitch-surface, in the manner described in Article 84, pages 70 to 73.

The conditions to be fulfilled by the *traces of the fronts and backs of the teeth on the hyperboloïdal pitch-surface* are:—A. That each of those traces shall be one of the generating straight lines of the hyperboloïd (Article 106, page 89); B. That the *normal pitch*, measured from front to front of the teeth along the *normal spiral* (Article 106, page 89), shall be the same in two wheels that gear together—(this second condition is always fulfilled if the two pitch-surfaces are correctly designed, and the numbers of teeth made inversely proportional to the angular velocities); and C. That the teeth, if in outside gearing, shall be *right-handed* on both wheels, or *left-handed* on both wheels; and if in inside gearing, contrary-handed on the two wheels.

Skew-bevel teeth may be said to be RIGHT-HANDED or LEFT-HANDED, according to the direction in which the generating lines of the teeth appear to deviate from the axis when looked at with the axis upright, as in fig. 103, page 147. For example, the wheel in that figure has left-handed teeth; for the generating line I′ I deviates to the left of the axis A′ A. The same rule applies to the direction in which the crests of the teeth appear to deviate from the radii of the wheel, when looked at as in the upper part of fig. 105, page 150.

Right-handed teeth have left-handed normal spirals, and left-handed teeth right-handed normal spirals.

146. **Skew-bevel Teeth—Rules.**—I. *Normal Section of a Tooth.*—In fig. 103, let A *a* A′ be the axis of a skew-bevel wheel: let *a* be the centre of the *throat* of its hyperboloïdal pitch-surface; let the dotted curve through I be the trace of that surface on a plane traversing the axis; and let C I = *ai* be the radius of the pitch-circle at the middle of the breadth of the rim of the intended

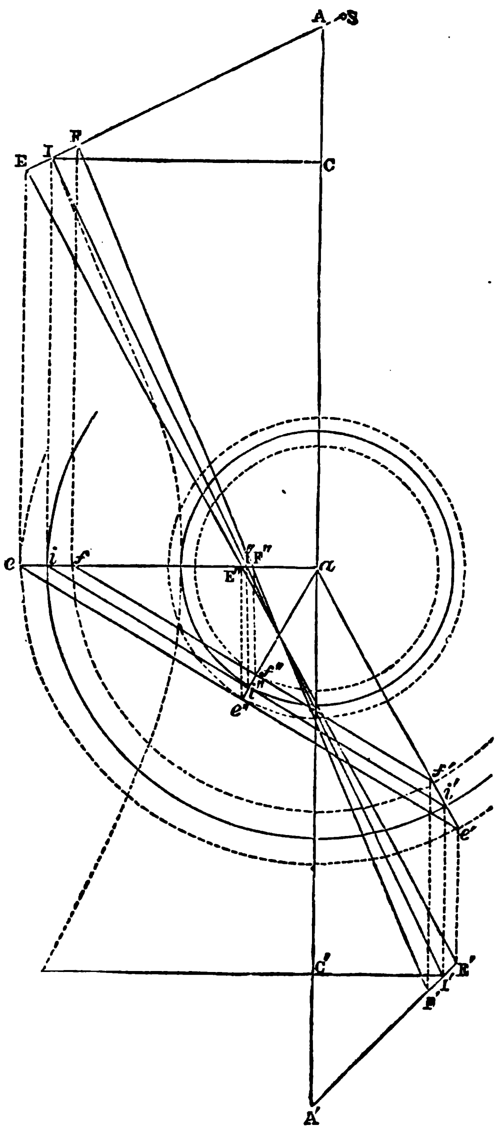

Fig. 103.

wheel, as found by Rule I. of Article 106, page 88. Draw by Rules II. and III. of that Article, pages 88, 89, the normal I A

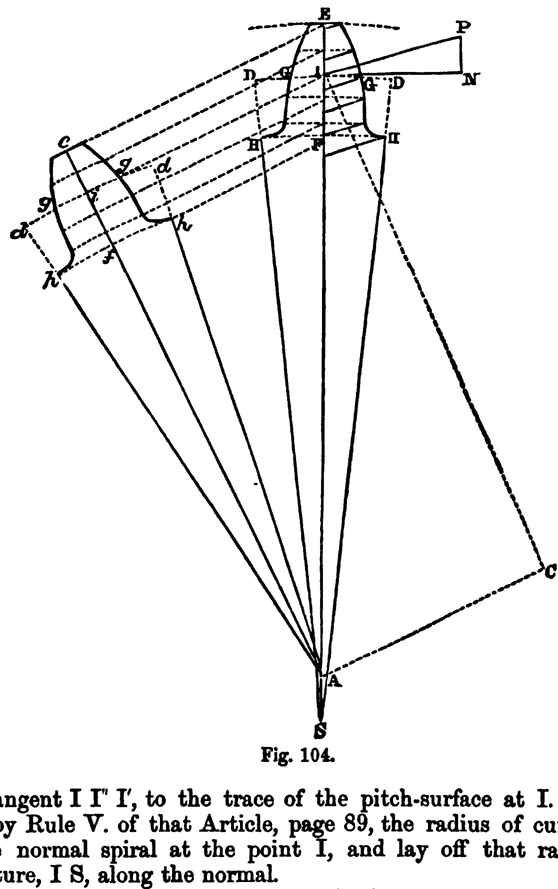

Fig. 104.

and tangent I I″ I′, to the trace of the pitch-surface at I. Then find, by Rule V. of that Article, page 89, the radius of curvature of the normal spiral at the point I, and lay off that radius of curvature, I S, along the normal.

In fig. 104 (which is on a larger scale than fig. 103, for the sake of distinctness), let A C, as before, be the axis of the wheel, C I the radius of the middle pitch-circle, I A the normal, and I S the radius of curvature of the normal spiral; draw I N perpendicular to I S. Then, by Rule V. of Article 106, page 89, find the angle (= O g F in fig. 68, page 88) which a tangent to the normal spiral makes with a tangent to the pitch-circle, and draw I P, making that angle with I N. Lay off I P equal to *the pitch as measured on the middle pitch-circle;* let fall P N perpendicular to I N; then

I N will be the *normal pitch at the middle pitch-circle.* About S, with the radius S I, draw a circular arc, and lay off on that arc the distance, D D, equal to the normal pitch, one-half to each side of I. Lay off the intended middle thickness, G G, of a tooth, one-half to each side of I. Then draw, by the rules for spur-wheels, the *normal section,* H G E G H, of a tooth, being its trace upon a surface which cuts it normally at the middle of the breadth of the rim of the wheel.

II. *Trace of a Tooth on the Normal Cone.*—Through A in fig. 104 draw A *i* parallel and equal to C I; and through I draw I *i* parallel and equal to C A. About A, with the radius A *i*, draw the circular arc *d d*, equal in length to I P, the pitch on the pitch-circle, and having the middle of its length at the point *i*. This will be the arc on the pitch-circle corresponding to the arc D D on the normal spiral.

Divide E F, the middle depth of the tooth, into any convenient number of intervals; and through E and F and the points of division draw straight lines parallel to I *i*, cutting A *i e* in a series of corresponding points. Through the points in E F draw circular arcs about S. Through the corresponding points of *e f* draw circular arcs about A. From the points where the arcs cut the trace E G H measure *oblique half-thicknesses* to the centre line, E F, of the tooth, *along oblique lines drawn parallel to* P I; and lay off those half-thicknesses at both sides of *e f*, along the arcs which cross it. Through the points thus found draw the curve *h g e g h;* this will be *the projection, on a plane perpendicular to the axis, of the trace of a tooth upon the normal cone of the pitch-surface at the middle of its breadth;* that is, upon the cone whose trace is A I in fig. 103. (If it be desired to draw the *development* of that trace, lay off the *oblique half-thicknesses* along arcs drawn about A, through the points of division of the radius A F I E. The result is the drawing of an outline outside of, and nearly parallel to, H G E G H. To prevent confusion, it is not shown in the figure.)

If the pitch-circle chosen is at the *throat* of the hyperboloïd, the normal cone becomes simply the plane of that circle; and in fig. 104, A *f i e* coincides with A F I E.

III. *Projections of the Middle Lines of a Tooth.*—In fig. 103, let F I E and *f i e*, as before, represent the projections of the central depth of a tooth, being part of a normal (E I F A, *e i f a*) to the pitch-surface at a point, I *i*, in the middle pitch-circle, whose radius is C I $=$ *a i;* so that F, *f*, I, *i*, and E, *e* are the projections of the *middle points* of the tooth at the root, at the pitch-surface, and at the crest respectively; and let it be required to find the projections of the *middle lines* of that tooth at the root, pitch-surface, and crest respectively.

About *a* draw the circles *f f'*, *i i'*, and *e e'*; being the projections, on a plane perpendicular to the axis, of the root-circle through F, the pitch-circle through I, and the addendum-circle through E. Draw also about *a* the pitch-circle at the throat of the hyperboloïd, and let *a i''* be its radius. Through *i* draw a straight line, *i i'' i'*, so as to touch this *throat pitch-circle*, and let that straight line cut the circle *i i'* in *i* and *i'*. Draw the straight lines *f f'' f'* and *e e'' e'* parallel to *i i'' i'*. Then these three parallel lines will be the *projections of the three middle lines* before mentioned, *on a plane perpendicular to the axis.*

Describe about *a* two circles touching *f f'' f'* and *e e'' e'* respectively. These will be respectively *the root-circle and the addendum-circle at the throat of the hyperboloïd.* The roots and crests of all the teeth lie in a pair of hyperboloïdal surfaces traversing this pair of circles, and traversing also the pair of circles through F and E.

The projection, on a plane traversing the axis, of the middle line of the tooth on the pitch-surface is the tangent I I'' already found, the points I'' and *i''* being in one straight line parallel to *a* A. To find the corresponding projections of the other two middle lines, there are two methods.

First Method.—From the points of contact *f''* and *e''*, parallel to *a* A, draw *f''* F'' and *e''* E'', cutting *a i* in F'' and E'' respectively. Join F F'' and E E''. These will be the required projections.

Second Method.—Lay off on the axis, *a* C' = *a* C, and *a* A' = *a* A, and draw C' I' parallel to C I: then C' I' will be part of the projection of a pitch-circle equal to C I. From *i'*, parallel to A *a* A', draw *i'* I', cutting C' I' in I'. Then *i'* and I' will be the two projections of one pitch-point, and II'' I' will be one straight line. Join A' I'. This will be the projection of a normal to the pitch-surface at I'. Through *f'* and *e'* (which lie in one radius, *a f' i' e'*) draw *f'* F' and *e'* E' parallel to A *a* A', cutting A' I' in F' and E' respectively. Join F F' and E E'. These will be the required projections of the middle lines at the root and crest of the tooth respectively.

IV. *Complete Projection of a Tooth on a Plane Normal to the Axis.*—Let the plane of projection in fig. 105 be normal to the axis of the wheel, and (as in fig. 103) let *a* be the axis; let the circles *e e'*, *i i'*, and *f f'*, be the projections of the middle addendum-circle, middle pitch-circle, and middle root-circle of the intended wheel; let the circles through *e''*, *i''*, and *f''* be the corresponding circles at the throat of the pitch-surface; and let the parallel straight lines *e e'' e'*, *i i' i''*, *f f'' f'*, be the projections of the middle lines of a tooth at the crest, pitch-surface, and root, drawn according to the preceding rules.

At the end of the radius *a f i e* construct, by the rules already given, the projection of the trace of the tooth upon the middle normal cone, being the curve marked *h g e g h* in fig. 104; and at the end of the radius *a f′ i′ e′* construct a similar and equal figure. From a series of points in the figure at *f i e* draw straight lines to the corresponding points in the figure at *f′ i′ e′*; each of those straight lines will be the projection of a *generating line* of the surface of the tooth. For example, the straight lines from the corners of the crest at *e* to the corresponding corners of the crest at *e′* (both of which lines touch the circle through *e″*) will be the projections of the two edges of the crest; the straight lines from the pair of points where the curve at *f i e* cuts the pitch-circle to the corresponding pair of points near *f′ i′ e′* (both of which lines touch the circle through *i″*) will be the projections of the lines in which the front and back of the tooth respectively cut the pitch-surface; and the straight lines from the bottoms of the clearing curves near *f* to the corresponding points near *f′* (both of which lines touch the circle through *f″*) will be the projections of the lines marking the bottoms of the hollows of which these curves are the traces.

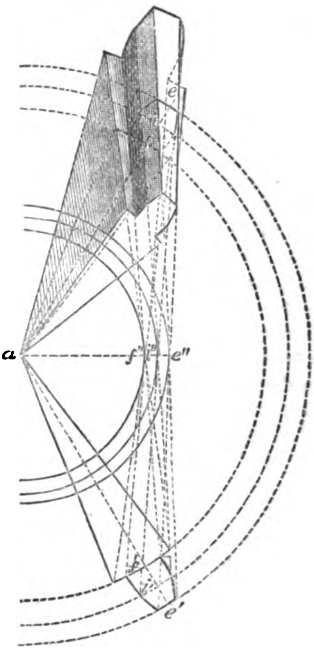

Fig. 105.

The projections of the outer and inner sides of the tooth (being portions of the outer and inner sides of the rim of the wheel) are figures similar to the curve at *f i e*, and constructed by the same method; the dimensions of the former being larger and those of the latter smaller than the dimensions of that middle figure, in the proportion in which the radii of the outer and inner pitch-circles are respectively greater and smaller than those of the middle pitch-circle. (As to those two circles, see Article 106, page 90.) The tooth in fig. 105 is drawn of an exaggerated breadth, in order to show more clearly the construction of the figure.

V. *Modelling Skew-bevel Teeth.*—Construct a frame of rods to represent the axis A A′ in fig. 103, the equal radii C I and C′ I′, the equal normals A F I E and A′ F′ I′ E′, and the generating line II′. Make a pair of equal and similar templets, each of the shape and dimensions of the normal section of a tooth, H G E G H, fig. 104.

Fix those two templets to the frame at F I E and F′ I′ E′, fig. 103, with their flat surfaces parallel to each other and normal to the rod II′. Then a straight edge or a stretched wire, made to touch the edges of the templets at a pair of corresponding points, will mark one of the generating lines of a tooth; and by the help of this apparatus, teeth may be modelled suitable either for the pitch-circle through C, or for that through C′, or for the pitch-circle at the throat of the hyperboloïd, or for any other pitch-circle on the same hyperboloïd.

147. The **Transverse Obliquity of Skew-bevel Teeth** is the angle, P I N, fig. 104, page 148 (equal to O g F′ in fig. 68, page 88), which the normal spiral makes with the pitch-circle; or it may be otherwise defined as the angle which the generating line of a tooth on the pitch-surface makes with the generating line of a tangent cone at a given point, I. From the rule for finding that angle (Article 106, page 89), it is evident that, with a given hyperboloïdal pitch-surface, the transverse obliquity of the teeth is greatest at the throat, and is the less the farther the middle pitch-circle of the wheel is removed from the throat. Hence, it is generally advisable, in designing skew-bevel wheels, to place the pitch-circles as far as practicable from the throats of the hyperboloïds, because obliquity of action tends to increase friction.

148. **Skew-bevel Wheels in Double Pairs.**—Skew-bevel wheels possess a property which ordinary bevel wheels do not—viz., that of being capable of combination by *double pairs*, as in fig. 106. The upper part of the figure represents a projection on a plane parallel to the line of contact, I I′, and to the common perpendicular of the axes F G. The lower part of the figure represents a projection on a plane normal to the common perpendicular. Small letters in the second projection correspond to capital letters in the first projection.

B and B′ are two equal and similar wheels fixed on the shaft A A′, with pitch-surfaces forming parts of the same hyperboloïd, and at equal distances from its throat. They have equal and similar teeth, with equal obliquities in the same direction; and, in short, both wheels may be cast from the same pattern. In the example given, the teeth of both wheels are right-handed. In like manner, D and D′ are two equal and similar wheels fixed on the shaft C C′; B gears with D, and B′ with D′.

This arrangement may be useful where it is desired, for the sake of strength or of steadiness of motion, to divide the force exerted in transmitting the motion between two pairs of wheels.

149. **Teeth with Sloping Backs.**—The teeth described in the preceding Articles of this Section have their backs similar to their fronts, so that the motion of the wheels may be reversed, the backs then acting as the fronts did during the forward motion. There are many cases in mechanism in which it is not necessary that the

motion of the wheels should ever be reversed; and in such cases the backs of the teeth of a pair of wheels are required simply to be

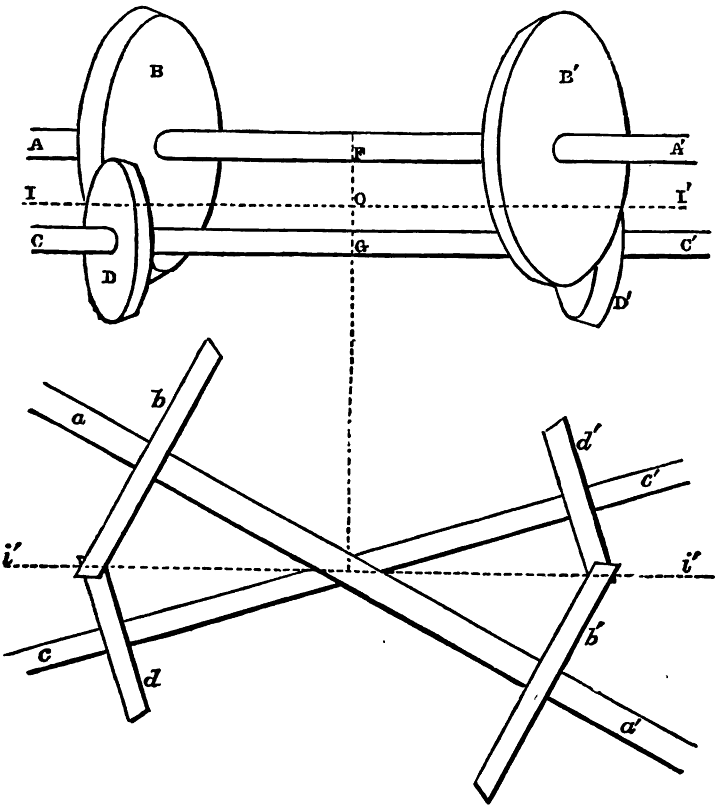

Fig. 106.

of such shapes as to clear each other, without reference to the transmission of motion. The consequence of this is, that although the traces of the backs must still belong to the same class of curves with the traces of the fronts, their *obliquity* may be considerably increased, the effect being to strengthen the teeth at their roots.*

The most convenient curves for the traces of the backs of teeth under those circumstances are involutes of a circle, for which there may be substituted in practice circular arcs approximating to them;

* This was first pointed out by Professor Willis.

and the method of drawing those arcs is as follows:—Let fig. 107 represent the trace of part of a wheel with its teeth, that wheel being the smallest wheel of a set that are to be capable of gearing together; because the smallest wheel of such a set requires the greatest addendum: let C be the centre, A A the addendum-circle,

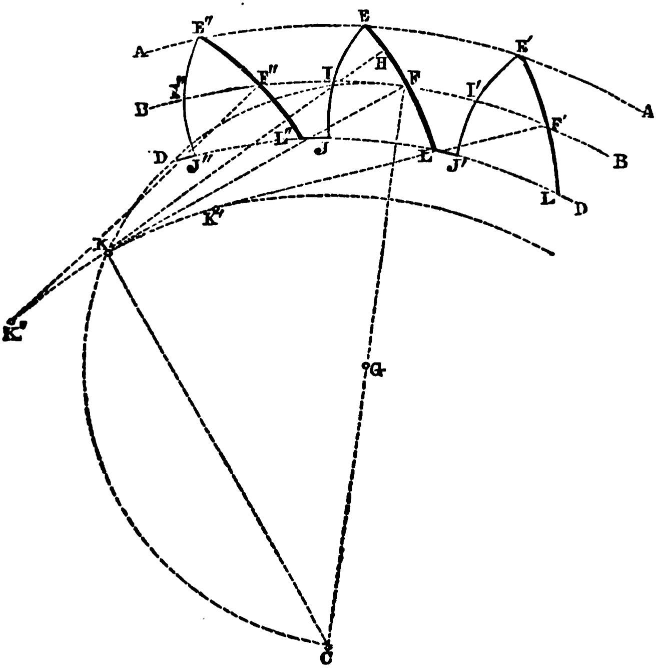

Fig. 107.

B B the pitch-circle, D D the root-circle, and let $E'' I'' J''$, E I J, $E' I' J'$, be the fronts of teeth designed according to the proper rules, and F'', F, F', the pitch-points of the backs of those teeth. To any one of those *back pitch-points*, as F, draw the radius C F; bisect C F in G, and about G draw the semicircle F K C. Draw a straight line, H K, perpendicular to and bisecting the distance, E F, between the crest E and back pitch-point F; and let that straight line cut the semicircle in K. About the centre C, with the radius C K, draw the circle $K'' K K'$; this will be the *base-circle* of the required involutes (see Article 131, page 121).

To draw the circular arcs approximating to those involutes, lay off, from the back pitch-points to the base-circle, the equal distances $F'' K'' = F' K' = F K$, &c; and about the respective centres, K, K′, K″, &c., draw the circular arcs E F L, E′ F′ L′, E″ F″ L″, &c.

In each of the larger wheels of the set, the radius of the base-circle for the backs is to bear to the radius of the pitch-circle the constant proportion $\frac{\text{C K}}{\text{C F}}$, in order that the backs of the teeth of all the wheels of the set may have the same obliquity—viz., the angle K C F.

In a straight rack capable of gearing with any wheel of the set, the traces of the backs of the teeth are to be straight lines, making with the pitch-line an angle equal to C F K.

150. **Stepped Teeth.**—In order to increase the smoothness of the action of toothed wheels, Dr. Hooke invented the making of the fronts of teeth in a series of steps, as shown in fig. 108, where the

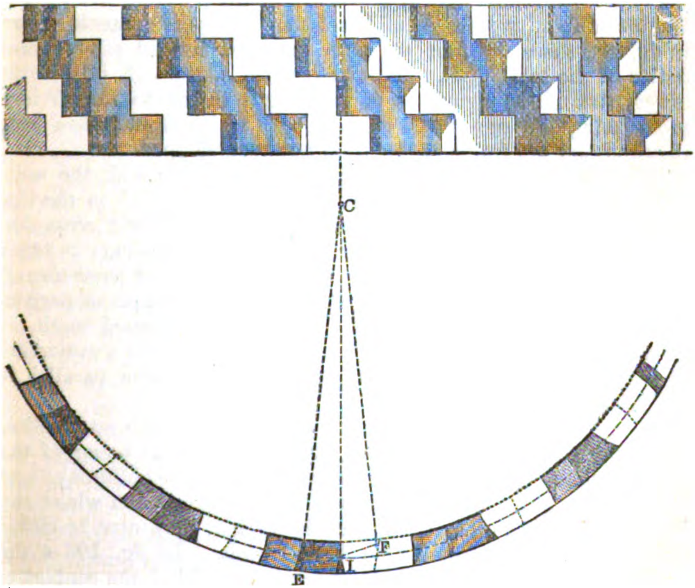

Fig. 108.

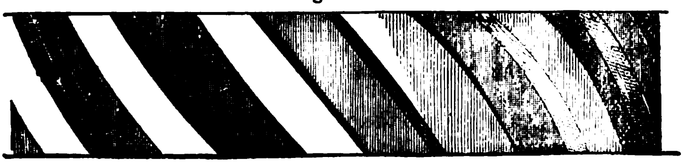

Fig. 108 A.

upper part of the figure is a projection of the rim of a wheel with stepped teeth on a plane parallel to the axis, and the lower part is a projection on a plane perpendicular to the axis. A wheel thus formed resembles in shape a series of equal and similar toothed discs placed side by side, with the teeth of each a little behind those of the preceding disc. In such a wheel, let p be the circular pitch, and n the number of steps. Then the path of contact, the addendum, and the extent of sliding, are those due to the *divided* pitch $\frac{p}{n}$, while the strength of the teeth is that due to the thickness corresponding to the *total* pitch p; so that the smooth action of small teeth and the strength of large teeth are combined. The action of small teeth is smoother and steadier than that of large teeth, because they can be made to approximate more closely to the exact theoretical figure; and also because the sliding motion of one tooth upon another is of less extent. In the example shown in fig. 108 there are four steps, so that the divided pitch is one-fourth of the total pitch; and the path of contact (E I F, in the lower part of the figure) is of the length suited to the divided pitch, being only one-fourth of the length which would have been required had the fronts of the teeth not been stepped.

151. **Helical Teeth,** also invented by Dr. Hooke with the same object, are teeth whose fronts, instead of being parallel to the line of contact of the pitch-cylinders of a pair of spur-wheels, cross that line obliquely, so as to be of a screw-like or helical form: in other words, they are teeth of the figure of short portions of *screw-threads* (Article 58, page 36); the trace of each thread on a plane perpendicular to the axis being similar to that of a stepped tooth, as shown in the lower part of fig. 108. Fig. 108 A shows a projection of the rim of a wheel with helical teeth on a plane parallel to the axis.

In order that a pair of wheels with parallel axes and helical teeth may gear correctly together, the teeth, besides being of the same circular pitch, must have the same transverse obliquity; and if in outside gearing, they must be right-handed on one wheel and left-handed on the other. If in inside gearing, they must be either right-handed or left-handed on both wheels. In fig. 108 A the teeth are left-handed. In wheel-work of this kind the contact of each pair of teeth commences at the foremost end of the helical fronts, and terminates at the aftermost end; and the rims of the wheels are to be made of such a breadth that the contact of one pair of teeth shall not terminate until that of the next pair has commenced.

Helical teeth are open to the objection that they exert a laterally oblique pressure, which tends to increase friction.

When, in designing a skew-bevel wheel, a portion of the tangent cylinder at the throat of the hyperboloïd (Article 106, page 87;

and Article 85, page 73) is used as an approximation to the true pitch-surface, the teeth of that wheel become screw-threads, having a transverse obliquity determined by the principles of Article 147, page 152; and, as has been already stated in the article referred to, they are either right-handed or left-handed in both wheels.

152. **Screw and Nut.**—The figure of a true screw, external or internal, and the motion of a screw working in a corresponding screw-shaped bearing, have been described in Articles 57 to 66, pages 36 to 42. In the elementary combination of an *external and internal screw*, more commonly called a *screw and nut*, the two pieces have threads, one external and the other internal, of similar figures and equal dimensions, so as to fit each other truly; and one of them turns about their common axis without translation, while the other slides parallel to that axis without rotation. The best form of section for the threads is rectangular. The comparative motion is, that the sliding piece advances through a distance equal to the pitch (viz., the "*total axial pitch*") during each revolution of the turning piece. If the threads are $\left\{\begin{matrix}\text{right-handed,}\\ \text{left-handed,}\end{matrix}\right\}$ the sliding piece is made to move towards an observer at one end of the axis by $\left\{\begin{matrix}\text{right-handed}\\ \text{left-handed}\end{matrix}\right\}$ rotation, and to move from him by $\left\{\begin{matrix}\text{left-handed}\\ \text{right-handed}\end{matrix}\right\}$ rotation, of the turning piece. The combination belongs to Mr. Willis's Class A, because the velocity-ratio is constant; and the extent of the motion is limited by the length of the screw.

153. **Screw Wheel-Work in General.**—Screw wheel-work consists of wheels with cylindrical pitch-surfaces, having screw-threads or helical teeth instead of ordinary teeth. One case of screw-gearing has been described in Article 151, page 156—viz., that in which the axes are parallel. The cases to which this and the following articles relate are those in which the axes are not parallel; so that the pitch-surfaces in an elementary combination are a pair cylinders touching each other in *one pitch-point*, like those represented in Article 85, fig. 55, page 73. The pitch-point (O′, fig. 55) is obviously in the common perpendicular of the two axes (F′ G′, fig. 55); and there is one straight line traversing the pitch-point (O′ C′, fig. 55), which is a tangent at once to the two pitch-cylinders and to the acting surfaces or fronts of each pair of threads at the instant when those surfaces touch each other at the pitch-point: that straight line may be called the LINE OF CONTACT. The *angles of inclination* of the screw-threads to the two axes (see Article 63, page 40) are equal respectively to the angles made by the line of contact with those axes. The PITCH-CIRCLES of the two screws are the two circular sections of the pitch-cylinders which traverse the pitch-point. The PLANE OF CONNECTION, OR PLANE OF

ACTION, is a plane traversing the pitch-point normal to the line of contact: that plane, of course, traverses the common perpendicular of the axes.

When the line of contact is found by the rule given in Article 84, page 71, the cylindrical pitch-surfaces represent the tangent-cylinders at the throats of a pair of hyperboloïds; and the screw-threads are approximations to the skew-bevel teeth suited for that combination, as already stated in Article 151, page 156. But in many cases the line of contact has positions greatly differing from this; and then the comparative motion becomes different from that of a pair of skew-bevel wheels; the object of screw-gearing in such cases being to obtain, with a given pair of cylindrical pitch-surfaces, a velocity-ratio of rotation independent of the radii of those surfaces; and such is the difference between approximate skew-bevel gearing and screw gearing in general.

In every elementary combination in screw wheel-work, each of the two pieces is at once a screw and a wheel; but it is customary, when their diameters are very different, to call that which has the smaller diameter the ENDLESS SCREW, or WORM, and that which has the greater diameter the WORM-WHEEL. For example, in fig. 111 (farther on) a' is the worm, or endless screw, and A' the worm-wheel. The word "endless" is used because of the extent of the motion being unlimited.

Screw wheel-work belongs to Mr. Willis's Class A, the velocity-ratio being constant.

The following are the general principles of elementary combinations in screw wheel-work:—

I. The angular velocities of the two screws are inversely, and their times of revolution directly, as the numbers of threads; whence it follows that the angular velocity-ratio must be expressible in whole numbers, as in the case of ordinary toothed wheels.

II. The *divided normal pitch* (see Article 66, page 42), as measured on the pitch-cylinders, must be the same in two screws that gear together.

III. The *common component* of the velocities of a pair of points in the two screws at the instant when those two points touch each other and pass the pitch-point, is perpendicular to the line of contact and to the common perpendicular of the axes; in other words, it coincides with the intersection of the plane of connection and the common tangent-plane of the two pitch-cylinders.

IV. The *circular* or *circumferential pitches* of the two screws (Article 42, page 66), as measured on their pitch-cylinders, are proportional to the total velocities of points (called the *surface velocities*) in those cylinders; and they bear the same proportion to the divided normal pitch that those total velocities bear to their common component.

V. The *relative transverse sliding* of a pair of threads that are in action takes place along the line of contact.

It will be shown in the next article that for a given pair of axes and a given angular velocity-ratio the relative transverse sliding is least when the pitch-cylinders are the tangent-cylinders at the throats of a pair of skew-bevel hyperboloïds.

154. **Screw Wheel-Work—Rules for Drawing.**—In figs. 109 and 110 the plane of projection is supposed to be the common tangent-plane of the two pitch-cylinders; and I represents the pitch-point; which is also the trace and projection of the common perpendicular of the two axes.

I. *Given, the projections of the two axes, the angular velocity-ratio, and the radii of the two pitch-cylinders, to find the proportionate values of their surface-velocities, and the proportionate value and direction of the velocity of transverse sliding.* The two cylinders may be called respectively A and *a*.

Fig. 109.

In fig. 109, let I A and I *a* represent the projections of the two axes. Along those projections lay off lengths I A, I *a*, proportional to the two angular velocities of rotation, and pointing in the direction in which an observer must look from I in order to make both rotations seem right-handed. Draw the straight line A *a*, and divide it at K into two parts inversely proportional to the radii of the two pitch-cylinders; in other words, let B and *b* denote the radii of the cylinders A and *a* respectively, so that B + *b* is the length of the common perpendicular, or line of centres; and let K divide A *a* in the following proportion:—

$$B + b : B : b$$
$$:: A a : K a : K A$$

Complete the parallelogram I V K *v*; then I V, I *v*, and the diagonal I K, will be respectively proportional and perpendicular to the surface velocity of the cylinder A, the surface velocity of the cylinder *a*, and the velocity of relative transverse sliding at the pitch-point I.

Or otherwise, by calculation; let $\frac{a}{A}$ be the ratio of the angular velocities, and $\frac{b}{B}$ that of the radii; then $\frac{a\,b}{A\,B}$ is obviously the ratio of the surface velocities.

It is obvious that for a given pair of axes and a given pair of angular velocities the velocity of transverse sliding is least when I K is perpendicular to A *a*. But A *a* is parallel to the line of contact of a pair of hyperboloïdal pitch-surfaces for skew-bevel wheels having the given velocity-ratio; and this is the demonstration of the statement in the preceding article, that screws which coincide approximately with skew-bevel wheels give the least possible transverse sliding of the threads for a given pair of axes and a given velocity-ratio (see page 159).

The proportionate value of the *common component of the surface velocities* may be represented by the length of a perpendicular let fall from either V or *v* upon I K; but the next rule gives a more convenient way of representing both it and the transverse sliding.

II. ***To draw the line of contact, and to find the proportions borne to the surface velocities by their common component, and by the transverse sliding; also the proportions borne to each other by the circular pitches, the divided axial pitches, and the divided normal pitch.***

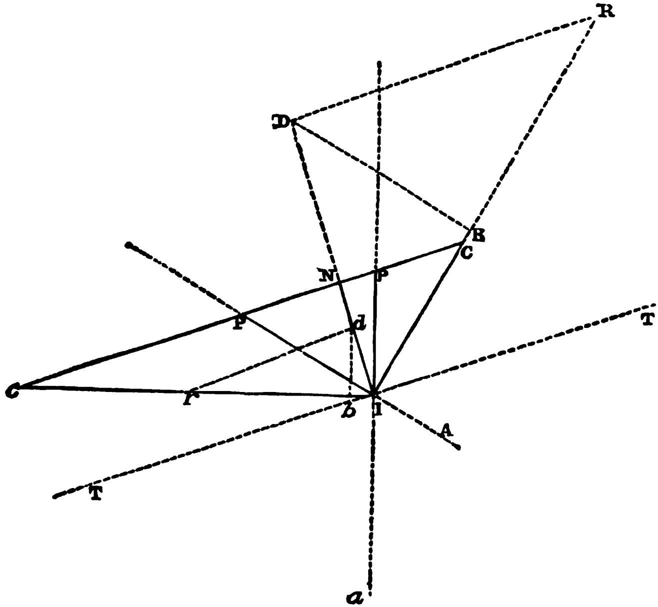

Fig. 110.

In fig. 110 (as in fig. 109), let I represent the pitch-point, and I A and I *a* the projections of the two axes. Perpendicular to I A and I *a* respectively, draw I C and I *c*, of the proper lengths, and in the proper directions, to represent the surface velocities of the two pitch-cylinders at the point I; draw the straight line C *c*, cutting the projections of the two axes in P and *p* respectively;

and upon C *c* let fall the perpendicular I N (which will obviously be parallel to I K in fig. 109). Through I draw T I T parallel to C *c*.

Then T I T will be *the line of contact;* I N will represent *the common component of the surface velocities* (and will also be the trace of the plane of connection); C *c* will represent *the velocity of transverse sliding;* and the proportions of the several divided pitches will be as follows:—

I C : circular pitch of A.
: : I *c* : circular pitch of *a*.
: : I P : divided axial pitch of A.
: : I *p* : divided axial pitch of *a*.
: : I N : divided normal pitch of both screws.

The figure may be regarded as part of the *development* of both screws upon the common tangent plane of their pitch-cylinders. (See Article 63, page 40. As to Racks, see Addendum, page 289.)

The *absolute lengths* of the circular pitches are found by dividing the pitch-circles into suitable numbers of equal parts, precisely as in the case of spur-wheels (see Articles 112 to 121, pages 103 to 114); and from them, by the aid of the proportions given by fig. 110, the absolute lengths of the divided axial pitches and of the divided normal pitch are easily found. For the total axial pitch of either screw, multiply the divided axial pitch by the number of threads.

III. *To find the radii of curvature of the normal screw-lines.* The normal helix, or normal screw-line (see Article 65, page 41), of each of the two screws touches I N at the pitch-point I; and the plane of connection of which I N is the trace is the common osculating plane of the two normal screw-lines at I. Their radii of curvature at that point both coincide with the common perpendicular of the axes. The rule for finding such radii (Articles 64 and 65, page 41), when applied to this case, takes the following form:—On I C lay off I B to represent the radius of the pitch-cylinder A; then perpendicular to I C draw B D parallel to I A, cutting I N in D; then perpendicular to I N draw D R, cutting I C in R; I R will be the radius of curvature of the normal helix of the screw A. A similar construction, substituting small for capital letters, serves to find I *r*, the radius of curvature of the normal helix of the screw *a*.

Fig. 111 represents two projections of the pitch-cylinders of a pair of screws designed by the rules which have just been given, and shows also the helical lines in which the fronts of the threads cut those pitch-cylinders. The upper part of the figure is a projection on the plane of action, whose trace, in fig. 110, is I N. A′ *a*′ is the common perpendicular of the two axes, and I′ the pitch-point; N′N′ is the trace of the common tangent plane of the two pitch-cylinders; and the arrow shows the direction of the common component of their surface velocities at the point I′. R and r are the centres of curvature of the two normal screw-lines at the point I′; and S S and s s, described about R and r respectively, are their two osculating circles, whose radii, I′ R and I′ r, are found by Rule III.

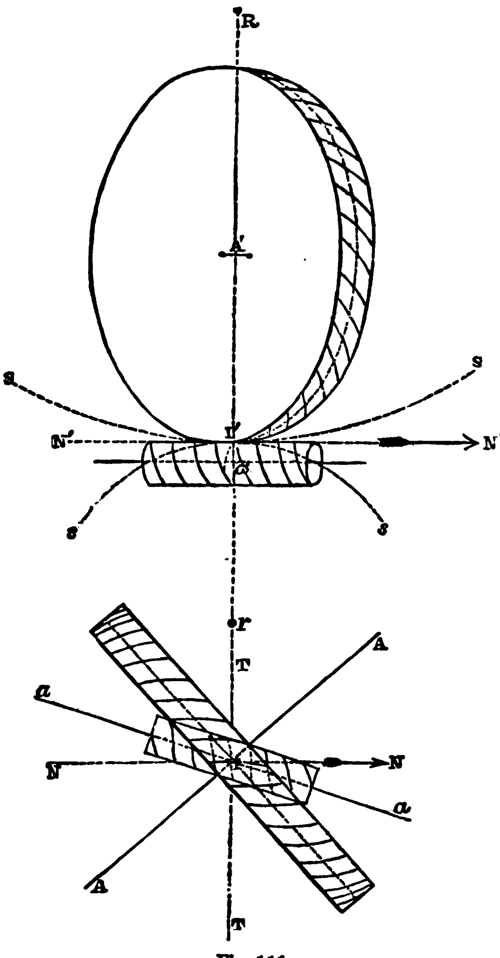

Fig. 111.

The lower part of the figure is a projection on the common tangent plane of the pitch-cylinders. A A and a a are the projections of their two axes; T I T is the line of contact; N I N is the trace of the plane of action; and the arrow marks the direction of the common component of the surface velocities at the pitch-point I.

In the particular example represented by figs. 109, 110, and 111, the following are the principal data and proportions:—

Velocity-ratio $\frac{a}{A} = 20$;

Number of threads of A, 40; of a, 2;

Ratios of radii and line of centres,

$$B + b : B : b$$
$$:: \; 11 \; : 10 \; : 1$$

Both screws right-handed.

155. **Figures of Threads found by Means of Normal Screw-Lines.**—By the following process threads may be designed for any gearing screw, so that they shall gear correctly with threads designed on the same principle for any other screw of the same normal pitch.

Let the screw to be provided with threads be, for example, the screw A of fig. 111. Draw, by Rule III. of Article 154, page 161, the osculating circle, S I′ S, of its normal screw-line. Lay off the normal pitch upon that osculating circle, and design the figure of a tooth and two half-spaces of that pitch, with the proper addendum and depth, as if the osculating circle were the pitch-circle of a spur-wheel; the figure so drawn will be the *normal section* of a thread, being the trace of the thread upon a surface which cuts it at right angles; and by the help of that section the threads may be made of the correct figure.

The normal sections of the acting surfaces of a thread may be either involutes of circles (Articles 131, 133, pages 120 to 128), or epicycloïds (Articles 136 to 140, pages 130 to 137). All screws with *involute threads* of the same divided normal pitch gear correctly together, and may be said to belong to *one set;* and they have the same property with involute toothed wheels, of admitting of some alteration of the distance between the axes. All screws of the same divided normal pitch having epicycloïdal teeth described by the same rolling circle gear correctly together, and may be said to belong to *one set.*

This method of designing the threads of gearing screws is believed to be now published for the first time.

156. **Figures of Threads designed on a Plane Normal to one Axis.**—In many cases which occur in practice the axes of the two screws are perpendicular to each other; so that, in fig. 110, page 160, A I P and a I p are at right angles, I C coincides with I p, and I c coincides with I P; and therefore the *divided axial pitch of either screw is equal to the circular pitch of the other.* In such cases, and especially where the diameters of the pitch-cylinders are very unequal, so that the larger screw is called a *worm-wheel,* and the smaller an *endless screw,* it is often convenient to design the traces of the threads on a plane normal to the axis of the worm-wheel, and traversing the axis of the endless screw; and then it is evident (as Mr. Willis appears to have been the first to show) that if the traces of the threads of the worm-wheel be made like those of a spur-wheel of the same radius and pitch, and those of the threads of the screw like the traces of the teeth of a rack suited to gear

with that spur-wheel, the worm-wheel and screw will gear correctly together.

Fig. 112 represents a worm-wheel and endless screw.

The lower part of the figure is a diagram drawn on the common tangent plane of the pitch-cylinders. I is the pitch-point; I C is

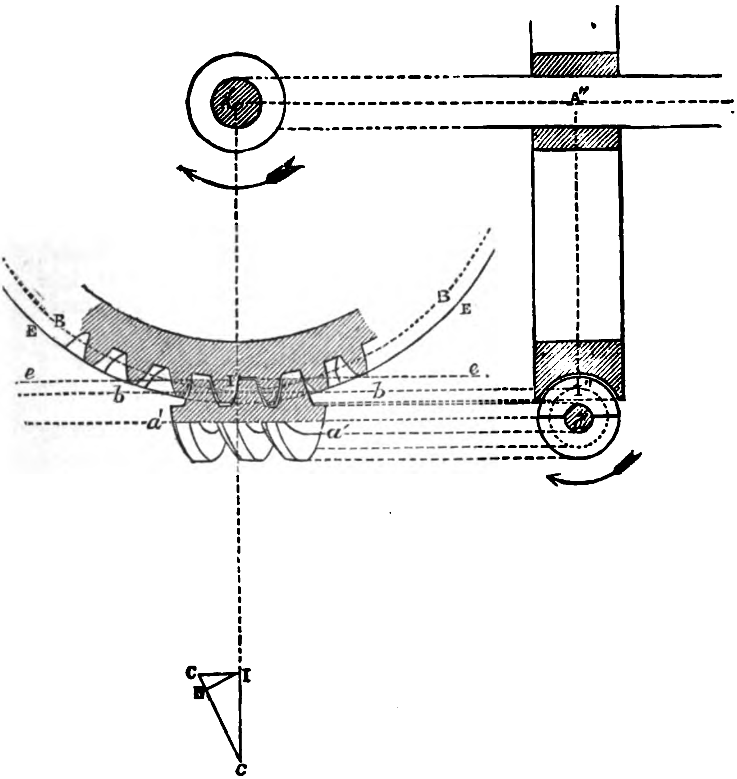

Fig. 112.

the divided axial pitch of the endless screw, being also the development of the circular pitch of the worm-wheel; I c is the divided axial pitch of the worm-wheel, being also the development of the circular pitch of the endless screw. I N, perpendicular to C c, is the development of the divided normal pitch of both screws; and C c is the extent of transverse sliding which takes place while an arc equal to the pitch passes the pitch-point.

In the left-hand division of the upper part of the figure the plane of projection is normal to the axis, A′, of the worm-wheel, and traverses the axis, a' a', of the endless screw. The circle B B is the trace of the pitch-cylinder of the wheel; the straight line b b is the trace of the upper side of the pitch-cylinder of the screw; and those traces touch each other in the pitch-point I′. The threads of the wheel, and those at the upper side of the screw, are shown in section; the traces of the threads of the wheel are like those of the teeth of a spur-wheel having the same circular pitch, and B B for a pitch-circle; the traces of the threads of the screw are like those of the teeth of a rack suited to gear with that spur-wheel, and having b b for its pitch-line. The addendum-circle, E E, of the worm-wheel, and the addendum-line, e e, of the endless screw, are drawn as for a spur-wheel and rack. The lower parts of the threads of the endless screw are shown in projection. In the example given, both wheel and screw have right-handed threads; the number of threads of the screw is two; of the wheel, 40; and the screw is represented as driving the wheel. The right-hand division of the upper part of the figure shows the wheel in section and the screw in projection; and the plane of projection traverses the axis, A″, of the wheel, and is normal to the axis, a'', of the screw; I″ is the pitch-point.

The traces of the threads of the wheel in the left-hand division of the upper part of the figure are involutes of a circle, and those of the threads of the screw are straight lines. That shape, as in the case of spur-wheels, enables the distance between the axes to be varied to a certain extent without affecting the accuracy of the action. But any shapes suited for the teeth of wheels and racks may be employed.

If a set of worm-wheels be made of the same circular pitch and obliquity of thread, and having the traces of the threads all involutes or all epicycloïds, traced by the same rolling circle; and if a set of endless screws be made, all of the same divided axial pitch, equal to the circular pitch of the wheels, and of an obliquity of thread equal to the complement of the obliquity of the threads of the wheels, and having the traces of the teeth, as the case may be, all straight lines of the proper obliquity, or all epicycloïds traced by the same rolling circle that is used to trace the threads of the wheels, then any one of the wheels will gear correctly with any one of the screws.

157. **Close-Fitting Tangent Screws.**—In many cases the object of screw-gearing is not the economical transmission of motive power, but the production of small angular motions with great accuracy: as, for example, when the principal wheel of a dividing engine, or that of a machine for pitching and cutting the teeth of wheels, or the wheel or sector which adjusts the direction of stroke of a

cutting tool in a shaping machine, is driven by a "tangent-screw" situated relatively to the wheel in the manner already shown in fig. 112. In such cases the screw has not only to move the wheel into any required position, but to hold it there; and therefore it is essential that there should be no back-lash. In order to ensure this, together with the requisite precision of action, an exact copy of the tangent-screw is made of steel, the edges of its thread are notched, and it is hardened, so that it becomes a cutting tool: it is then mounted in a suitable frame, so as to gear with the roughly formed teeth or threads of the wheel, and turned so as to drive them; in the course of which operation it cuts them to the proper figure. The axis of the cutting screw is placed at first at a distance from the axis of the wheel somewhat greater than the intended permanent distance; and after each complete revolution of the wheel the axes are brought a little nearer together, until the permanent distance is attained; and by turning the screw in this last position the shaping of the teeth or wheel-threads is finished. From the property of threads with traces similar to those of involute teeth, which has already been mentioned in Article 156, page 165, it is evident that this class of figures is peculiarly well suited to cases in which the tangent-screw is made to cut the wheel, because of the gradual diminution of the distance between the axes which takes place during the process of cutting.

158. **Oldham's Coupling.**—A *coupling* is a mode of connecting a pair of shafts so that they shall rotate in the same direction, with the same mean angular velocity. If the axes of the shafts are in the same straight line, the coupling consists in so connecting their contiguous ends that they shall rotate as one piece; but if the axes are not in the same straight line, combinations of mechanism are required. Various sorts of couplings will be described and compared together in a later division of this treatise. The present Article relates to a coupling for parallel shafts, invented by Oldham, which acts by *sliding contact*. It is represented in fig. 113. C_1, C_2 are the axes of the two parallel shafts; D_1, D_2, two discs facing each other, fixed on the ends of the two shafts respectively; E_1 E_1, a bar sliding in a diametral groove in the face of D_1; E_2 E_2, a bar sliding in a diametral groove in the face of D_2: those bars are fixed together at A, at right angles to each other, so as to form a rigid cross. The angular velocities of the two discs and of the cross are all equal at every instant; the middle point of the cross, at A, revolves in the dotted

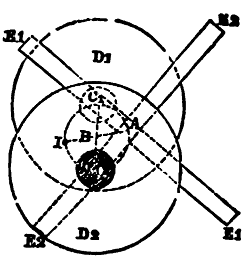

Fig. 113.

circle described upon the line of centres, C_1 C_2, as a diameter, twice for each turn of the discs and cross; the instantaneous axis of rotation of the cross at any instant is at I, the point in the circle C_1 C_2 diametrically opposite to A; and each arm of the cross slides in its groove through a distance equal to *twice the line of centres* during each half revolution, or twice the line of centres and back again—that is, four times the line of centres—during each revolution.

Oldham's coupling belongs to Mr. Willis's Class A. The cross may be strengthened by making its two bars take the form of projecting diametral ridges on opposite sides of a third circular disc Or the cross may consist of two grooves in the opposite sides of such a disc, and instead of grooved discs, the two shafts may carry cross bars fitting the grooves of the cross.

159. **Pin and Straight Slot.**—The communication of a uniform velocity-ratio by the sliding contact of a round pin with the sides of a slot or groove has already been described in Article 141, page 137. A velocity-ratio varying in any manner may be communicated by making the slot of a suitable figure, the principle of the combination being, that the line of connection is a normal to the centre line of the slot, traversing the centre line of the pin. The present Article relates to cases in which the slot is straight and the velocity-ratio variable. Three such cases are illustrated by figs. 114, 115, and 116, further on. Fig. 114 represents a *coupling*, belonging to Mr. Willis's Class B, where two shafts turn about the parallel axes A and B with equal mean angular velocities, though the angular velocity-ratio at each instant is variable. Fig. 115 shows a crank turning continuously about the axis A, and carrying a pin, C, which, by means of the slot F G, drives a lever which rocks or oscillates about the axis B. Fig. 116 shows a crank turning continuously about the axis A, and carrying a pin, C, which, by means of the slot F G in the cross-head of the rod B, gives a reciprocating sliding motion to that rod. The last two combinations belong to Mr. Willis's Class C.

In practice, for the purpose of diminishing friction and preventing back-lash, it is usual to make the pin turn in a bush which slides in the slot; but that bush is not shown in the figures.

The following are the principles of the action of those three combinations:—

I. *Coupling* (fig. 114).—In order that the directional relation of the rotations may be constant, the *crank-arm*, A C, must be greater than the line of centres, A B.

With a given crank-arm, A C, to find *the position of the axis* B *of the slot-lever*, so that the crank and slot-lever shall alternately overtake and fall behind each other by a given angle:—With the radius A C describe the circle D C E, and draw the diameter D A E, with which the line of centres is to coincide. Lay off

E A H = E A h = the complement of the given angle, and draw H B h perpendicular to D A E. B will be the trace of the required axis.

At the instant when the centre of the pin is at H or h, the angular velocities are equal; and A H B = A h B is the given angle beforementioned.

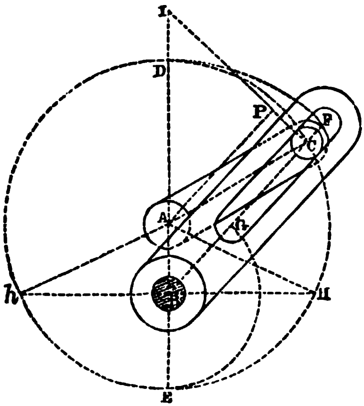

Fig. 114.

With a given position, C, of the centre of the pin, to find the *angular velocity-ratio*:—From C, perpendicular to the centre line, B C, of the slot, draw the *line* of connection, C I, cutting the line of centres in I; then

$$\frac{\text{Angular velocity of B}}{\text{Angular velocity of A}} = \frac{\text{A I}}{\text{B I}};$$

or otherwise: draw A P parallel to B C and perpendicular to C I; then

$$\frac{\text{Angular velocity of B}}{\text{Angular velocity of A}} = \frac{\text{A P}}{\text{B C}}.$$

The $\left\{\begin{matrix}\text{greatest}\\\text{least}\end{matrix}\right\}$ values of this ratio occur when the pin is at $\left\{\begin{matrix}\text{E}\\\text{D}\end{matrix}\right\}$ respectively; and they are as follows:—

$$\text{Greatest, } \frac{\text{A E}}{\text{B E}} = \frac{\text{A C}}{\text{A C} - \text{A B}};$$

$$\text{Least, } \frac{\text{A D}}{\text{B D}} = \frac{\text{A C}}{\text{A C} + \text{A B}}.$$

The *travel* or *length of sliding of the pin in the slot* is

$$\text{F G} = \text{B F} - \text{B G} = \text{B D} - \text{B E};$$

and this takes place twice in each revolution.

II. *Crank and Slotted Lever* (fig. 115).—As the crank-arm, A C, in fig. 115, is shorter than the line of centres, A B, the slotted lever, B G F, has a reciprocating or rocking motion.

With a given line of centres, A B, and a given *semi-amplitude* or angular half-stroke of the rocking motion of the lever, A B K = A B k, to find *the length of crank-arm*:—From A let fall A K perpendicular to B K, or A k perpendicular to B k; A K = A k will be the required crank-arm.

K and k will be the two *dead points;* that is to say, the positions of the centre of the pin at the two instants when the lever has no velocity, having just ceased to move in one direction, and being just about to begin to move in the opposite direction.

To find the *angular velocity-ratio* at the instant when the centre of the pin is in a given position, C:—Draw the corresponding position, B C F, of the centre line of the slot, and perpendicular to it draw C I, cutting the line of centres in I; then

$$\frac{\text{Angular velocity of lever}}{\text{Angular velocity of crank}} = \frac{\text{A I}}{\text{B I}}.$$

To find the travel of the pin in the slot, lay off B G = B E, and B F = B D; G and F will be the two ends of the travel of the centre of the pin; and F G = D E = 2 A C will be the length of travel.

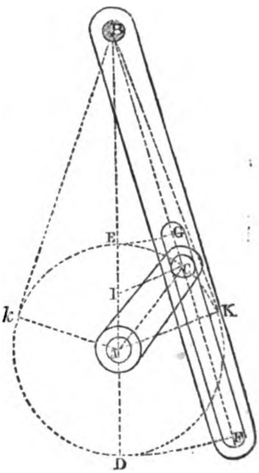

Fig. 115.

III. *Crank and Slot-headed Sliding Rod* (fig. 116).—The crank-arm, A C, in this case is to be made equal to one-half of the intended *length of stroke* of the sliding rod, B. Draw the circle described by C, the centre of the pin, and let *k* A K be the diameter of that circle which is parallel to the direction of motion of the rod; then K and *k* will be the *dead points*, or positions of the centre of the pin at the two instants when the rod has no velocity. To find the *velocity-ratio* of the rod and crank-pin when the centre of the crank-pin is in a given position, C: perpendicular to the direction of motion of the rod draw the diameter D A E; this line will correspond to the line of centres in the preceding problems; then through C, and perpendicular to the centre-line, F G, of the slot, draw the line of connection, C I, cutting D A E in I; the following will be the required velocity-ratio:—

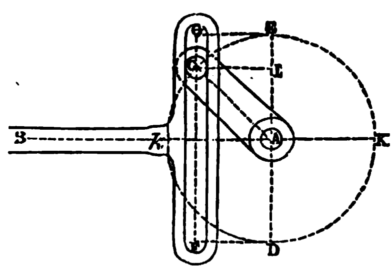

Fig. 116.

$$\frac{\text{Velocity of rod, B}}{\text{Velocity of centre of pin, C}} = \frac{\text{A I}}{\text{A C}}.$$

The *extent of travel of the pin in the slot* is F G = D E = 2 A C.

160. **Cams and Wipers in General.**—Cams and wipers are those primary pieces, with curved acting surfaces, which work in sliding contact without being related to imaginary pitch-surfaces, as the teeth of wheels and threads of screws are. The distinction between a cam and a wiper is, that a cam in most cases is continuous in its action, and a wiper is always intermittent; but a wiper is sometimes called a cam notwithstanding. A cam is often like a non-circular sector or wheel in appearance; a wiper is often like a solitary tooth. (As to "rolling cams," see Article 110, page 99.)

The solutions of all problems respecting the velocity-ratio and directional relation in the action of cams and wipers are obtained by properly applying the general principle of Article 122, page 114.

In most cases which occur in practice, the condition to be fulfilled in designing a cam or a wiper does not directly involve the velocity-ratio, but assigns a certain series of definite positions which the follower is to assume when the driver is in a corresponding series of definite positions. Examples of such problems will be given in the following Articles.

161. **Cam with Groove and Pin.**—Throughout the present Article it will be supposed that the acting surface of the follower, which is to be driven by the cam, is the cylindrical surface of a pin. It is easy to see that without in any respect altering the action, a cylindrical roller turning about a smaller pin may be substituted for a pin in order to diminish friction. If the pin is to be driven by the cam in one direction only, being made to return at the proper time by the force of gravity or by the elasticity of a spring, the cam may have only one acting edge; but if the pin is to be driven back as well as forward by the cam, the cam must have two acting edges, with the pin between them, so as to form a groove or a slot of a uniform width equal to the diameter of the pin, with clearance just sufficient to prevent jamming or undue friction. The centre of the pin may be treated as practically coinciding at all times with the centre-line of such a groove, which centre-line may be called the *pitch-line* of the cam. The most convenient way to design a cam is usually to draw, in the first place, its pitch-line, and then to lay off the half-breadth of the groove on both sides of the pitch-line. When one acting edge only is required, it is to be laid off on one side of a groove, the other side being omitted.

The *line of connection* at any instant is a straight line normal to the pitch-line at the centre of the pin.

The surface in which the groove is made may be either a plane or a surface of revolution; a plane for a *cam-plate* which either turns about an axis normal to its own plane or slides in a straight line, and acts upon a pin whose centre moves in a plane parallel to that of the cam-plate; a solid of revolution, being either a cylinder, a cone, or a hyperboloïd, for a cam which turns about an axis, and acts on a pin whose centre has a reciprocating motion in a straight line coinciding with a generating line of the surface of revolution.

The following example is a case of a rotating plane cam, giving motion through a pin and lever to a rocking shaft whose axis is parallel to the axis of rotation of the cam.

In fig. 117 the plane of projection is that of the cam-plate, and is normal to the axes of the cam and of the lever. In the lower part of the figure, A′ represents the trace of the axis of the rocking

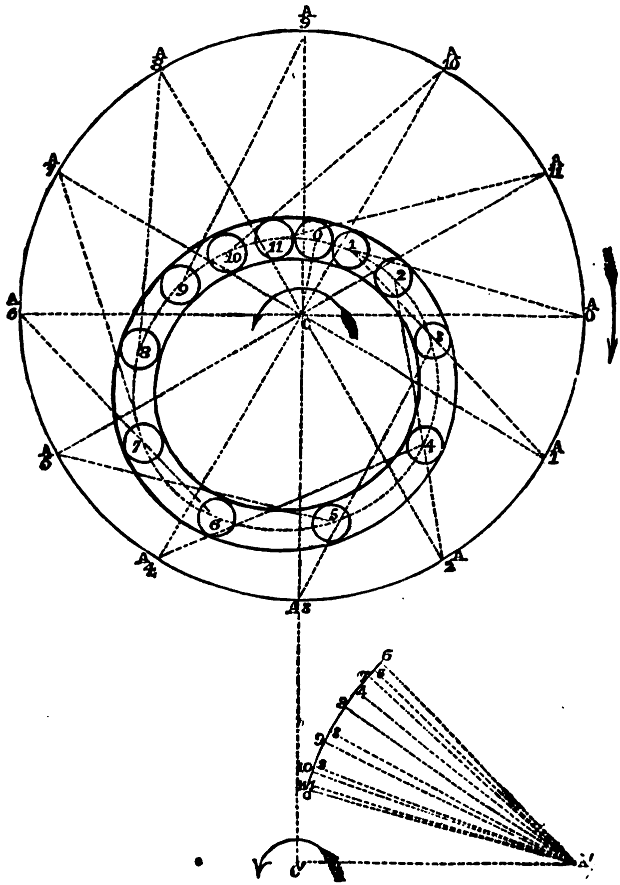

Fig. 117.

shaft, and C' the trace of the axis of the cam, so that $A' C'$ is the line of centres. The direction of rotation of the cam is shown by an arrow. In the example, the direction is left-handed. The circular arc, 0 6, described about A' with the radius A' 0, is the path to be described by the centre of the pin; and the twelve points in that arc, marked with numbers from 0 to 11, are twelve positions which the centre of the pin is to occupy at the end of twelve equal divisions of a revolution of the cam. It is required to find the form of the cam which will produce that motion in the pin.

In the upper part of the figure, let C represent the axis of the cam; suppose that the cam is fixed, and that the line of centres, C A, rotates about C, carrying the axis, A, of the rocking shaft along with it, with an angular velocity equal and contrary to the actual angular velocity of the cam. That supposition will not alter the relative motions of the working pieces. With the radius C A describe a circle to represent the supposed path of A relatively to C; divide its circumference into twelve equal parts, and to the points of division draw radii, $C A_0$, $C A_1$, $C A_2$, &c., to represent twelve successive positions of the line of centres relatively to the cam, as supposed to be fixed. Lay off the angles $C A_0$ 0, $C A_1$ 1, $C A_2$ 2, &c., in the upper part of the figure respectively, equal to the angles $C' A'$ 0, $C' A'$ 1, $C' A'$ 2, &c., in the lower part of the figure; and make each of the straight lines A_0 0, A_1 1, A_2 2, &c., equal to the lever arm A' 0. The points thus found, 0, 1, 2, &c., will be points in the pitch-line of the cam, and a curve drawn through them will be the required pitch-line.

About each of the points 0, 1, 2, &c., draw a circle of a radius equal to that of the pin: a pair of curves touching those circles so as to be parallel to the pitch-line will mark the two sides of the groove, without allowance for clearance. Clearance may be provided either by slightly diminishing the diameter of the pin or by slightly increasing the width of the groove. If the lever is to be raised by the cam, but brought down again by gravity, the outer side of the groove may be omitted, and the cam will become a disc bounded by the innermost of the three parallel curves shown in the figure.

The number of parts into which the revolution of the cam is divided may be made more or less numerous according to the degree of precision required.

It is easy to see how a similar method may be applied to the designing of a cam-disc which shall produce a given motion in a follower whose acting surface is of any given form. A figure is to be constructed like the upper part of fig. 117, on the supposition that the cam is fixed, and that the frame of the machine rotates about the axis of the cam with an angular velocity equal and contrary to the actual angular velocity of the cam. Then, just as the pin in the upper part of fig. 117 is drawn in its several positions, 0, 1, 2, &c., the trace of the acting surface of the follower is to be drawn in its several successive positions; and a line touching that trace in all its positions will be the trace of the required cam-disc.

The *dead points* of a cam are the points in its pitch-line which are at the greatest and least distances from its axis. In the example shown in fig. 117 the dead points are 0 and 6. When the centre of the pin is at those points it has no velocity. Any part of the pitch-line which is an arc of a circle about C corresponds to a *pause* in the motion of the pin.

162. **Drawing a Cam by Circular Arcs.**—In many cases in which cams have to be designed, the dead points alone are given by the conditions of the problem, leaving the parts of the pitch-line between those points to be drawn according to convenience. For example, in fig. 118,

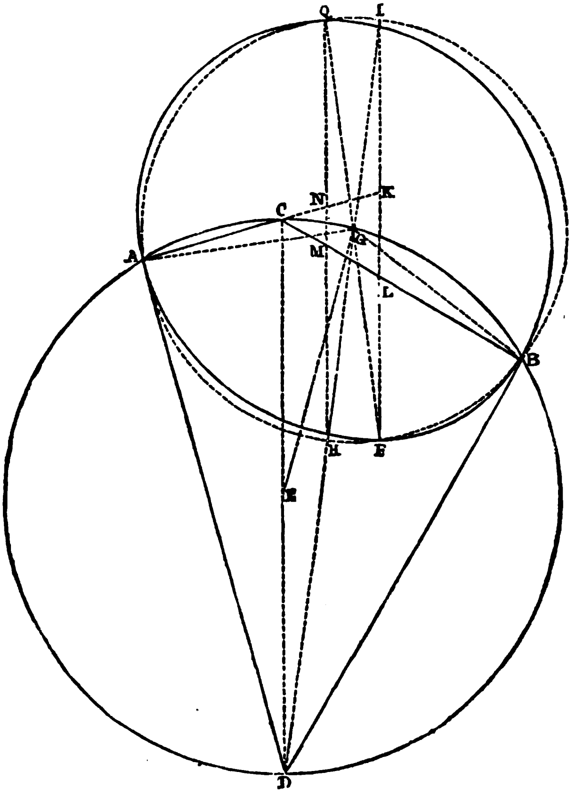

Fig. 118.

C is the axis of the cam, and A and B are dead points; so that C B and C A are respectively the least and greatest radii drawn from the axis to the pitch-line; and the pitch-line at A and B is normal to those radii respectively. The intermediate arcs of the pitch-line are to be drawn of any convenient form, so as to traverse A and B, and be normal to C A and C B.

The easiest way to draw such curves is by means of arcs of circles.

The simplest case is when C A and C B are parts of one straight line. The required pitch-line is then an *eccentric circle*, described upon the straight line A C B as a diameter.

When C A and C B, as in the figure, are not parts of one straight line, the following method may be used, being an extension of Rule IV. of Article 79, page 61, and having the effect of giving a pitch-line made up of four circular arcs, whose radii deviate less from equality than those of any other combination of four circular arcs which would answer the same purpose.

From A and B, perpendicular to A C and B C respectively, draw A D and B D, cutting each other in D. These will be tangents to the required pitch-line. Join C D; bisect it in E; and about E, with the radius E C = E D, describe a circle which will traverse the four points A, C, B, D. Bisect the arc A C B in G. About G, with the radius G A = G B, describe a circle; and draw the straight line D H G I, cutting that circle in H and I. Through the points H and I, and parallel to D C, draw the straight lines H Q and I P, cutting the circle A I B H in P and Q (the ends of one diameter), and cutting also the straight line C B in M and L, and the straight line A C produced in N and K. Then draw four circular arcs, as follows:—

The arc	A P,	described about	the point	K,
"	P B,	"	"	L,
"	B Q,	"	"	M,
"	Q A,	"	"	N;

and those arcs will make up a pitch-line having C B and C A for its greatest and least distances from the axis C, as required; and also having its radii of curvature less unequal than is possible with any other combination of four circular arcs, and no more, fulfilling the required conditions.

When a cam is to have more than two dead points, each pair of adjacent dead points are to be connected with each other by means of two circular arcs, drawn according to Rule IV. of Article 79, pages 61 and 62, fig. 48.

163. **Many-coiled Cams; Spiral and Conoidal Cams.**—When the complete series of movements of a piece that is to be driven by a cam extends over more than one revolution of the cam, there are

cases in which the required result may be effected by means of a groove in a cam-plate having a pitch-line of more than one coil; but difficulties in working may arise from the fact that the coils of the groove must intersect each other. There are other cases in which the motion required in the follower is of a kind that may be produced by means of a spiral cam, such as that shown in fig. 119. The upper part of the figure is a projection on a plane normal to the axis; the lower part, a projection on a plane parallel to the axis. A A′ is the spiral cam; B, a screw of an axial pitch exactly equal to the axial pitch of the cam. This screw, resting in a fixed nut, forms one of the bearings of the cam-shaft, and causes the shaft and cam together to advance along the axis at each revolution through a distance equal to the pitch, thus bringing a new coil of the cam into action. The cam, A, may also be made with a continuous conoidal surface, of which different parts are brought into action at each revolution by the advance caused by the screw B. It is evident that in spiral and conoidal cams the extent of the motion is limited.

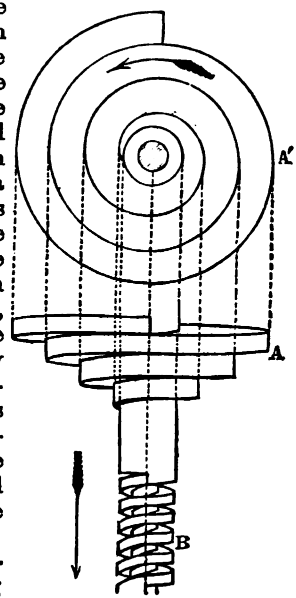

Fig. 119.

164. **Wipers and Pallets — Escapements.**—In fig. 120 a shaft rotating about the axis A is provided with one or more solitary teeth called *wipers*, such as E. The action of the wipers upon the projecting parts of the piece that they drive (which, for the sake of a general term, may be called *pallets*) may be either *intermittent* or *reciprocating*.

I. As an example of *intermittent* action, one of the wipers represented in fig. 120, in moving from the position H to the position E, is supposed to have driven before it a pallet from the position G to the position F. The pallet projects from a vertical sliding bar, or *stamper*, C.

B B is the addendum-circle of the wipers, and D D the addendum-line of the pallets. Those lines cut each other at the *point of escape*, E; and just at that point the pallet *escapes* from the wiper, and the stamper, with its pallet, falls back to its original position, and is ready to be lifted again by the next wiper.

The stamper and pallet referred to in this case are shaded.

II. As an example of *reciprocating* action, the sliding bar, C, of the preceding example is supposed to have attached to it a frame,

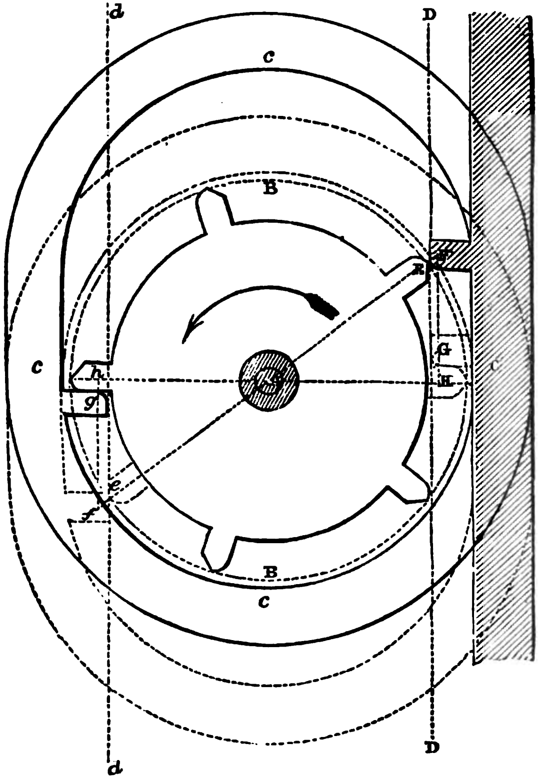

Fig. 120.

c c c, at the opposite side of which is another pallet, *g*; and this pallet is so placed that immediately after the escape of the former pallet, F, from the wiper at E, another wiper at *h* begins to act upon the pallet *g*, and so to produce the *return stroke* of the frame, C *c c c*. The point, *e*, where the addendum-line, *d d*, of the pallet *g* cuts the addendum-circle, B B, of the wipers, is the *point* of *escape* of the second pallet (whose position at the instant of escape is marked *f*); and immediately afterwards a third wiper, arriving at the position H, begins to produce a new forward stroke.

The *length of stroke* is represented in the figure by F G = *f g*. It is evident that the number of wipers must be odd.

This is the combination already referred to in Article 142, page 141. It belongs to a class of contrivances called *escapements*, because of the *escape* of the follower from the action of the driver at certain instants. There are many escapements which do not belong to the subject of pure mechanism; and amongst these are found most of the escapements that are used in clocks and watches, as being well suited to the regulation of those machines; for in such escapements the driver and follower are disconnected from each other during the greater part of the movement. Only two more escapements, therefore, will be described here.

III. *Anchor Recoil Escapement.*—This escapement, though not well suited to the exact keeping of time, is used in old clockwork. It is also used in vertical roasting jacks. The driver is a wheel called the *scape wheel*, and the trace of its axis is represented by the point A, fig. 121. E I F is its pitch-circle, cutting the line

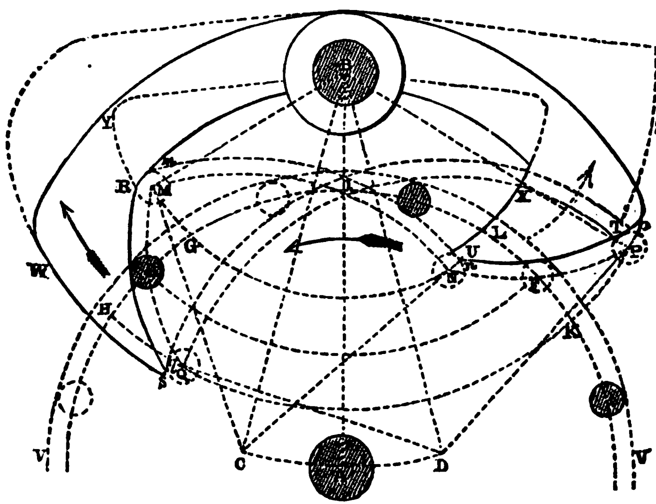

Fig. 121.

of centres, A B, in I. V V is its addendum-circle. In the figure the teeth are represented as cylindrical pins; in any case their acting surfaces may be regarded as parts of cylinders, which, if the teeth are sharp-pointed, are of insensible diameter. The arrow near I shows the direction of rotation of the wheel. The point B is the trace of the axis of the *verge*, or rocking shaft, to which a reciprocating movement is to be given through the alternate action of the teeth on the *pallets*, R S and T U, which are the acting surfaces of the *crutch*, S R T U. At the instant represented in the figure, the crutch is at the middle of its swing, and in the act of moving towards the left, through the action of the tooth E on the pallet R S. The swing of the crutch takes place while the wheel moves through half the pitch; at the end of which interval the tooth E and pallet R S *escape* from each other, and another tooth begins to act on the pallet T U, so as to make the crutch swing towards the right, and so on alternately. The dotted circle at F represents a tooth in the act of driving the pallet T U, at the middle of the swing, towards the right.

To design the figures of the pallets, a method is to be employed analogous to that described in Article 161, page 172; that is to say, the crutch is to be supposed fixed, and the line of centres, B A, is to be supposed to swing to and fro about the axis B, carrying with it the axis A, through an angle equal to the angle through which the crutch is actually to swing.

Lay off the angles A B C = A B D = the *semi-amplitude*, or half angle of swing; and make B C = B D = B A. Then C and D are the two extreme positions of the axis A in its supposed swinging motion. With a radius equal to that of the pitch-circle, draw the arcs M N about C, and P Q about D; and with a radius equal to that of the addendum-circle, draw the arcs *m n* about C, and *p q* about D. From the point I lay off upon the pitch-circle the arcs I E = I F = *an odd number of times the quarter-pitch;* so that E I F shall be an odd number of half pitches. The points E and F should be as near as practicable to the points where two straight lines from B touch the pitch-circle. About E and F draw circles to represent the traces of the acting surfaces of the pins or teeth. Lay off, on the pitch-circle, the arcs E G = F K = the quarter-pitch, with the radius of the acting surface of a tooth deducted: this deduction is to ensure that between the escape of a tooth from one pallet and the commencement of the action of another tooth on the opposite pallet there shall be an interval sufficient to enable the tooth that has just escaped to move clear of the pallet which it has quitted.

About the centre B, through the point G, draw the circular arc M G N, cutting the arc M N, already described about C, in the points M and N. About the centre B, through the point K, draw the circular arc P K Q, cutting the arc P Q, already described about D, in the points P and Q. Through M, E, and Q draw a continuous curve; this will be the pitch-line of the pallet R S. Through N, F and P draw a continuous curve: this will be the pitch-line of the pallet T U. Then, parallel to those pitch-lines respectively, and at a distance from them equal to the radius of the acting surface of a tooth, draw the traces, R S and T U, of the acting surfaces of the pallets.

The points of the pallets, at S and U, are to be cut off, so as not to project within the circles *q p* and *n m* respectively. The traces of the backs of the pallets, S W and U X, are to be circular arcs described about B.

IV. *Dead-beat Escapement.*—In the dead-beat escapement the crutch swings each way through an arc of indefinite extent, in addition to that through which it is driven by the action of the teeth of the scape wheel; and the scape wheel is made to pause in its motion during each such additional swing, by its teeth bearing against parts of the pallets whose surfaces are cylinders described about the axis of the verge. The traces of these may be called the *dead arcs* of the pallets. Tho recoil escapement shown in fig. 121, may be converted into a dead-beat escapement, as follows:— About B, with a radius equal to B M added to the radius of the acting surface of a tooth, draw the circular arc R Y; and also about B, with a radius equal to B P, deducting the radius of the acting surface of a tooth, draw the circular arc T Z: those two arcs will be the required dead arcs of the pallets.

In order that a dead-beat escapement may go on working, there must be a force, such as gravity or the elasticity of a spring, continually tending to bring the crutch to its middle position, at and near which the pallets are driven by the teeth; hence its principles are to a certain extent beyond the province of pure mechanism.

In the dead-beat escapements of accurate clocks, the angle through which the crutch swings is very small, and the angle through which the teeth act on the pallets is still smaller; so that in fig. 121 those angles may be looked upon as greatly exaggerated, for the sake of distinctly showing the geometrical principles of the combination.

SECTION V.—*Connection by Bands.*

165. **Bands and Pulleys Classed.** (*A. M.*, 478.)—The word *bands* may be used as a general term to denote all kinds of flexible connecting pieces; and the word *pulleys*, when not otherwise qualified, to denote all kinds of rotating pieces which are connected with each other by means of bands. Bands may be classed in the following manner; which also involves a classification of the pulleys to which the bands are suited:—

I. *Belts*, which are made of leather, gutta percha, woven fabrics, &c., are flat and thin, and require nearly cylindrical pulleys with smooth surfaces. A belt tends to move towards that part of a pulley whose radius is greatest. Pulleys for belts, therefore, are slightly swelled in the middle, in order that the belt may remain on the pulley unless forcibly shifted, and are in general without ledges. A belt when in motion is shifted off a pulley, or from one pulley on to another of equal size alongside of it, by pressing against the "advancing side" of the belt; that is, that part of the belt which is moving *towards* the pulley. Amongst belts may be classed *flat ropes.*

II. *Cords*, made of catgut, leather, hempen or other fibres, or wire, are nearly cylindrical in section, and require either drums with ledges, or grooved pulleys.

III. *Chains*, which are composed of links or bars jointed together, require wheels or drums, grooved, notched, and toothed, so as to fit the links of the chains. Chains suited for this purpose are called *gearing chains.*

Bands for communicating motion of indefinite extent are *endless.*

Bands for communicating reciprocating motion have usually their ends made fast to the pulleys or drums which they connect, and which, when the extent of motion is less than a revolution, may be sectors.

166. **Principles of Connection by Bands.**—The *line of connection* of a pair of pulleys connected by means of a band is the central line or axis of that part of the band whose tension transmits the motion.

The *pitch-surface* of a pulley over which a band passes is the surface to which the line of connection is always a tangent; that is to say, an imaginary surface whose distance from all parts of the acting surface of the pulley that the band touches is equal to the distance from the acting surface of the band to its centre line. The pitch-surface of a pulley cannot be anywhere concave; for where the acting surface is concave, the band stretches in a straight line across the hollow, and the pitch-surface is plane. In ordinary pulleys for communicating a constant velocity-ratio the pitch-surface is a circular cylinder; and its radius (called the *effective radius*) is equal to the real radius of the pulley added to half the thickness of the band.

The *pitch-line* of a pulley is the line on its pitch-surface in which the centre-line lies of that part of the band which touches the pulley. The line of connection is a tangent to the pitch-line. When the line of connection is in a plane perpendicular to the axis of the pulley, the pitch-line is the trace of the pitch-surface on that plane: for example, the circular section of a cylindrical pulley. When the line of connection is oblique to the axis, the pitch-line is *helical*, or screw-like.

Problems respecting the comparative motion of pieces connected by bands are solved by applying the principles of Article 91, page 78, taking A B in fig. 58 of that Article to represent the centre line of that part of the band whose tension transmits the motion, and A A′ and B B′ to represent the common perpendiculars from that line to the axes of the pulleys. When the pitch-surfaces of the pulleys are circular cylinders, A A′ and B B′ represent their effective radii. Rule II. of Article 91 shows how to find the angular velocity-ratio of two pulleys whose proportionate dimensions are given. The following is the converse rule for finding the proportionate radii of two pulleys which are to transmit a given angular velocity-ratio. In fig. 58, page 78, draw A *a* to represent the projection of the axis of one pulley upon a plane parallel to that axis traversing the line of connection, A B; and draw B *b* to represent a similar projection of the axis of the other pulley. Lay off the distances A *a* and B *b* to opposite sides of A B, to represent the intended angular velocities of the two pulleys. Draw A *c* and B *d* perpendicular to A B; and draw *a c* and *b d* parallel to A B, cutting A *c* and B *d* in *c* and *d* respectively. Then the lengths A *c* and B *d* will represent the component angular velocities of the pulleys about axes perpendicular to the line of connection, A B. (In most cases which occur in practice, both the axes lie in planes perpendicular to the line of connection; and then A *a* and B *b* coincide with A *c* and B *d* respectively.)

Draw the straight line *c d*, cutting the line of connection, A B, in K. Then we have the proportion

B K : A K

: : effective radius of A : effective radius of B;

and if one of those radii—for example, that of A—is given, the other is found as follows:—From A lay off A I = B K (or otherwise, from B lay off B I = A K). Perpendicular to A B draw A A′ and B B′; lay off A A′ = the given radius of the pulley A, and draw the straight line A′ I B′, cutting B B′ in B′; B B′ will be the required radius of B.

In the ordinary case, in which both axes lie in planes perpendicular to the line of connection, it is evident that the velocities of a pair of circular pulleys are *inversely as their effective radii.*

It is to be borne in mind that, especially as regards cases in which the axes do not both lie in planes perpendicular to the line of connection, everything stated in the present Article is based on the supposition that *the band is perfectly flexible in all directions.* In the case of flat belts connecting pulleys whose axes are not both in planes perpendicular to the line of connection, there are certain effects of the lateral stiffness of the belt which will be considered farther on.

The *velocity of the band* is equal to that of a point revolving at the end of the radius A A′, fig. 58, page 78, with the angular velocity represented by A *c*, and also to that of a point revolving at the end of the radius B B′, with the angular velocity represented by B *d*. When a band connects a pulley with a sliding piece, the comparative motion is given by Rule III. of Article 91, page 79.

Smooth bands, such as belts and cords, are not suited to communicate a velocity-ratio *with precision*, as teeth are, because of their being free to slip on the pulleys; but the freedom to slip is advantageous in swift and powerful machinery, because of its preventing the shocks which take place when mechanism which is at rest is suddenly *thrown into gear*, or put in connection with the prime mover. A band at a certain tension is not capable of exerting more than a certain definite force upon a pulley over which it passes; and therefore occupies, in communicating its own speed to the rim of that pulley, a certain definite time, depending on the masses that are set in motion along with the pulley and the speed to be impressed upon them; and until that time has elapsed the band has a slipping motion on the pulley; thus avoiding shocks, which consist in the too rapid communication of changes of speed. This will be further considered under the head of the Dynamics of Machines.

167. **Pulleys with Equal Angular Velocities.**—When a pair of pulleys turn about parallel axes in the same direction, with equal angular velocities, their pitch-lines may be of any figure whatsoever, curved or polygonal, provided they are equal and similar, and not concave. Each of the two straight parts of the band is equal and parallel to the line of centres; and those parts, if the pulleys are circular and not eccentric, remain at a constant distance from the line of centres; but have a reciprocating motion towards and from that line if the pulleys are either eccentric or non-circular. A *reel* is virtually a pulley whose pitch-line is a polygon with rounded angles; and such is also the case with the *expanding pulley*, consisting of four quadrants of a circle, which can be separated to a greater or less distance from each other by means of screws.

168. **Bands and Pulleys for a Constant Velocity-Ratio.**—In order that the velocity-ratio of a pair of pulleys may be constant, their pitch-lines must be circular (except in the particular case specified in the preceding Article, when the figure is not restricted to the circle alone).

The band may be *open* or *uncrossed*, as in fig. 122; or it may be *crossed*, as in fig. 123. With an open band the directions of rotation

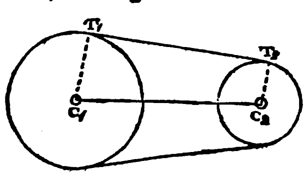

Fig. 122.

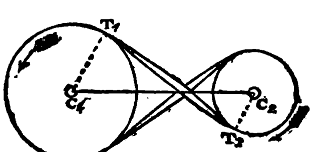

Fig. 123.

are the same; with a crossed band, contrary. In each of these figures, 1 denotes the driving pulley, and 2 the following pulley; $C_1 C_2$ is the line of centres, and $T_1 T_2$ the line of connection; and the angular velocity-ratio is expressed by

$$\frac{a_2}{a_1} = \frac{C_1 T_1}{C_2 T_2}.$$

169. The **Length of an Endless Band**, such as those shown in figs. 122 and 123, consists of two straight parts, each equal to the line of connection, and two circular arcs. When the band is crossed, as in fig. 123, the circular arcs are of equal angular extent; when the band is open, as in fig. 122, the angles subtended by the two arcs make up one revolution. When the length of a band is to be measured on a drawing, the circular parts may be rectified graphically by Rule I. or Rule II. of Article 51, page 28.

To find the length of an endless band by calculation, let the line of centres, $\overline{C_1 C_2} = c$, and the *effective radii* of the pulleys, $\overline{C_1 T_1} = r_1$; $\overline{C_2 T_2} = r_2$; r_1 being the greater. Then each of the two equal straight parts of the band is evidently of the length

$$\left.\begin{aligned} T_1 T_2 &= \sqrt{c^2 - (r_1 + r_2)^2} \text{ for a crossed band;} \\ T_1 T_2 &= \sqrt{c^2 - (r_1 - r_2)^2} \text{ for an open band.} \end{aligned}\right\} \ldots\ldots(1.)$$

Let i_1 be the arc to radius unity of the greater pulley, and i_2 that of the less pulley, with which the band is in contact; then for a crossed band

$$i_1 = i_2 = \pi + 2 \operatorname{arc} \cdot \sin \frac{r_1 + r_2}{c};$$

and for an open band

$$i_1 = \pi + 2 \operatorname{arc} \cdot \sin \frac{r_1 - r_2}{c}; \; i_2 = \pi - 2 \operatorname{arc} \cdot \sin \frac{r_1 - r_2}{c}; \quad (2.)$$

and the addition of the lengths of the straight and curved parts gives the following total length:—

For a crossed band,

$$L = 2\sqrt{c^2 - (r_1 + r_2)^2} + (r_1 + r_2) \cdot \left(\pi + 2 \operatorname{arc} \cdot \sin \cdot \frac{r_1 + r_2}{c}\right);$$

and for an open band,

$$L = 2\sqrt{c^2 - (r_1 - r_2)^2} + \pi(r_1 + r_2) + 2(r_1 - r_2) \operatorname{arc} \cdot \sin \cdot \frac{r_1 - r_2}{c}. \quad (3.)$$

As the last of these equations would be troublesome to use in a practical application to be mentioned in Article 171, an approximation to it, sufficiently close for practical purposes, is obtained by considering, that if $r_1 - r_2$ is small compared with c, $\sqrt{c^2 - (r_1 - r_2)^2} = c - \frac{(r_1 - r_2)^2}{2c}$ nearly, and $\text{arc} \cdot \sin \cdot \frac{r_1 - r_2}{c} = \frac{r_1 - r_2}{c}$ nearly; whence, for an open band,

$$\text{L nearly} = 2c + \pi (r_1 + r_2) + \frac{(r_1 - r_2)^2}{c}, \ldots\ldots (3\text{ A.})$$

in which it is sufficiently accurate for practical purposes to make $\pi = 3\frac{1}{7}$.

170. **Pulleys with Flat Belts.**—It has already been stated in Article 165, page 179, that a flat belt tends to move towards that part of the pulley whose radius is greatest, or to "climb," as the phrase is; and that pulleys for such belts are therefore made without ledges, and with a slight swell or convexity at the middle of the rim, in order that the belt may tend to remain there. The swell usually allowed in the rims of pulleys is *one twenty-fourth part of the breadth.*

The tendency to climb is produced by the lateral stiffness of the belt, in the following manner:—When the part of the belt which touches the pulley deviates towards one side, as in fig. 124, the part which is approaching the pulley is made to deviate towards the opposite side; and thus, after the pulley has turned through a small angle, the deviation of the belt is corrected.

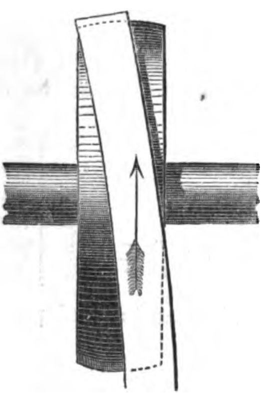

Fig. 124.

A crossed belt is twisted half round in passing from one pulley to another, as shown in fig. 123, so as to bring the same side of the belt into contact with both pulleys. The principal object of this is, that the two straight parts of the belt may pass each other flatwise where they cross, so as not to resist each other's motion. Another object, in the case of leather belts, is to bring the rougher side of the leather into contact with both pulleys.

It has already been stated that the position which a belt assumes upon a pulley is determined by the position of its *advancing side;* that is, of the part of the belt which is approaching the pulley. In the contrivance called the "*fast and loose pulley,*" for engaging and disengaging machinery, a belt driven by a suitable driving pulley is provided with two similar and equal following pulleys, mounted side by side upon one axis; one of these pulleys is made fast to the shaft; the other turns loosely upon it. The belt, when in motion, can be shifted by means of a fork, that guides its advancing side to the fast pulley or to the loose pulley at will, so as to engage or disengage the shaft on which those pulleys are fitted. The driving pulley is made of a breadth equal to the breadths of the fast and loose pulleys together.

The lateral stiffness of a belt is also made available for the purpose of keeping it in its place on the pulleys when their axes are not parallel, as in fig. 125, which is sketched in isometrical perspective. C_1 C_1 and C_2 C_2 are the axes; E_1 E_2, their common

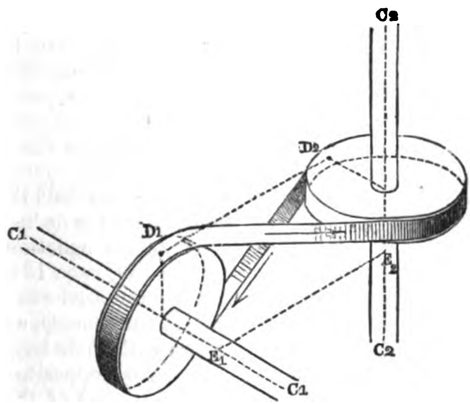

Fig. 125.

perpendicular. In order that the belt may remain on the pulleys, *the central plane of each pulley must pass through the* point of delivery *of the other* pulley—that is, the point where the belt leaves the other pulley; or, in other words, *the central planes of the two pulleys should intersect in the straight line which connects the two points of delivery.* In fig. 125, D_1 and D_2 are the two points of delivery; and the pulleys are so placed that D_1 D_2 is the line of intersection of their central planes. It is easy to see that this arrangement does not admit of the motion being reversed; for when that takes place, D_1 and D_2 cease to be the points of delivery, and become the points where the belt is received; and it is at once thrown off the pulleys.

171. **Speed Cones** (*A. M.*, 483) are a contrivance for varying and adjusting the velocity-ratio communicated between a pair of parallel shafts by means of a belt, and may be either continuous cones or conoids, as in fig. 126, A, B, whose velocity-ratio can be

varied gradually while they are in motion by shifting the belt; or sets of pulleys whose radii vary by steps, as in fig. 126, C, D—in which case the velocity-ratio can be changed by shifting the belt from one pair of pulleys to another while the machine is at rest.

In order that the belt may be equally tight in every possible position on a pair of speed-cones, the quantity L in the equations of Article 169, pages 183, 184, must be constant.

For a *crossed* belt, as at A and C, L depends solely on the line of centres, c, and on the sum of the radii, $r_1 + r_2$. Now c is constant because the axes are parallel; therefore the *sum of the radii* of the pitch-circles connected in every position of the belt is to be constant. That condition is fulfilled by a pair of continuous cones, generated by the revolution of two straight lines inclined opposite ways to their respective axes at equal angles, and by a set of pairs of pulleys in which the sum of the radii is the same for each pair.

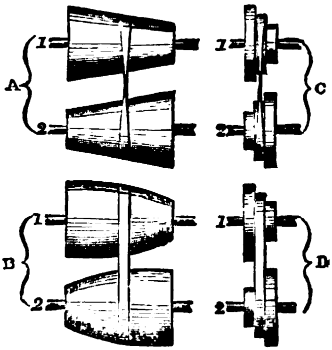

Fig. 126.

For an *open* belt the following practical rule is deduced from the approximate equation (3 A.) of Article 169, page 184:—

Let the speed-cones be equal and similar conoids, as in fig. 126, B, but with their large and small ends turned opposite ways. Let r_1 be the radius of the large end of each, r_2 that of the small end, r_0 that of the middle; and let y be the *swell* or *convexity*, measured perpendicular to the axis, of the arc by whose revolution each of the conoids is generated; then

$$y = \frac{(r_1 - r_2)^2}{2 \pi c}; \qquad (1.)$$

and

$$r_0 = \frac{r_1 + r_2}{2} + y; \qquad (2.)$$

$\pi = 3\frac{1}{7}$ nearly enough for the present purpose.

To find the swell, y, by graphic construction: in fig. 126 E, draw A B = $3\frac{1}{7}$ times the line of centres; from B, perpendicular to A B, draw B C = the difference between the greatest and least radii; join A C, and cut off from it A D = A B; D C will be the required swell.

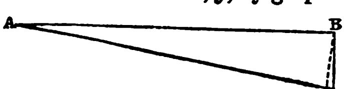

Fig. 126 E.

The radii at the middle and ends being thus determined, make the generating curve an arc either of a circle or of a parabola.

For a pair of stepped cones, as in fig. 126 D, let a series of *differences* of the radii, or values of $r_1 - r_2$, be assumed; then, for each pair of pulleys, the half-sum of the radii is to be computed from the difference by the formula—

$$\frac{r_1 + r_2}{2} = r_0 - y; \qquad \text{(3.)}$$

r_0 being the value of that half-sum when the radii are equal; and finally, the radii are to be computed from their half-sum and half-difference, as follows:—

$$\left. \begin{aligned} r_1 &= \frac{r_1 + r_2}{2} + \frac{r_1 - r_2}{2}; \\ r_2 &= \frac{r_1 + r_2}{2} - \frac{r_1 - r_2}{2} \end{aligned} \right\} \qquad \text{(4.)}$$

172. **Pulleys for Ropes and Cords** require ledges to prevent the band from slipping off; for even flat ropes have not sufficient lateral stiffness to make them remain, of themselves, on the convexity of a pulley. A cord, in passing round a pulley, lies in a groove, sometimes called the *gorge* of the pulley; if the object of the pulley is merely to support, guide, or strain the cord, the gorge may be considerably wider than the cord; if the pulley is to drive or to be driven by the cord, so as to transmit motive power, the gorge must in general fit the cord closely, or even be of a triangular shape, so as to hold it tight. Sometimes the gorge of a pulley which is to be driven by a cord at a low speed has radial ribs on its sides, in order to give it a firmer hold of the cord.

The groove of a pulley for a wire rope should not grasp it tightly, lest the rope be injured; and the motion must be communicated by means of the ordinary friction alone. M. C. F. Hirn has introduced, with good success, the practice of filling the bottoms of the grooves of iron pulleys for wire ropes moving at a high speed with gutta percha, jammed in tight. This will be again referred to in treating of the dynamics of machinery, and of its construction.

When a cord does not merely pass over a pulley, but is made fast to it at one end, and wound upon it, the pulley usually becomes what is called a *drum* or a *barrel*. A drum for a round rope is cylindrical, and the rope is wound upon it in helical coils. Each layer of coils increases the effective radius of the drum by an amount equal to the diameter of the rope. A drum for a flat rope is of a breadth simply equal to the breadth of the rope, which is wound upon it in single coils, each of which increases the effective radius by an amount equal to the thickness of the rope; and instead of ledges it often has pairs of arms, forming as it were skeleton ledges.

173. **Guide Pulleys.**—A guide pulley merely changes the direction of a band on the way from the pulley which drives the band to the pulley which is driven by it. Guide pulleys are useful chiefly to change the direction of a round cord which communicates motion between two other pulleys whose pitch-circles are not in the same plane. In a case of that kind the following is the rule for finding a proper position for a guide pulley:—By the Rule of Article 27, page 10, find the line of intersection of the planes of the pitch-circles of the driving and following pulley respectively. From any convenient point in that line draw tangents to the proper sides of the two pitch-circles, to represent the centre-lines of two straight parts of the band; then, by the rule of Article 22, page 8, draw the rabatment of the angle which these straight lines make with each other. Let A C B in fig. 127 represent that rabatted angle; draw a straight line, C D, bisecting it; and about any convenient point, D, in that straight line describe a circle touching the two straight lines, C A, C B: this will be the pitch-circle of a suitable guide pulley.

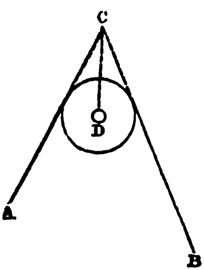

Fig. 127.

174. **Straining Pulleys.**—A straining pulley is used to bring a band to the degree of tension which is necessary in order to enable it to transmit motion from a driving pulley to a following pulley. A straining pulley, as applied to a flat belt, is usually pressed, by means of a lever, against one of the parts of the belt which extends between the driving and following pulleys, so as to push that part of the belt towards the line of centres. The effect of this is to tighten the belt and increase the friction exerted between it and the pulleys which it connects. This is one of the contrivances used for engaging and disengaging machinery. The straining or tightening pulley is usually applied to the *returning* part of the belt; that is, the part which moves from the driving pulley towards the following pulley.

Sometimes a straining pulley hangs in a loop or bight of a cord, and is loaded with a weight, as in fig. 128, farther on.

175. **Eccentric and Non-Circular Pulleys** are used for transmitting a varying velocity-ratio. For example, in fig. 128 the pitch-line of the pulley A is an eccentric circle, and might be a curve of any figure presenting no concavity; the pitch-line of B is circular and centred on its axis in the figure; but it, too, might be eccentric or non-circular. D E is the line of connection, being the centre-line

of the driving part of the cord, and a tangent to both pitch-lines; and the cord is kept tight by a loaded straining pulley at C. The angular velocities of the pulleys A and B at any given instant are inversely as the perpendicular distances A D and B E of their axes from the line of connection; or in symbols, let a and b be those angular velocities; then

$$\frac{b}{a} = \frac{\text{A D}}{\text{B E}}.$$

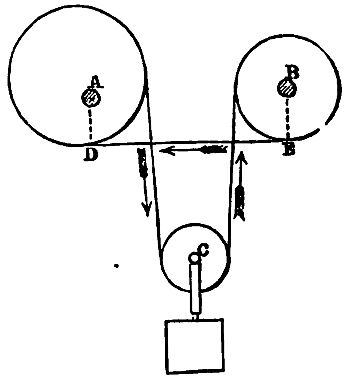

Fig. 128.

There is one instance in which no straining pulley is required; and that is when the pitch-lines of the driving pulley and of the following pulley are a pair of equal and similar ellipses, centred on two of their foci, A, A′, as shown in fig. 129, and connected by means of a crossed cord. The *mean angular velocities* are equal and opposite, each entire revolution being performed in the same

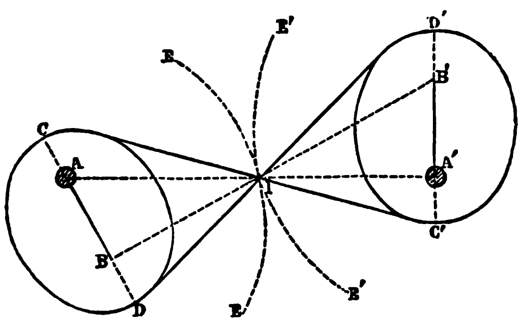

Fig. 129.

time by both pulleys; and the velocity-ratios at different instants are the same as in a combination of a pair of elliptic wheels having the same foci and the same line of centres. In the figure, E I E and E′ I E′ represent the pitch-lines of such a pair of elliptic wheels: the pitch point being always at the intersection, I, of the two straight parts of the cord.

To design such a pair of elliptic pulleys, the data required are the line of centres, A A′, and the angle by which each pulley is alternately to overtake and to fall behind the other pulley. Then, by Rule I. of Article 108, page 95, find the foci; and about those foci draw any ellipse that is not larger than the ellipse suited, according to the same rule, for the pitch-line of a wheel to work in rolling contact; the ellipse so drawn will be suitable for the pitch-lines of both pulleys, C D and C′ D′. The pulleys, like the wheels described in Article 108, will rotate in the same manner as if the revolving foci were connected with each other by a straight link, B B′, equal to the line of centres, A A′; and their corresponding positions and velocity-ratio at any given instant may be found by Rules II. and III. of Article 108, pages 96, 97.

Amongst non-circular pulleys are *fusees*, used in watch-work; in which the pitch-line is a spiral described on a conoidal surface.

Non-circular pulleys may be indefinitely varied in figure without difficulty; for the possibility of keeping the band tight by means of a straining pulley removes the necessity of preserving certain relations between the pitch-lines, as in rolling contact.

176. **Chain Pulleys and Gearing Chains.**—A chain pulley in some cases is merely a circular grooved pulley for guiding a chain: or a cylindrical barrel on which a chain is wound, being made fast at one end to the barrel, as in cranes; and those need no special description. But when a chain is to drive or to be driven by a pulley to which it is not made fast, the acting surface of the pulley must be adapted to the figure of the chain, so as to insure a sufficient hold between them. Amongst chain pulleys of this kind are included *capstans and windlasses*.

The pitch-line of a true chain pulley is a polygon, as exemplified in figs. 130 and 131, in each of which figures the angles of the pitch polygon are marked by black spots, and its sides by dotted lines. Each side of the pitch polygon is equal to what may be called the *pitch*, or *effective length*, of a link of the chain. When the links consist of flat bars of equal length, connected by means of cylindrical pins, as in fig. 130, the pitch of each link is the same, being the distance between the centres of two pins; and the pitch-line accordingly is an equilateral polygon (in the figure a regular hexagon). When the chain consists of oval links, like those of a chain-cable, as in fig. 131, the pitch of a link which lies flatwise on the rim of the pulley is equal to its longer internal diameter *plus* the diameter of the iron, and the pitch of a link which stands edgewise on the rim of the pulley is equal to its longer interval diameter *minus* the diameter of the iron; so that the pitch polygon has long and short sides alternately (in the figure there are twelve sides—six long and six short; and the length of a long side is to that of a short side as 5 to 3). In fig. 130 the pulley is simply a polygonal prism; in fig. 131 it has hollows to fit those links which stand edgewise.

Each of the pulleys shown in these figures has teeth; and the traces of the acting surfaces of the teeth are circular arcs, described about the adjacent angles of the pitch polygons. In fig. 130 the

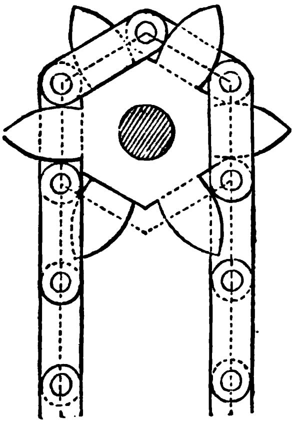

Fig. 130.

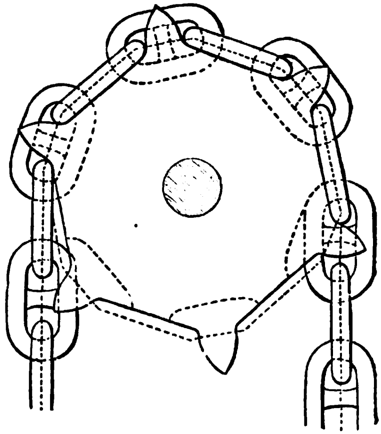

Fig. 131.

chain consists of double and single links alternately; and the sides of the pulley are provided alternately with single teeth and with pairs—a single tooth where each double link lies, and a pair of teeth for each single link to lie between. Sometimes the pulley is provided with single teeth only—one in the middle of each side on which a double link lies. Chains of the shape shown in fig. 130 are made with various numbers of parallel and similar bars in each link, according to the strength required. Of course, the number of bars in a link is even and odd alternately. Such chains are also sometimes made with links of leather, connected together by brass pins, and are used to communicate motion between cylindrical drums. The object of this is to have greater flexibility than is possessed by a flat leather belt. In fig. 131 each short side only of the polygon is provided with a pair of teeth, which receive a link standing edgewise between them, and press against the end of a link that lies flatwise.

Sometimes a chain pulley consists of a number of radiating forks, forming as it were a reel; this is called a *sprocket-wheel.* Sometimes it has a triangular gorge, with radiating ribs on the inner surface of each of the ledges.

177. **Suspended Pulleys.**—When rotation is transmitted, by means of two pairs of pulleys connected by cords, from one shaft through

an intermediate shaft to a third shaft, having its axis in one straight line with the first shaft, the waste of work in overcoming friction may be diminished by supporting the intermediate shaft without bearings: its weight being simply hung by means of the cords from the pulleys on the other two shafts; and care being taken to load the intermediate shaft so as to produce the tension on the cords which is required in order to transmit the motion. What that tension ought to be is a question belonging to the dynamics of machinery. This contrivance appears to have been first introduced by Sir William Thomson. In fig. 132 A is the first and C the third shaft, and B is the intermediate shaft, suspended by means of the cords that pass round its pulleys; D, D are heavy round discs, of the weights required in order to give sufficient tension to the cords. The shaft B, and all the pieces which it carries, should be very accurately balanced.

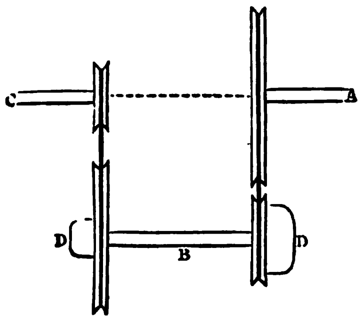

Fig. 132.

Section VI.—*Connection by Linkwork.*

178. **Definitions.** (*A. M.*, 484.)—The pieces which are connected by linkwork, if they rotate or oscillate, are usually called *cranks*, *beams*, and *levers*. The *link* by which they are connected is a rigid bar, which may be straight or of any other figure: the straight figure, being the most favourable to strength, is used when there is no special reason to the contrary. The link is known by various names under various circumstances, such as *coupling-rod*, *connecting-rod*, *crank-rod*, *eccentric-rod*, &c. It is attached to the pieces which it connects by two pins, about which it is free to turn. The effect of the link is to maintain the distance between the centres of those pins invariable; hence the line joining the centres of the pins is *the line of connection;* and those centres may be called the *connected points*. In a turning piece the perpendicular let fall from its connected point upon its axis of rotation is the *arm* or *crank-arm*. If the motions of the pieces are performed parallel to one plane, or about one central point, the pins are almost always cylindrical, with their axes perpendicular to the plane, or traversing the point, as the case may be. In all other cases the acting surfaces of the pins must be portions of spheres described about the connected points—making what are called *ball-and-*

socket joints; unless *universal joints* are used, which will be described further on.

179. **Principles of Connection.** (*A. M.* 485.)—All questions as to the comparative motions of a pair of pieces connected by a link may be solved by means of the general principles and rules given in Article 91, pages 78 to 80, and illustrated by figs. 57 and 58. The axes of rotation of a pair of turning pieces connected by a link are almost always parallel to each other, and perpendicular to the line of connection; in which case the angular velocity-ratio at any instant is the reciprocal of the ratio of the common perpendiculars let fall from the line of connection upon the axes of rotation.

Another method of treating questions of linkwork is to find, by the principles of Article 69, pages 46 to 50, the instantaneous axis of the link; for the two connected points move in the same manner with two points in the link, considered as a rigid body.

If a connected point belongs to a turning piece, the direction of its motion at a given instant is perpendicular to the plane containing the axis and crank-arm of the piece. If a connected point belongs to a shifting piece, the direction of its motion at any instant is given, and a plane can be drawn perpendicular to that direction.

The line of intersection of the planes perpendicular to the paths of the two connected points at a given instant is the *instantaneous axis of the link* at that instant; and the *velocities of the connected points are directly as their distances from that axis.*

In drawing on a plane surface, the two planes perpendicular to the paths of the connected points are represented by two lines (being their traces on a plane normal to them), and the instantaneous axis by a point; and should the length of the two lines render it impracticable to produce them until they actually intersect, the velocity-ratio of the connected points may be found by the principle, that it is equal to the ratio of the segments which a line parallel to the line of connection cuts off from any two lines drawn from a given point perpendicular respectively to the paths of the connected points. Examples will be given further on.

180. **Dead Points.** (*A. M.*, 486.)—If at any instant the plane traversing one of the crank-arms and its axis of rotation coincides with the line of connection, the common perpendicular of the line of connection and the axis of that crank-arm vanishes, and the directional relation of the motions becomes indeterminate. The position of the connected point of the crank arm in question at such an instant is called a *dead point.* The velocity of the other connected point at that instant is null; unless it

also reaches a dead point at the same instant, so that the line of connection is in the plane of the two axes of rotation; in which case the velocity-ratio is indeterminate.

181. **Coupled Parallel Shafts.** (*A.M.*, 487.)—There are only two cases in which an uniform angular velocity-ratio (being that of equality) is communicated by linkwork. One of them is that in which two or more parallel shafts (such as those of the driving wheels of a locomotive engine) are made to rotate with constantly equal angular velocities, by having equal cranks, which are maintained parallel by a coupling rod of such a length that the line of connection is equal to the distance between the axes. The cranks pass their dead points simultaneously. To obviate the unsteadiness of motion which this tends to cause, the shafts are provided with a second set of cranks at right angles to the first, connected by means of a similar coupling rod, so that one set of cranks pass their dead points at the instant when the other set are farthest from theirs. (See fig. 32, page 44.)

This elementary combination belongs to Willis's Class A.

182. **Drag-Link.**—The term *drag-link* is applied to a link, as C D, fig. 133, which connects together two cranks, A C and B D, so as to make them perform a complete revolution in the same time and in the same direction. The cranks may be equal or unequal. If the two axes (whose traces in the figure are A and B) are parts of one straight line (that is, if the points A and B coincide), the angular velocities of the cranks are equal at every instant, and the combination belongs to Willis's Class A; and such is the action of the drag-link when used as a coupling. If the axes are not parts of one straight line (so that A and B are different points), the velocity-ratio varies, and the combination belongs to Class B.

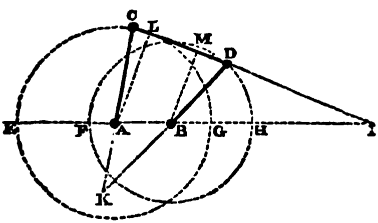

Fig. 133.

In most cases the crank which is the driver goes foremost, and pulls the follower after it; and hence the name of "drag-link."

The following are rules to be observed in determining the dimensions of the parts.

I. In order that the directional relation may be constant, *each of the crank-arms*, A C, B D, *should be longer than the line of centres*, A B.

II. For the same reason, and also in order that there may be no dead-points, *the length of the line of connection*, C D, *should be greater than the lesser segment*, E F, *and less than the greater* *segment*, F G, *into which the diameter*, E G, *of the greater of the two circles described by the connected points is divided by the other circle.* This principle holds also when those circles are equal, and is then applicable to the diameter of either of them. In other words, C D is to be made

Greater than A B + A C − B D,
and less than A C + B D − A B.

The comparative motions may be found by either of the following rules:—

III. *To find the angular velocity-ratio* in a given position of the cranks: on the line of connection, C D, let fall from the axes the perpendiculars, A L, B M; then

$$\frac{\text{Angular velocity of B D}}{\text{Angular velocity of A C}} = \frac{\text{A L}}{\text{B M}};$$

Or otherwise: produce the line of connection, C D, till it cuts the line of centres in I; then

$$\frac{\text{Angular velocity of B D}}{\text{Angular velocity of A C}} = \frac{\text{I A}}{\text{I B}}.$$

When C D is parallel to A B the angular velocities are equal.

IV. *To find the linear velocity-ratio of the connected points:* in a given position of the cranks produce the crank-arms until they intersect; their point of intersection, K, will be the trace of the instantaneous axis of the link; then

$$\frac{\text{Velocity of D}}{\text{Velocity of C}} = \frac{\text{K D}}{\text{K C}}.$$

The limits between which that velocity-ratio fluctuates are $\frac{\text{B D} - \text{A B}}{\text{A C}}$, when B D traverses A, and $\frac{\text{B D}}{\text{A C} - \text{A B}}$, when A C traverses B.

The two shafts, in their rotation, may be regarded as alternately overtaking and falling behind each other by an angle which we may call the *angular displacement.* The complete angular displacement is attained in two opposite directions alternately, at the two instants when the angular velocities of the shafts are equal: that is, when the line of connection is parallel to the line of centres. The following is a rule for designing a *drag-link motion with equal cranks, which shall produce a given angular displacement;* and although not the only rule by which that problem might be solved, it appears to be the simplest in its application.

V. In fig. 134 draw two straight lines, C O *c*, D O *d*, cutting each other at right angles in the point O; lay off along those lines the equal lengths O C = O D. From C and D draw the straight lines C A, D B, making the angles O C A = O D B = *half the given angular displacement*, and cutting O *d* and O *c* respectively in A and B. Join A B and C D. Then A B will represent the line of centres; A C and B D the two crank-arms; and C D the line of connection.

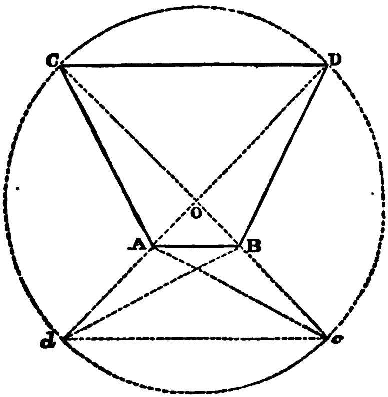

Fig. 184.

The position of the parts represented will be that in which the angle between the crank-arms is least. To show, if required, the position of the parts when that angle is greatest, lay off O *c* and O *d* equal to O C and O D, and join A *c*, B *d*, and *c d*.

183. **Link for Contrary Rotations.**—The only other elementary combination by linkwork which belongs to Willis's Class B is that in which two equal cranks, rotating about parallel axes in contrary directions, are connected by means of a link equal in length to the line of centres. This has been already described in Article 108, page 97, and represented in fig. 72, page 96, as a contrivance to aid the action of elliptic wheels. There are two dead points in each revolution which the pins pass at the instant when the line of connection coincides with the line of centres; consequently the link is not well adapted to act alone, and requires a pair of elliptic wheels, or of elliptic pulleys (Article 175, page 189), to ensure the accurate transmission of the motion.

184. **Linkwork with Reciprocating Motion—Crank and Beam—Crank and Piston-Rod.** (*A. M.*, 488.)—The following are examples of the most frequent cases in practice of linkwork belonging to Willis's Class C, in which the directional relation is reciprocating; and in determining the comparative motion, they are treated by the method of instantaneous axes, already referred to in Article 179, page 193:—

Example I. Two Turning Pieces with Parallel Axes, such as a beam and crank (fig. 135).—Let C_1, C_2, be the parallel axes of the pieces; T_1, T_2, their connected points; C_1 T_1, C_2 T_2, their crank arms; T_1 T_2, the link. At a given instant let v_1 be the velocity of T_1; v_2 that of T_2.

To find the ratio of those velocities, produce C_1 T_1, C_2 T_2, till they intersect in K; K is the instantaneous axis of the link or connecting-rod, and the velocity-ratio is

$$v_1 : v_2 :: K\,T_1 : K\,T_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Should K be inconveniently far off, draw any triangle with its sides respectively parallel to $C_1\,T_1$, $C_2\,T_2$, and $T_1\,T_2$; the ratio of the two sides first mentioned will be the velocity-ratio required. For example, draw C_2 A parallel to $C_1\,T_1$, cutting $T_1\,T_2$ in A; then

$$v_1 : v_2 :: C_2\,A : C_2\,T_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

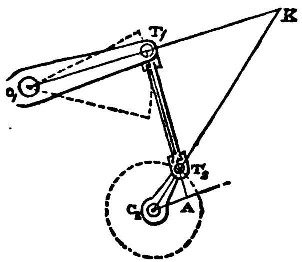

Fig. 135.

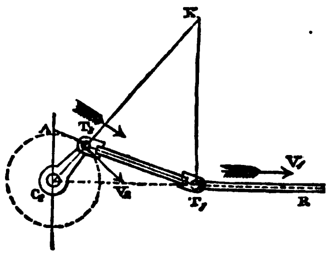

Fig. 136.

Example II. *Rotating Piece and Sliding Piece,* such as a piston-rod and crank (fig. 136).—Let C_2 be the axis of a rotating piece, and T_1 R the straight line along which a sliding piece moves. Let T_1, T_2, be the connected points; $C_2\,T_2$, the crank arm of the rotating piece; and $T_1\,T_2$, the link or connecting rod. The points T_1, T_2, and the line T_1 R, are supposed to be in one plane, perpendicular to the axis C_2. Draw T_1 K perpendicular to T_1 R, intersecting $C_2\,T_2$ in K; K is the instantaneous axis of the link; and the rest of the solution is the same as in Example I.

185. (*A. M.*, 489.) An **Eccentric** (fig. 137) being a circular disc keyed on a shaft, with whose axis its centre does not coincide, and used to give a reciprocating motion to a rod, is equivalent to a crank whose connected point is T, the centre of the eccentric disc, and whose crank arm is C T, the distance of that point from the axis of the shaft, called the *eccentricity.*

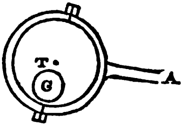

Fig. 137.

An eccentric may be made capable of having its eccentricity altered by means of an adjusting screw, so as to vary the extent of the reciprocating motion which it communicates, and which is called the *throw,* or *travel,* or *length of stroke.*

186. (*A. M.*, 490.) **The Length of Stroke** of a point in a reciprocating piece is the distance between the two ends of the path in

which that point moves. When it is connected by a link with a point in a continuously rotating piece, the ends of the stroke of the reciprocating point correspond with the *dead points* of the continuously rotating piece (Article 180, page 193).

I. *When the crank-arm and the path of the connected point in the reciprocating piece are given, to find the stroke and the dead points.* If the connected point in the reciprocating piece moves in a straight line traversing and perpendicular to the axis of the turning piece, the length of stroke is obviously twice the crank-arm. If that connected point moves in any other path, let F F, in fig. 138, represent that path, A the trace of the crank-axis, and A D = A E the crank-arm. From the point A to the path F F lay off the distances A B = the line of connection − the crank-arm, and A C = the line of connection + the crank-arm; then B C will be the stroke of the connected point in the reciprocating piece. Draw the straight lines C E A and B A D, cutting the circular path of the crank-pin in the points E and D: these will be the dead points.

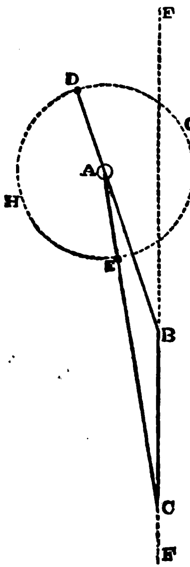

Fig. 138.

II. *When the crank-arm,* A D = A E, *the length of the line of connection, and the dead points,* D *and* E, *are given, to find the two ends of the stroke of the connected point in the reciprocating piece.* In D A and A E produced, make D B and E C each equal to the length of the line of connection; B and C will be the required ends of the stroke.

When the path of the connected point in the reciprocating piece is a straight line, the preceding principles may be thus expressed in algebraical symbols:—

Let S be the length of stroke, L the length of the line of connection, and R the crank-arm. Then, if the two ends of the stroke are in one straight line with the axis of the crank,

$$S = 2\,R; \quad \text{.........................(1.)}$$

and if their ends are not in one straight line with that axis, then S, L − R, and L + R, are the three sides of a triangle, having the angle opposite S at that axis; so that if θ be the supplement of the arc between the dead points,

$$\left.\begin{aligned} S^2 &= 2\,(L^2 + R^2) - 2\,(L^2 - R^2)\cos\theta; \\ \cos\theta &= \frac{2\,L^2 + 2\,R^2 - S^2}{2\,(L^2 - R^2)}. \end{aligned}\right\} \quad \text{........(2.)}$$

187. **Mean Velocity Ratio.**—In dynamical questions respecting machines, especially when the mode of connection is by linkwork, it is often requisite to determine the *mean ratio* of the linear velocities of a pair of connected points during some definite period; which mean ratio is simply the ratio of the distances moved through by those points in that period. Three cases may be distinguished, according as the combination of linkwork belongs to Willis's Class A, Class B, or Class C.

In Class A the mean velocity-ratio is identical with the velocity-ratio at each instant. For examples, see Article 181, page 194, and Article 182, page 194.

In Class B the mean velocity-ratio of the connected points during each complete revolution is that of the circumferences of the circles in which they move. For examples, see Article 182, page 194, and Article 183, page 196.

In Class C the mean velocity-ratio of the connected points may be taken either for a whole revolution of the revolving point and double stroke of the reciprocating point, or it may be taken separately for the forward stroke and return stroke of the reciprocating point, where it has different values for these two parts of the motion. In the former case it is expressed by the ratio of twice the length of stroke of the reciprocating point to the circumference of the circle described by the revolving point; that is to say, for example, in fig. 138, page 198, by the ratio

$$\frac{2\,\mathrm{B\,C}}{\text{Circumference } \mathrm{D\,G\,E\,H}}.$$

In the latter case, the two mean velocity-ratios are expressed by the proportions borne by the length of stroke of the reciprocating point, to the two arcs into which the dead points divide the path of the revolving point. For example, in fig. 138, those two ratios are respectively—

$$\frac{\mathrm{B\,C}}{\text{Arc } \mathrm{D\,G\,E}}, \text{ and } \frac{\mathrm{B\,C}}{\text{Arc } \mathrm{E\,H\,D}}.$$

The most frequent case in practice is that represented in fig. 136, page 197, where the reciprocating point moves in a straight line traversing the axis about which the revolving point moves; and in that case the mean velocity-ratio for each single stroke and for a whole revolution is

$$\frac{2}{\pi} = 0{\cdot}63662 \text{ nearly.}$$

188. **Extreme Velocity-Ratios.**—In those cases in which one of the points connected by a link revolves continuously, while the other has a reciprocating motion, it is often desirable to determine the *greatest* value of the ratio borne by the velocity of the recipro-

cating point to that of the revolving point. The general principle upon which that greatest ratio depends is shown in fig. 139, in which T′ represents the reciprocating point, and T the revolving point; T T′, the line of connection; and C T, the crank-arm. Let C A be perpendicular to the direction of motion of the reciprocating point T′, and let A be the point where the line of connection cuts C A; then, as has been already shown in Article 184, page, 196,

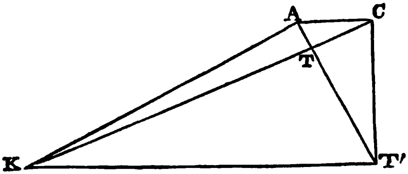

Fig. 139.

$$\frac{\text{Velocity of T}'}{\text{Velocity of T}} = \frac{\text{C A}}{\text{C T}};$$

and at the instant when that ratio is greatest, A is at its greatest distance from C; therefore, at that instant the direction of motion of the point A in the line of connection is along that line itself. Draw T′ K parallel to C A, produce C T till it cuts T′ K in K, the instantaneous centre of motion of the link, and join K A; then the direction of motion of the point A in the line of connection at any instant is perpendicular to A K; and therefore, at the instant when C A is greatest, A K is perpendicular to A T′. Upon this proposition depends the determination of the greatest value of the ratio $\frac{\text{C A}}{\text{C T}}$; but that determination cannot be completed by geometry alone; for it requires the solution of a cubic equation, as stated in the footnote.*

* In fig. 139, let the crank-arm C T $= a$; let the line of connection T T′ $= b$; these two quantities being given; and when the ratio of the velocity of T′ to that of T is a maximum, let the angle C T′ T $= \theta$, and the angle A C T $= \phi$.

Solve the following cubic equation:—

$$\sin^6\theta - \sin^4\theta - \sin^2\theta + \frac{a^2}{b^2} = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

so as to determine the value of $\sin^2\theta$, which is the only root of that equation that is positive and less than 1. Next, calculate the value of the angle ϕ, or those of its trigonometrical functions, by the help of one or more of the following equations (each of which implies the others):—

$$\left.\begin{array}{l} \tan\phi = \cos\theta\sin\theta = \frac{\sin 2\theta}{2}; \\ \sin\phi = \sqrt{\left\{\frac{\sin^2\theta - \sin^4\theta}{1 + \sin^2\theta - \sin^4\theta};\right\}} \\ \cos\phi = \frac{1}{\sqrt{(1 + \sin^2\theta - \sin^4\theta)}}; \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

An *approximate solution* of this question may, however, be obtained by plane geometry, when the line of connection, T T', is not less than about twice the crank-arm, C T. It consists in treating the angle at T as if it were a right angle (from which it differs by the angle A K T); and thus we obtain

$$\frac{C\,A}{C\,T} = \text{(nearly)}\ \frac{C\,T'}{T\,T'},\ = \frac{\sqrt{(C\,T^2 + T\,T'^2)}}{T\,T'}.$$

When T T' is great as compared with C T, the error of this solution is inappreciable, or nearly so; when T T' = 2 C T, the approximate solution is too small by about one per cent., and is therefore near enough for practical purposes; when T T' becomes less than 2 C T, the error rapidly increases, so as to make the approximate solution inapplicable; but cases of this last kind are very uncommon in practice.

189. **Doubling of Oscillations by Linkwork.**—When two reciprocating pieces are connected by means of a link, the follower may be made to perform two oscillations or strokes for one of the driver, in the following manner:—In fig. 140, let the driver be an arm or lever, A B; A its axis of motion, and B its connected point.

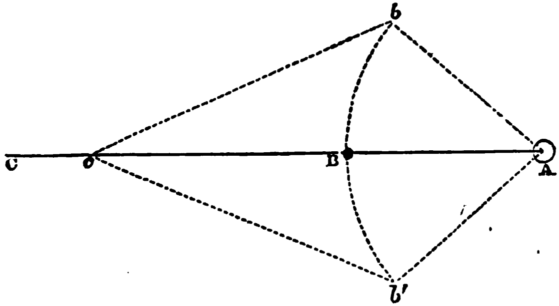

Fig. 140.

Let C be the connected point of the follower, and B C the link. Then the parts of the combination are to be so arranged that the straight line C c, which traverses the two ends of the stroke of the point C, shall traverse also the axis A, and shall bisect the arc of

and finally, calculate the required greatest velocity-ratio by the following formula:—

$$\frac{C\,A}{C\,T} = \frac{\cos(\theta - \phi)}{\cos\theta} \quad \text{.............................(3.)}$$

In the two extreme cases the values of that ratio are as follows:—When b is immeasurably longer than a, C A ÷ C T sensibly = 1; when $b = a$, C A ÷ C T = 2.

motion, b B b', of the connected point B. The result will be, that while the point B performs a single stroke, from b to b', the point C will perform a double stroke, from c to C and back again.

If C is a point in a second lever, that second lever may, by means of a similar arrangement, be made to drive a third lever, so as again to double the frequency of the strokes; and thus, by a train of linkwork, the last follower may have the frequency of its strokes increased, as compared with those of the first driver, in a ratio expressed by any required power of 2.

190. **Slow Motion by Linkwork.**—As has been already explained in Article 180, page 193, when the connected point in the driver of an elementary combination by linkwork is at a dead point, the velocity of the connected point of the follower is nothing; and when the connected point of the driver is near a dead point, the motion of the connected point of the follower is comparatively very slow, and gradually increases as the connected point of the driver moves away from the dead point. When, therefore, it is desired that the motion of a follower shall, at and near a particular position of the combination, be very slow as compared with that of the driver, or as compared with that of the follower itself when in other positions, arrangements may be used of the class which is exemplified in fig. 141 and fig. 141 A.

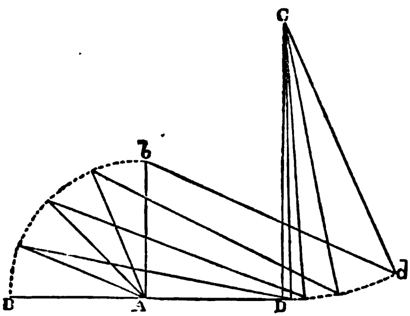

Fig. 141.

In fig. 141 the lever A B, turning about an axis at A, drives, by means of the link B D, the lever C D, which turns about an axis at C. When the driving lever is in the position marked A B, it is in one straight line with the link B D; so that B is a dead point, and the velocity of the follower is null. As the connected point of the driver advances from B towards b, the connected point of the follower advances from D towards d, with a comparative velocity which is at first very small, and goes on increasing by degrees. When the motion is reversed, the comparative velocity of the latter point gradually diminishes as it returns from d towards D, and finally vanishes at the last-named point. Motions of this kind are useful in the opening and closing of steam-valves, in order to prevent shocks.

Fig. 141 A shows a train of two elementary combinations of the same kind with that just described; the effect being to make the motion of a third connected point, E, quite insensible during a certain part of the motion of the first connected point, B.

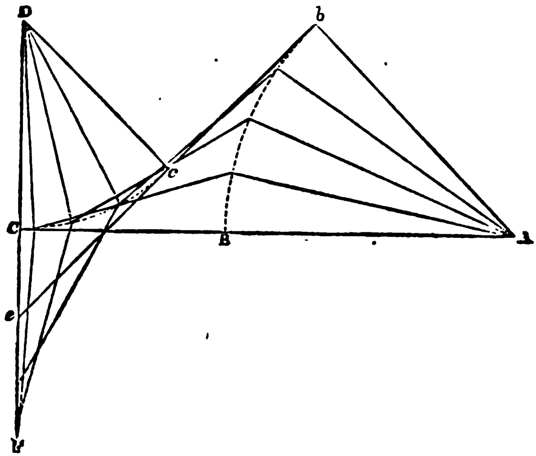

Fig. 141 A.

191. (*A. M.*, 491.) **Hooke's Coupling, or Universal Joint** (fig. 142), is a contrivance for coupling shafts whose axes intersect each other in a point.

Let O be the point of intersection of the axes O C_1, O C_2, and i their angle of inclination to each other. The pair of shafts C_1, C_2, terminate in a pair of forks, F_1, F_2, in bearings at the extremities of which turn the pivots at the ends of the arms of a rectangular cross having its centre at O. This cross is the link; the connected points are the centres of the bearings F_1, F_2. At each instant each of those points moves at right angles to the central plane of its shaft and fork; therefore the line of intersection of the central planes of the two forks, at any instant, is the instantaneous axis of the cross; and the *velocity-ratio* of the points F_1, F_2 (which, as the forks are equal, is also the *angular velocity-ratio* of the shafts), is equal to the ratio of the distances of those points from that instantaneous axis. The *mean* value of that velocity-ratio is that of equality; for each successive *quarter turn* is made by both shafts in the same time; but its instantaneous value fluctuates between the limits,

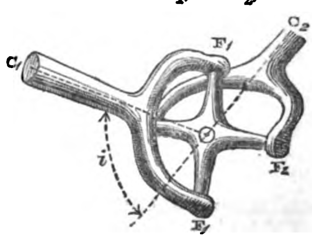

Fig. 142.

$$\left.\begin{array}{l}\dfrac{a_2}{a_1} = \dfrac{1}{\cos i},\ \text{when } F_1 \text{ is in the plane of the axes;}\\[2ex] \dfrac{a_2}{a_1} = \cos i \text{ when } F_2 \text{ is in that plane.}\end{array}\right\} \ldots (1.)$$

The following is the geometrical construction for finding the position of one of the shafts which corresponds to any given position of the other; also the velocity-ratio corresponding to that position:—Let the shaft whose position is given be called the *first* shaft, and the other the *second* shaft; and let the corresponding arms of the cross be called the first and second arms respectively.

In fig. 143, let O be the point of intersection of the axes of the two shafts, and let the plane of projection be a plane traversing O,

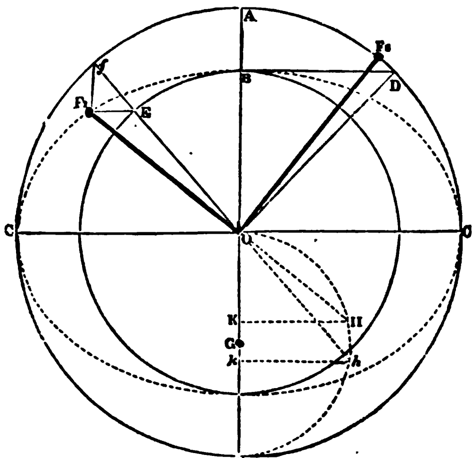

Fig. 143.

and normal to the axis of the *second* shaft. Let A O *a* be the trace of the plane of the two axes, and C O C, perpendicular to A O *a*, a normal to that plane. With any convenient radius, O A, describe a circle about O. Lay off the angle A O D equal to the angle *i*, which the axes of the shafts make with each other. Through D, parallel to C C, draw D B, cutting O A in B; then $\frac{\text{O B}}{\text{O A}} = \cos i$ is the velocity-ratio of the second to the first axis, when the first arm coincides with O C and the second with O A; and $\frac{\text{O A}}{\text{O B}} = \frac{1}{\cos i} =$ sec *i* is the velocity-ratio, when the first arm coincides with O A, and the second with O C.

About O, with the radius O B, describe a circle. Draw the radius O E *f*, cutting the two circles in E and *f* respectively, and making the angle A O *f* = the given angle which the *first arm* makes with the plane of the axes:—in other words, let O *f* be the *rabatment* of the first arm, made by rabatting a plane normal to the first axis upon the plane of projection. Through E, parallel to O C, draw E F_1; and through *f*, parallel to O A, draw *f* F_1; the point F_1 will be the projection of the point whose rabatment is *f*. Draw the straight line O F_1; this will be the *projection of the first arm* on a plane normal to the second axis. Then perpendicular to O F_1 draw O F_2; this will be *the required position of the second arm*.

The projection of the path of the point F_1 is the ellipse C B C.

To find the angular velocity-ratio corresponding to the given position of the arms; about any convenient point, G, in A O *a*, describe a circle through O, cutting F_1 O and *f* O (produced if required) in H and *h* respectively; from which points draw H K and *h k* parallel to O C, and cutting A O *a* in K and *k* respectively. Then we have

$$\frac{a_2}{a_1} = \frac{\mathrm{K\,H}}{k\,h}.\text{*} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

The particular form of universal joint shown in fig. 142 is chosen in order to exhibit all the parts distinctly. In practice, the joint is often made much more compact, the forks not having more space between them and the cross than is necessary in order to admit of the required extent of motion of the cross-arms, and the cross being sometimes made in the form of a circular disc, or of a ring, or of a ball, with four pivots projecting from its circumference. Where the angle of obliquity of the two shafts (i) is small, each of the forks is often made in the form of a round disc on the end of the shaft, having a pair of projecting horns or lugs to carry the bearings of the pivots.

The universal joint belongs to Willis's Class B. When used as a coupling, it is liable to the objection, that although the mean velocity-ratio is uniform, being that of equality, the velocity-ratio at each instant fluctuates, and thus gives rise to vibratory and unsteady motion.

192. (*A. M.*, 492.) The **Double Hooke's Joint** (fig. 144) is used to obviate the vibratory and unsteady motion caused by the fluctuation

* In algebraical symbols, let ϕ_1 = A O *f*, and ϕ_2 = A O F_2, be the angles made by the first and second arm respectively at a given instant with the plane of the axes of the shafts; then

$$\tan\phi_1 \cdot \tan\phi_2 = \cos i; \text{ and}$$
$$\frac{a_2}{a_1} = -\frac{d\,\phi_2}{d\,\phi_1} = \frac{\sin 2\,\phi_2}{\sin 2\,\phi_1} = \frac{\tan\phi_1 + \mathrm{cotan}\,\phi_1}{\tan\phi_2 + \mathrm{cotan}\,\phi_2}.$$

of the velocity-ratio which has already been mentioned. Between the two shafts to be connected, C_1, C_3, there is introduced a short intermediate shaft, C_2, making equal angles with C_1 and C_3, connected with each of them by a Hooke's joint, and having both its own forks in the same plane. The effect of this combination is, that the angular velocities of the *first* and *third* shafts are equal to each other at every instant; and that the planes of the first and third forks make, at every instant, equal angles with the plane of the three axes. Hence, as regards the comparative motion of the first and third shafts, the double Hooke's joint belongs to Class A; but as regards the motion of the second or intermediate shaft, it belongs to Class B.*

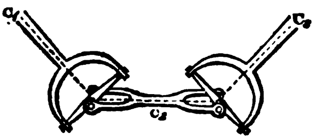

Fig. 144.

The double Hooke's joint works correctly when the third shaft is *parallel* to the first, as well as in the position shown in the figure.

193. **Hooke-and-Oldham Coupling.**—This name may be given to an universal joint in which the pivots of the cross are capable of sliding lengthwise as well as of turning in their bearings in the horns of the forks. It combines the properties of Hooke's coupling with that of Oldham's coupling, formerly described (Article 158, page 166); that is to say, it is capable of transmitting motion between shafts whose axes are neither parallel nor intersecting. It acts by sliding contact and linkwork combined: when single, it belongs to Class B; and when double, with the axes of the three shafts in parallel planes, and the first and third making equal angles with the intermediate axis, to Class A.

194. **Intermittent Linkwork—Click and Ratchet.**—A *click* or *catch*, being a reciprocating bar (such as B C in figs. 145 and 146) acting upon a ratchet wheel or rack, which it pushes or pulls through a certain arc at each forward stroke, and leaves at rest at each backward stroke, is an example of intermittent linkwork. During the forward stroke, the action of the click is governed by the principles of linkwork; during the backward stroke, that action ceases. A *fixed catch*, or *pall*, or *detent* (such as *b c* in fig. 145), turning on a fixed axis, prevents the ratchet wheel or rack from reversing its motion.

* Let i be the angle of inclination of C_1 and C_2, and also that of C_2 and C_3. Let ϕ_1, ϕ_2, ϕ_3, be the angles made at a given instant by the planes of the forks of the three shafts with the plane of their axes, and let a_1, a_2, a_3, be their angular velocities. Then

$$\tan \phi_2 \cdot \tan \phi_3 = \cos i = \tan \phi_1 \cdot \tan \phi_2;$$

whence $$\tan \phi_3 = \tan \phi_1; \text{ and } a_3 = a_1.$$

The *effective stroke*, being the space through which the ratchet is driven by each forward stroke of the click, is necessarily once, or a

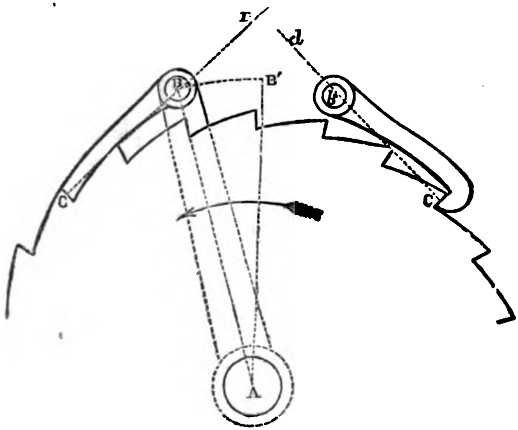

Fig. 145.

whole number of times, the pitch of the teeth of the ratchet; and it is obvious that the length of the total stroke of the click must

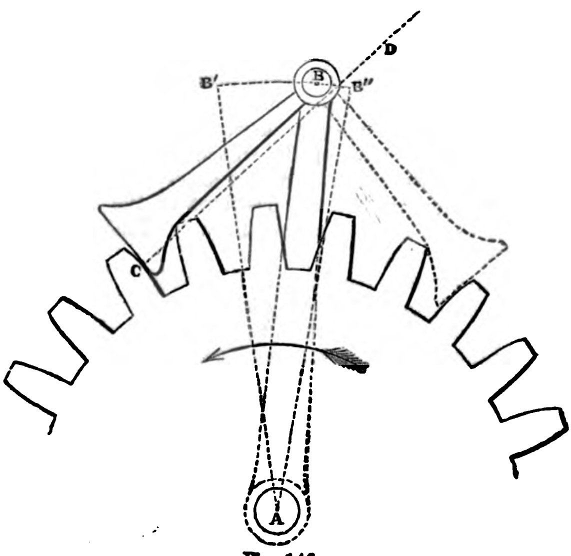

Fig. 146.

be greater than the effective stroke, and less than the next greater whole number of times the pitch. It is advisable, when practicable, to make the excess of the total above the effective stroke no greater than is just sufficient to ensure that the click shall clear each successive tooth of the ratchet. In figs. 145 and 146 the effective stroke is once the pitch of the ratchet; in fig. 147, twice the pitch.

A catch may be made to drop into its place in front of each successive tooth either by gravity or by the pressure of a spring, according to the circumstances of the case.

Some clicks act by thrusting, as B C in fig. 145, and B C in fig. 146; others by pulling, as *b c* in fig. 145.

The direction of the pressure between a click and a tooth is nearly a normal to the acting surfaces of the click and tooth at the centre of their area of contact; for example, in fig. 145, the dotted lines marked C D, *c d*, and in fig. 146, the dotted line marked C D. In order that a click may be certain not to lose its hold of the tooth, that normal *ought to pass inside the axis of motion of a thrusting click, and outside the axis of motion of a pulling click.* For example, in fig. 145, C D passes inside the axis B, and *c d* passes outside the axis *b*; the words "inside" and "outside" being used to denote respectively nearer to and further from the ratchet.

It is convenient, though not essential, that a click for driving a wheel should be carried by an arm concentric with the wheel; such as the arms A B in fig. 145, and A B in fig. 146. In such cases the *total angular stroke* of the click-arm (marked B A B′ in fig. 145, and B′ A B″ in fig. 146) must be a little greater than the effective angular stroke, which is once, or a whole number of times, the pitch-angle of the teeth of the wheel. The axis of motion of the click-arm may, however, be placed elsewhere if necessary, provided care is taken that in all positions of the arm the line of pressure passes to the proper side of the axis of motion of the click. (See figs. 148, 149, further on.)

Fig. 146 represents a *tumbling* or *reversible click*, shaped so as to act upon the teeth of an ordinary toothed wheel. In its present position it drives the wheel in the direction pointed out by the arrow: by throwing it over into the position marked with dotted lines, it is made, when required, to drive the wheel the contrary way.

It is easy to see that the acting surfaces of clicks, and the teeth of ratchets on which they act, may be shaped in a variety of ways besides those exemplified in the figures.

195. **Silent Click.**—This is a contrivance for avoiding the noise and the tear and wear which arise from the sudden dropping of the common click into the space between the teeth of the ratchet-

wheel. The wheel is like an ordinary toothed wheel. B C is the click, which, in the example, is made to push the teeth. It is carried by one branch, A B, of a bell-crank lever, which has a rocking motion about the same axis with the wheel. The other

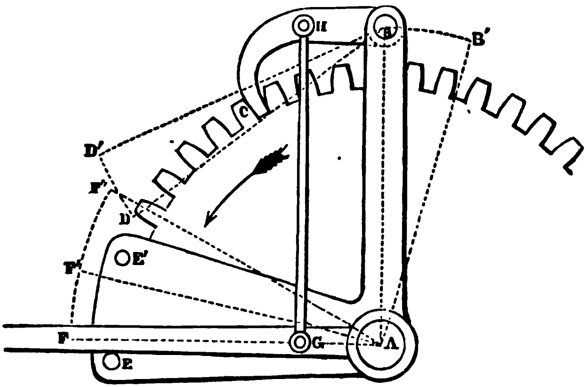

Fig. 147.

branch of the bell-crank lever has two studs or pins in it, E and E′. Between these pins is the driving arm, A F, which has a reciprocating motion about the same axis, and is connected by a link, G H, with the click.

B A B′ is the total angular stroke of the bell-crank lever; D B D′ is the angle through which the click must be moved in order to lift it clear of the teeth. The sum of these angles, B A B′ + D B D′, is = F A F″, the angular stroke of the driving arm. The positions of the studs, E and E′, are so adjusted, that the driving arm in passing from the one to the other moves through the angle F A F′ = D B D′; being the angular motion that lifts the click clear of the teeth before the return stroke, or makes it take hold before the forward stroke. During those parts of the motion of the driving arm and click, the bell-crank lever stands still: its forward and return strokes are made by the driving arm pressing against the studs E and E′ respectively.

196. **Double-Acting Click.**—This is the contrivance sometimes called, from its inventor, "the lever of La Garousse." It consists of two clicks making alternate strokes, so as to produce a nearly continuous motion of the ratchet which they drive; that motion being intermitted for an instant only at each reversal of the direction of movement of the clicks. In fig. 148 the clicks act by pushing; in fig. 149, by pulling. The former arrangement is on the whole the best adapted to cases in which the mechanism

requires considerable strength; such as windlasses on board ship. Each single stroke of the click-arms advances the ratchet through one-half of its pitch.

Corresponding points in the two figures are marked with the same letters; and as fig. 148 contains some parts which do not

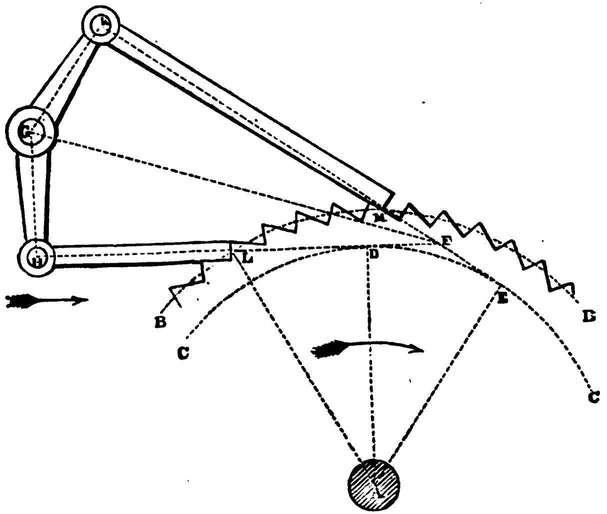

Fig. 148.

occur in fig. 149, the former will, in the first place, be referred to in explaining the principles to be followed in designing such combinations.

Let the figure and dimensions of the ratchet-wheel be given, and let A be its axis, and B B its pitch-circle; that is, a circle midway between the points and roots of the teeth.

Having fixed the mean obliquity of the action of the clicks—that is, the angle which their lines of action, at mid-stroke, are to make with tangents to the pitch-circle—draw any convenient radius of the pitch-circle, as L A, and from it lay off the angle L A D, equal to that obliquity. On A D let fall the perpendicular L D, and with the radius A D describe the circle C C; this will be the *base-circle*, to which the lines of action of the clicks are to be tangents. (As to base-circles, see also Article 131, page 121.) Lay off the angle D A E equal to *an odd number of times half the pitch-angle;* then through the points D and E in the base-circle draw two tangents, cutting each other in F. Draw F G, bisecting the angle at F, and take any convenient point in it, G, for the trace of the axis of motion of the rocking-shaft which carries the click-arms. From G let fall G H and G K perpendicular to the tangents F D H and E F K; then H and K will be the positions of the centres of motion of the two clicks at mid-stroke; and G H and F K will represent the click-arms. Let L and M be the points where D H and E K respectively cut the pitch-circle; then H L and K M will be the lengths of the two clicks. The *effective stroke* of each click will be equal to half the pitch, *as measured on the base-circle* C C; and the total stroke must be as much greater as is necessary in order to make the clicks clear the teeth.

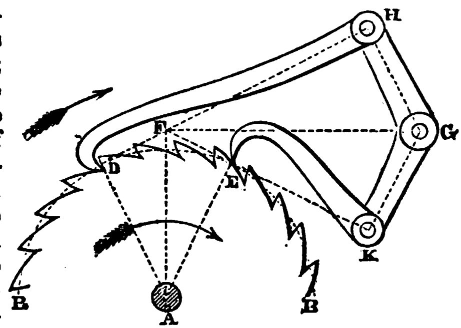

Fig. 149.

In fig. 149, where the clicks pull instead of pushing, the obliquity is nothing; and the consequence is that the base-circle, C C, coincides with the pitch-circle, B B, and that the points L and M coincide respectively with D and E.

197. **Frictional Catch.**—The frictional catch (called sometimes the "silent feed-motion") is a sort of intermittent linkwork, founded on the dynamical principle, that two surfaces will not slide on each other so long as the angle which the direction of the pressure exerted between them makes with their common normal at the place where they touch each other is less than a certain angle called the *angle of repose*, which depends on the nature of the surfaces, and their state of roughness or smoothness, and of lubrication. The smoother and the better lubricated the surfaces, the smaller is the angle of repose.

In trigonometrical language, the angle of repose is the angle whose *tangent is equal to the co-efficient of friction:* that is, to the ratio which the friction between two surfaces, being the force which resists sliding, bears to the normal pressure; or, what is the same thing, it is the angle whose *sine is equal to the ratio that the friction bears to the resultant pressure* when sliding takes place. The subject of friction, and of the angle of repose, properly belong to the dynamical part of this treatise, and will be mentioned in greater detail further on. For the present purpose it is sufficient to state that the sine of the angle of repose for metallic surfaces in a moderately smooth state, and not lubricated, as deduced from the experiments of Morin, ranges from 0·15 to 0·2, or thereabouts; so that an angle whose sine is *one-seventh* of radius may be considered to be less than the angle of repose of any pair of metallic surfaces which are in the above-mentioned condition.

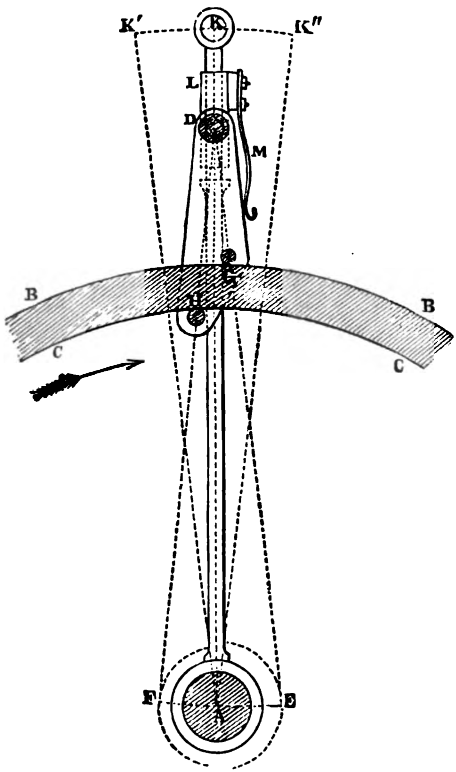

Fig. 150.

The frictional catch, though always depending on the principle just stated, is capable of great variety in detail. The arrangement represented in fig. 150 is constructed in the following manner:—

The shaft and rim of the wheel to be acted upon are shown in section. A K is the catch-arm, having a rocking motion about the axis A of the wheel; the link by which it is driven is supposed to be jointed to it at K; and K′ K″ represents the stroke, or arc of motion, of the point K; so that K′ A K″ is the angular stroke of the catch-arm. L is a socket, capable of sliding longitudinally on the catch-arm to a small extent; a shoulder for limiting the extent of that sliding motion is marked by dotted lines. The socket and the part of the arm on which it slides should be square, and not round, to prevent the socket from turning. From the side of the socket there projects a pin at D, from which the catch D G H hangs. M is a spring, pressing against the forward side of the catch. G and H are two studs on the catch, which grip and carry forward the rim, B B C C, of the wheel during the forward stroke, by means of friction, but let it go during the return stroke.

A similar frictional catch, not shown in the figure, hanging from a socket on a fixed instead of a moveable arm, at any convenient part of the rim of the wheel, serves for a detent, to hold the wheel still during the return stroke of the moveable catch-arm.

The following is the graphic construction for determining the proper position of the studs G and H:—Multiply the radii of the outer and inner surfaces, B B and C C, of the rim of the wheel by a co-efficient a little less than the sine of the angle of repose—say $\frac{1}{7}$—and with the lengths so found as radii describe two circular arcs about A; the greater (marked E) lying in the direction of forward motion, and the less (marked F) in the contrary direction. From D, the centre of the pin, draw D E and D F, touching those two arcs. Then G, where D E cuts B B, and H, where D F cuts C C, will be the proper positions for the points of contact of the two studs with the rim of the wheel. For the force by which the catch is driven during the forward stroke acts through D; that force is resolved into two components, acting along the lines D G E and F H D respectively; and those lines make with the normals to the rim of the wheel, at G and H respectively, angles less than the angle of repose of a pair of metallic surfaces that are not lubricated. Should it be thought desirable, the positions of the holding studs, or of one of them, may be made adjustable by means of screws or otherwise.

The stiffness of the spring M ought to be sufficient to bring the catch quickly into the holding position at the end of each return stroke.

The length of stroke of a frictional catch is arbitrary, and may, by suitable contrivances, be altered during the motion. Contrivances for that purpose will be described further on.

A pair of frictional catches may be made double-acting, like the double-acting clicks of the preceding Article.

198. **Slotted Link.**—A slotted link is connected with a pin at one of its ends, not by a round hole fitting the pin closely, but by an oblong opening or slot with semicircular ends. This is an example of intermittent linkwork; the intermission in its action taking place during the middle part of each stroke, while the pin is shifting its position relatively to the link from the one end of the slot to the other. That intermission takes effect by producing a pause in the motion of that piece which is the follower, and which may be either the link or the pin; and the stroke of the follower is shorter than that of the driver by an extent corresponding to the length of the slot, as measured from centre to centre of its two semicircular ends.

199. **Band Links.**—Where tension alone, and not thrust, is to act along a link, it may be flexible, and may consist either

of a single band, or of an endless band passing round a pair of pulleys which turn round axes traversing and moving with the connected points. For example, in fig. 151, A is the axis of a rotating shaft, B that of a crank-pin, C the other connected point, and B C the line of connection; and the connection is effected by means of an endless band, passing round a pulley which is centred upon C, and round the crank-pin itself, which acts as another pulley. The pulleys are of course secondary pieces; and the motion of each of them belongs to the subject of aggregate combinations, being compounded of the motion which they have along with the line of connection, B C, and of their respective rotations relatively to that line as their line of centres; but the motion of the points B and C is the same as if B C were a rigid link, provided that forces act which keep the band always in a state of tension.

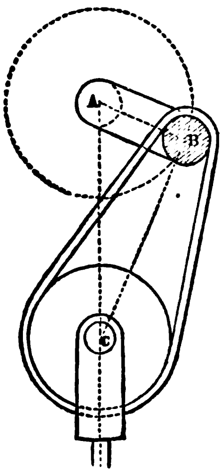

Fig. 151

This combination is used in order to lessen the friction, as compared with that which takes place between a rigid link and a pair of pins; and the band employed is often a leather chain, of the kind already mentioned in Article 176, page 191, because of its flexibility.

Section VII.—*Connection by Plies of Cord, or by Reduplication.*

200. **General Explanations.** (*A. M.*, 494.)—The combination of pieces connected by the several plies of a cord, rope, or chain, consists of a pair of cases or frames called *blocks*, each containing one or more pulleys called *sheaves*. One of the blocks (A, figs. 152, 153), called the *fixed block*, or *fall-block*, is fixed; the other, called the *fly-block*, or *running block*, B, is moveable to or from the fall-block, with which it is connected by means of a rope, or *fall*, of which one end is fastened either to a fixed point or to the running block, while the other end, C, called the *hauling part*, is free; and the intermediate portion of the rope passes alternately round the pulleys in the fixed block and running block. The several plies of the rope are called by seamen *parts;* and the part which has its end fastened is called the *standing part*. The whole combination is called a *tackle* or *purchase*. When the hauling part is the driver, and the running block the follower, the two blocks are being drawn

together; when the running block is the driver, and the hauling part the follower, the two blocks are being pulled apart.

201. **Velocity-Ratios.** (*A. M.*, 495, 496.)—The *velocity-ratio* chiefly considered in a purchase is that between the velocities of the running block, B, and of the hauling part, C. That ratio is expressed by the *number of plies* of rope by which the running

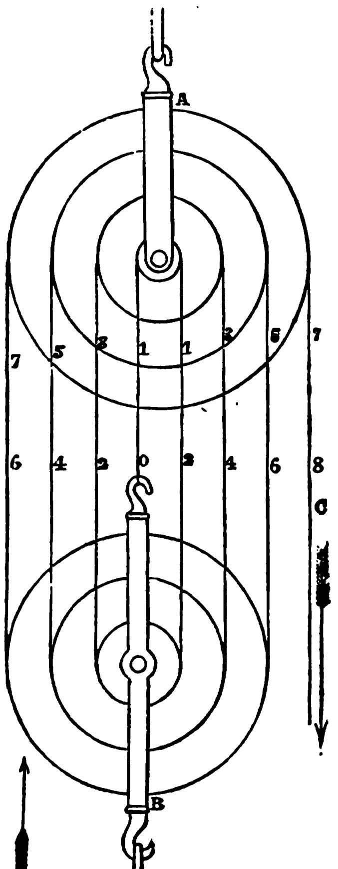

Fig. 152.

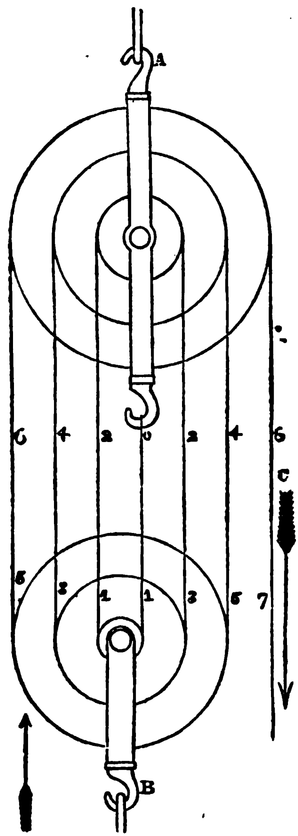

Fig. 153.

block is connected with the fall-block. Thus, in fig. 152, $C \div B = 7$; and in fig. 153, $C \div B = 6$. A tackle is called a *twofold* *purchase*, a *threefold purchase*, and so on, according to the value of the velocity-ratio C ÷ B. For example, fig. 152 is a sevenfold purchase, and fig. 153 a sixfold purchase.

The *velocity of any ply or part* of the rope is found in the following manner:—For a ply on the side of the fall-block, A, next the hauling-part, C, it is to be considered what would be the velocity of that ply if it were itself the hauling part: that is to say, the ratio of its velocity to that of the running block is expressed by the number of plies *between* the ply in question and the point of attachment of the standing part. For a ply on the side of the fall-block furthest from the hauling part, the velocity is equal and contrary to that of the next succeeding ply, with which it is directly connected over one of the sheaves of the fall-block. If the standing part is attached to a fixed point, as in fig. 153, its velocity is nothing; if to the running block, as in fig. 152, its velocity is equal to that of the block. The comparative velocities of the several parts of the ropes are expressed by the upper row of figures. The lower row of figures express the velocities of the several parts relatively to the running block.

202. **Ordinary Form of Pulley-Blocks.**—A block, as used on board ship, consists of an oval *shell*, usually of elm or metal, containing one or more pulleys, called *sheaves*, of lignum-vitæ or metal, turning about a cylindrical wrought-iron *pin*. The round hole in the centre of a wooden sheave is lined with a gun-metal tube called the *bushing*. The part of the sheave-hole through which the rope or chain reeves is called the *swallow*. In the bottom and sides of a block is a groove called the *score*, into which fits the *strop* or *strapping* of rope or iron by which the block is hung or secured to its place. Ordinary blocks containing one pin are called *single*, *double*, *treble*, &c., according to the number of sheaves that turn about that pin side by side. Each sheave turns in a separate hole in the shell. Fig. 154 shows examples of the forms of iron pulley-blocks commonly used in machinery on land. A is a treble block; B, a double block. The block B has an eye for the attachment of the standing part of the rope.

Fig. 154.

203. **White's Pulleys.**—When the sheaves of a block, as in the ordinary form, are all of the same diameter, they all turn with different angular velocities, because of the different velocities of the plies of rope that pass over them. But by making the effective radius of each sheave proportional to the velocity, *relatively to the block*, of the ply of rope which it is to carry, the angular velocities of the sheaves in one block may be rendered equal; so that the

sheaves may be made all in one piece, having two journals which turn in fixed bearings.

These are called "White's Pulleys," from the inventor; and they are represented in figs. 152 and 153, page 215: having been chosen to illustrate the general principles of the action of blocks and tackle, because of the clearness with which they show the positions of all the parts of the rope. They are not, however, much used in practice, because the unequal stretching of different parts of the cord prevents the combination from working with that degree of accuracy which is necessary in order that any advantage may be obtained by means of it over the common construction.

204. **Compound Purchases.**—A compound purchase consists of a *train* of simple purchases; that is to say, the hauling part of one tackle is secured to the running block of another, and so on, for any number of tackles. In practice, however, the number of tackles in a compound purchase is almost always two; and then the rope that has the running block secured to it is usually called the *pendant*, and the rope that is directly hauled upon by hand, the *fall*.

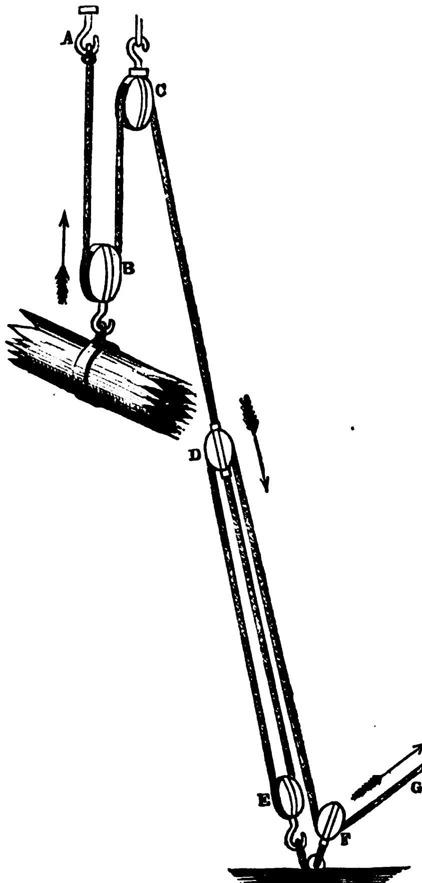

Fig. 155.

The velocity-ratio is, as in other trains of elementary combinations, the product of the velocity-ratios belonging to the elementary or simple tackles of which the compound purchase consists.

For example, in fig. 155, A B C is a twofold purchase; and at D, its pendant is secured to the fly-block of a threefold purchase, D E F, whose hauling part is F G. The velocity-ratio of D to B is 2, and that of G to D is 3; so that the velocity-ratio of G to B is 2 × 3 = 6; and the compound purchase is sixfold.

205. **Rope and Space Required for a Purchase.**—An elementary or simple purchase requires no more space to work in than the greatest distance from outside to outside of the fixed and running blocks. The least length of rope sufficient for it may be found as follows:—To the greatest distance between the centres of the blocks add half the *effective* circumference of a sheave (see Article 166, page 180); multiply the sum by the number of plies of rope which connect the blocks with each other; and to the product add the least length of the hauling part required under the circumstances of the particular case.

A compound purchase requires a length of space to work in equal to the whole distance traversed by the fly-block of the last purchase in the train (viz., that whose hauling part is free), with a sufficient additional length added for the blocks and their fastenings.

206. **Obliquely-acting Tackle.**—The parts of the rope of a tackle, instead of being parallel to each other and to the direction of motion of the running block, may make various angles with that direction. For example, in fig. 156, B is the running block, and B *b*

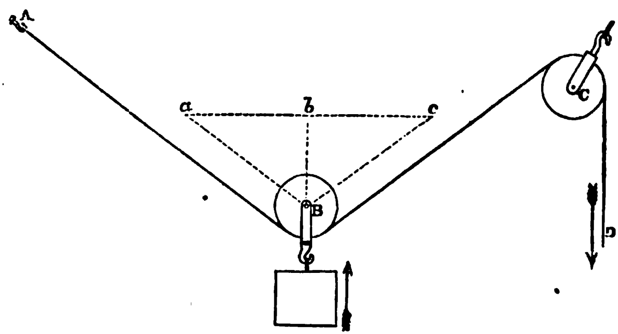

Fig. 156.

its line of motion; and in the case represented, that block hangs from two parts of a rope—the standing part, B A, and another part, B C. To find the velocity-ratio of the hauling part, D, to the running block, B: from the centre, B, of that block, draw straight lines, B *a*, B *c*, parallel to the parts of the rope by which it hangs; at any convenient distance from B, draw the straight line *a b c* perpendicular to B b, and cutting all the straight lines which diverge from B; then,

as B b : is to B a + B c,
: : so is the velocity of B
: to the velocity of D;

and the same rule may be extended to any number of parts, thus:

$$\frac{\text{velocity of D}}{\text{velocity of B}} = \frac{\text{sum of lengths cut off on lines diverging from B}}{\text{B}\,b}.$$

The combination belongs to Class B; because, owing to the continual variation of the obliquity of the parts of the rope, the velocity-ratio is continually changing.

206 A. **Tiller-Ropes.**—The *tiller* of a ship is a horizontal lever projecting from the rudder-head, by means of which the position of

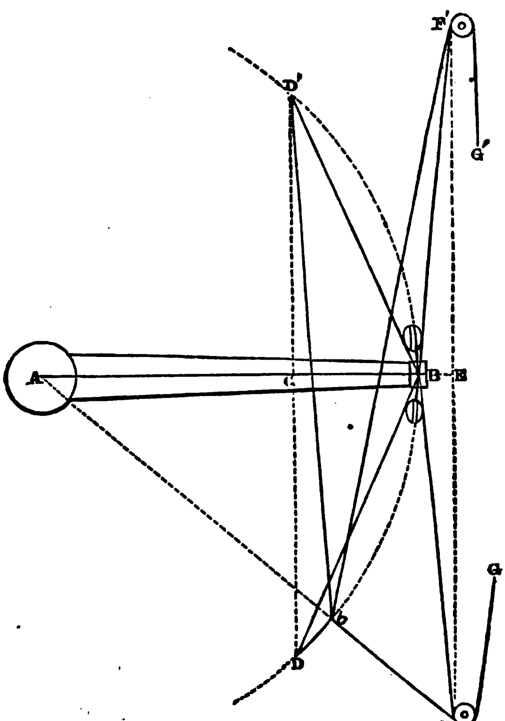

Fig. 156 A.

the rudder is adjusted. It usually points forward; that is, in the contrary direction to the rudder itself. In ships of war the tiller is usually *put over*, or moved to one side or to the other, by means of a pair of obliquely-acting twofold tackles, made of raw hide ropes, which haul it respectively *to starboard* (that is, towards the right) and *to port* (that is, towards the left), when required. The hauling parts of both tackles are guided by fixed pulleys so as to be wound in opposite directions round one barrel, which is turned by means of the steering-wheel.*

Fig. 156 A is a plan of this combination. A is the rudder-head; A B, the tiller, shown as amidships, or pointing right ahead; D B F G is the starboard tiller-rope; D′ B F′ G′, the port tiller-rope. These ropes are made fast to eye-bolts at D and D′; at B they are rove through blocks that are secured to the tiller; at F and F′ they are led round fixed pulleys; and G and G′ are their hauling parts, which are led, by means of pulleys which it is unnecessary to show in the figure, to the barrel of the steering-wheel.

A *b* is the position of the tiller when put over about 40° to starboard; and the corresponding positions of the tiller-ropes are D *b* F G and D′ *b* F′ G′.

In order that the tiller-ropes may never become too slack, it is necessary that the sum of the lengths of their several parts should be nearly constant in all positions of the tiller; that is to say, that we should have, in all positions,

$$\mathrm{D}\,b + b\,\mathrm{F} + \mathrm{D'}\,b + b\,\mathrm{F'} \text{ nearly } = 2\,(\mathrm{D\,B} + \mathrm{B\,F}).$$

That object is attained, with a rough approximation sufficient for practical purposes, by adjusting the positions of the points D, D′, and F, F′, according to the following rule:—

RULE.—About A, with the radius A B, describe a circle. Make $\mathrm{A\,C} = \frac{2}{3}\,\mathrm{A\,B}$; and through C, perpendicular to A B, draw a straight line cutting that circle in D and D′. These will be the points at which the standing parts of the ropes are to be made fast. Then produce A B to E, making $\mathrm{B\,E} = \frac{1}{12}\,\mathrm{A\,B}$; and through E, perpendicular to A B E, draw F′ E F, making $\mathrm{E\,F} = \mathrm{E\,F'} = \frac{5}{4}\,\mathrm{C\,D}$; F and F′ will be the stations for the fixed blocks.

When the angle B A *b* is about 40°, the sum of the lengths of the parts of the ropes is a little greater than when the tiller is amidships; but the difference (which is about one-50th part of the length expressed in the preceding equation) is not so great as to

* See Peake's *Rudimentary Treatise on Shipbuilding*, second volume, pp. 66, 162; also Watts, Rankine, Napier, and Barnes *On Shipbuilding*, p. 202.

cause any inconvenient increase of tightness. For angles not exceeding 30° the approximation to uniformity of tightness is extremely close.

SECTION VIII.—*Hydraulic Connection.*

207. **General Nature of the Combinations.**—The kind of combinations to which the present section relates are those in which two cylinders fitted with moveable pistons are connected with each other by a passage, and the space between the pistons is entirely filled with a mass of fluid of invariable volume.

Any liquid mass may be treated, in most practical questions respecting the transmission of motion, as if its volume were invariable, because of the smallness of the change of volume produced in a liquid by any possible change of pressure. For example, in the case of water, the compression produced by an increase in the intensity of the pressure to the extent of one atmosphere (or 14·7 lbs. on the square inch), is only one-20,000th part of the whole volume. (See Article 88, page 75.)

The volume, then, of the mass of fluid enclosed in the space between two pistons being invariable, it follows that if one piston (the driver) moves inwards, sweeping through a given volume, the other piston (the follower) must move outwards, sweeping through an exactly equal volume; otherwise the volume of the space contained between the pistons would change; and this is the principle upon which the comparative motion in hydraulic connection depends.

208. **Cylinders, Pistons, and Plungers.**—A piston is a primary piece, sliding in a vessel called a cylinder. The motion of the piston is most commonly straight; and then the bearing surfaces of the piston and cylinder are actually cylindrical, in the mathematical sense of that word.

When the motion of a piston is circular, the bearing surfaces of the piston, and of the vessel in which it slides, are surfaces of revolution described about the axis of rotation of the piston; but that vessel, in common language, is still called a *cylinder*, although its figure may not be cylindrical.

A *plunger* is distinguished from an ordinary piston in the following way:—The bearing surface of a cylinder for a plunger consists merely of a *collar*, of a depth sufficient to prevent the fluid from escaping; and the plunger slides through that collar, and has a bearing surface of a length equal to the depth of the collar added to the length of stroke; so that during the motion different parts of the surface of the plunger come successively into contact with the same surface of the collar. On the other hand, an ordinary piston has a bearing surface of a depth merely sufficient to prevent the fluid from escaping; and the cylinder has a bearing surface of a length equal to the depth of that of the piston added to the length of stroke; so that during the motion the same surface of the piston comes into contact successively with different parts of the surface of the cylinder. For example, in fig. 157, A is a plunger, working through the collar B in the cylinder C; and in fig. 158, A is an

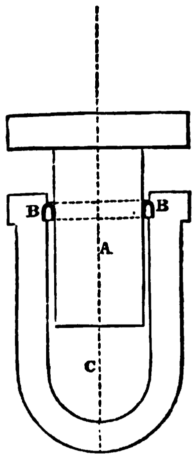

Fig. 157.

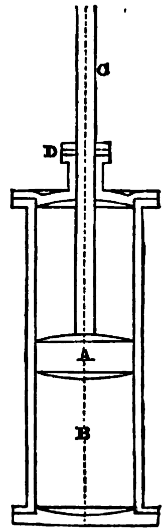

Fig. 158.

ordinary piston, working in the cylinder B. The action of plungers and of ordinary pistons in transmitting motion is exactly the same; and in stating the general principles of that action, the word *piston* is used to include plungers as well as ordinary pistons.

The *volume swept* by a piston in a given time is the product of two factors—transverse area and length. The *transverse area* is that of a plane bounded by the bearing surface of the piston and cylinder, and normal to the direction of motion of the piston, so that it cuts that surface everywhere at right angles. In a straight-sliding piston that plane is normal to the axis of the cylinder; in a piston moving circularly, it traverses the axis of rotation of the piston: in other words, the area is that of a projection of the piston on a plane normal to its direction of motion.

When the motion of the piston is straight, the *length* of the volume swept through is simply the distance moved by each point of the piston. When the motion is circular, that length is the distance moved through by the *centre of the area* of the piston.*

So long as the transverse area and length of the space swept by a piston are the same, it is obvious that the form of the ends of that piston does not affect the volume of that space.

When the space in the cylinder which contains the fluid acted on by a piston is traversed by a *piston-rod*, the effective transverse area is equal to the tranverse area of the piston, with that of the rod subtracted. For example, in fig. 158, the upper division of the cylinder is traversed by the piston-rod C, working through the stuffing-box D; hence the effective transverse area in that division of the cylinder is the difference between the transverse areas of the piston A and rod C. In the lower division of the cylinder, where there is no rod, the whole transverse area of the piston is effective. A *trunk* acts in this respect like a piston-rod of large diameter.

209. **Comparative Velocities of Pistons.**—From the equality of the volumes swept through by a pair of pistons that are connected with each other by means of an intervening fluid mass of invariable volume, it obviously follows that *the velocities of the pistons are inversely as their transverse areas.*

The transverse areas are to be measured, as stated in the preceding Article, on planes normal to the directions of motion of the pistons; and when the motion of a piston is circular, the velocity referred to in the rule is that of the centre of its transverse area.

Let A and A′ denote the transverse areas of the two pistons marked with those letters in fig. 159, page 224, and v and v' their velocities; then their velocity-ratio is $\frac{v'}{v} = \frac{A}{A'}$.

As the velocity-ratio of a given pair of connected pistons is constant, the combination belongs to Willis's Class A.

210. **Comparative Velocities of Fluid Particles.**—It may sometimes be required to find the comparative mean velocities with which

* To find the distance of the centre of a plane area from an axis in the plane of that area: divide the area, by lines parallel to that axis, into a number of narrow bands; let dx be the breadth of one of those bands, and y its length; then $y\,dx$ is the area of that band; and $\int y\,dx$ is the whole area. Let x be the distance from the axis to the centre of the band $y\,dx$; then $x\,y\,dx$ is the *geometrical moment* of that band, and $\int x\,y\,dx$ is the geometrical moment of the whole area relatively to the axis; which moment, being divided by the area, gives the required distance of the centre of the area from the axis, viz.,

$$x_0 = \frac{\int x\,y\,dx}{\int y\,dx}.$$ (See Article 293, page 334.)

the fluid particles flow through a given section of the passage which connects a pair of pistons; it being understood that the mean velocity of flow through a given section of the passage denotes the mean value of the component velocities, in a direction normal to that section, of all the particles that pass through it. From the fact that in a given time equal volumes of fluid flow through all sectional surfaces that extend completely across the passage, it follows that *the mean velocity of flow through any such section is inversely as its area* (a principle already stated in Article 88, page 76); and this principle applies to all possible sections, transverse and oblique, plane and curved.

For example, in fig. 159, let B denote the area of a transverse section, B B, of the passage which connects the two cylinders, and

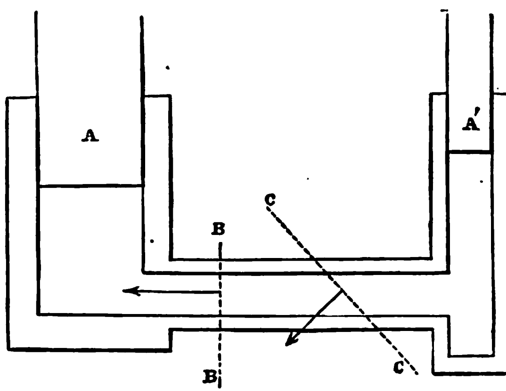

Fig. 159.

u the mean velocity with which the particles of fluid flow through that section; then v, as before, being the velocity of the piston whose transverse area is A, we have

$$\frac{u}{v} = \frac{A}{B}.$$

Also, let C denote the area of an oblique section, C C, of the passage, and w the mean component velocity of the fluid particles in a direction normal to that section; then

$$\frac{w}{v} = \frac{A}{C}; \text{ and } \frac{w}{u} = \frac{B}{C}.$$

211. **Use of Valves—Intermittent Hydraulic Connection.**—Valves are used to regulate the communication of motion through a fluid

by opening and shutting passages through which the fluid flows. For example, a cylinder may be provided with valves which shall cause the fluid to flow in through one passage, and out through another. Of this use of valves two cases may be distinguished.

I. *When the piston drives the fluid,* the valves may be what is called *self-acting;* that is, moved by the fluid. If there be two passages into the cylinder, one provided with a valve opening inwards, and the other with a valve opening outwards, then, during the outward stroke of the piston, the former valve is opened and the latter shut by the inward pressure of the fluid, which flows in through the former passage; and during the inward stroke of the piston the former valve is shut and the latter opened by the outward pressure of the fluid, which flows out through the latter passage. This combination of cylinder, piston, and valves constitutes a *pump.*

II. *When the fluid drives the piston,* the valves must be opened and shut by mechanism, or by hand. In this case the cylinder is a *working cylinder.*

It is by the aid of valves that *intermittent hydraulic connection* between two pistons is effected; and the action produced is analogous to that of the click, ratchet, and detent, in intermittent link-work.

For example, in the Hydraulic Press, the rapid motion of a small plunger in a pump causes the slow motion of a large plunger in a working cylinder; and the connection of the pistons is made intermittent by means of the discharge valve of the pump; being a valve which opens outwards from the pump and inwards as regards the working cylinder. The pump draws water from a reservoir, and forces it into the working cylinder: during the inward stroke of the pump plunger, the plunger of the working cylinder moves outward with a velocity as much less than that of the pump plunger as its area is greater. At the end of the inward stroke of the pump plunger, the valve between the pump and the working cylinder closes, and prevents any water from returning from the working cylinder into the pump; and it thus answers the purpose of the detent in ratchet-work (see page 206). During the outward stroke of the pump plunger that valve remains shut, and the plunger of the working cylinder stands still, while the pump is again filling itself with water through a valve opening inwards. When the piston of the working cylinder has finished its outward stroke, which may be of any length, and may occupy the time of any number of strokes of the pump, it is permitted to be moved inwards again by opening a valve by hand and allowing the water to escape.

A hydraulic press is often furnished with two, three, or more pumps, making their inward strokes in succession, and so producing a continuous motion of the working plunger. This is analogous to the double-acting click (page 209).

211 A. **Flexible Cylinders and Pistons.**—By an extension of the use of the word "cylinder," it may be made to include vessels made wholly or partly of a flexible material, which answer the purpose of a cylinder with its piston, by altering their shape and internal capacity; such as bellows. Questions as to this class of vessels may be approximately solved according to purely geometrical principles, by assuming the flexible material of which they are made to be inextensible.

In bellows, and pumps constructed on the principle of bellows, the vessel must have at least a pair of rigid ends, which, being moved alternately from and towards each other, answer the purpose of a piston. If those ends are equal and similar, and connected together by sides that may be assumed to be inextensible and perfectly flexible, the volume of fluid alternately drawn in and forced out may be taken as nearly equal to the area of one end multiplied by the distance through which the centre of area of one end moves alternately towards and from the other end.

Another example is furnished by a kind of pump, in which a circular orifice in one of the sides of a box is closed by a rigid flat disc of smaller diameter, and a bag in the form of a conical frustum of leather, or some other suitable material—the inner edge of the leather being made fast to the disc, and the outer edge to the circumference of the orifice. In working, the disc is moved alternately inwards and outwards, so as to draw the conical bag tight in opposite directions alternately. To find the *virtual area* of piston, add together the area of the disc, the area of the orifice, and four times the area of a circle whose diameter is the half-sum of the diameters of the disc and orifice, and divide the sum by six. That virtual area, multiplied by the length of stroke, gives nearly the volume of fluid moved per stroke.

In *Bourdon's pumps and engines* an elastic metal tube, of a flattened form of transverse section, is bent so as to present the figure of a circular arc. The internal capacity of the tube is varied by alternately admitting and expelling fluid; the effect of which is to flatten the curvature of the tube when its capacity is increased, and to sharpen that curvature when that capacity is diminished; so that if one end of the tube is fixed in position and direction, the other end has an oscillating motion.

In fig. 81, page 114, the arcs A D, A D′, A D″ may be taken to represent successive positions of the tube; A being its fixed end, and D its moveable end. The path of the moveable end, D D′ D″, is nearly an arc of a circle of the radius C G = $\frac{3}{4}$ of the length of the tube. The capacities of the tube in its several different positions, A D, A D′, A D″, &c., vary nearly *in the inverse ratio of the* *arcs* G D, G D′, G D″, &c.; so that if the capacity of the tube, when in a given position, is known, we can calculate its capacity in any other position, and the volume of fluid admitted or expelled in passing from any given position to any other.*

SECTION IX.—*Miscellaneous Principles respecting Trains.*

212. **Converging Trains.**—The essential principles of a train of mechanism have been stated in Article 93, page 80. Two or more trains may converge into one; that is to say, two or more primary pieces, which are followers in different trains, may all act as drivers to one primary piece. In such cases the comparative motion in each of the elementary combinations formed by the one follower with its several drivers is fixed by the nature of the connection; and thus the comparative motions of all the pieces are determined. As an example of converging trains, we may take a compound steam engine, in which two or more pistons drive one shaft, each by its own connecting-rod and crank.

213. **Diverging Trains.**—One train of mechanism may diverge into two or more; that is to say, one primary piece may act as driver to two or more primary pieces, each of which may be the commencement of a distinct train. In this case, as well as in that of converging trains, the comparative motions of all the pieces are determined.

Examples of diverging trains might be multiplied to any extent. One of the most common cases is that in which a number of different machines in a factory are driven by one prime mover: all those machines are so many diverging trains. In many instances there are diverging trains in one machine; thus in almost every

* Let A D′ be the position for which the capacity of the tube is known, and let V′ be that capacity. Let A D and A D″ be the positions of the tube at the two ends of its stroke; let V and V″ be the corresponding capacities; and let the lengths of the arcs G D, G D′, G D″ be denoted by s, s', s'' respectively. Then we have

$$V\,s = V'\,s' = V''\,s''; \text{ and } \frac{1}{s} : \frac{1}{s'} : \frac{1}{s''} :: V : V' : V'' \text{..........(1.)}$$

The volume of fluid admitted or expelled at each stroke is as follows:—

$$V'' - V = V'\,s'\left(\frac{1}{s''} - \frac{1}{s}\right) = \frac{V'\,s'\,(s - s'')}{s\,s''} \text{..............(2.)}$$

The length of stroke of the point D is $s - s''$; hence the apparatus may be regarded as equivalent to a cylinder and piston of that length of stroke, and of the following transverse area:—

$$\frac{V'' - V}{s - s''} = \frac{V'\,s'}{s\,s''} \text{..................................(3.)}$$

machine tool there are at least two diverging trains—one to produce the cutting motion, and the other the feed motion.

214. **Train for diminishing Fluctuations of Speed.**—The fluctuations in the velocity-ratio, when a revolving and a reciprocating point are connected by means of a link, have been stated in Article 184, pages 196, 197, and in Article 188, pages 199 to 201. In some cases it is desirable that the velocity-ratio of a reciprocating point to a revolving point should be more nearly uniform. For this purpose a train of two combinations may be used,—the first primary piece being a rotating shaft, which may be called A; the second, another rotating shaft, which may be called B; and the third, the reciprocating piece, C. The connection of A with B is by means of a pair of equal and similar two-lobed wheels (see Article 109, page 97); and a crank on B, by means of a connecting-rod, drives C. The two-lobed wheels are to be so placed that the shortest radius of the wheel on B shall be in gearing with the longest radius of the wheel on A at the instants when the crank is passing its dead-points. The result to be aimed at in the arrangement is, that each *quarter-stroke* of C shall be made as nearly as possible in the time of *one-eighth of a revolution of* A; and in order that this may be the case, the following should be the angles moved through by the two shafts respectively in given times:—

Shaft A,..................	0°	45°	90°	135°	180°
Shaft B, commencing at a dead-point of the crank,.................	0°	60°	90°	120°	180°

Hence it appears that B is alternately to overtake and to fall behind A by 15°. This angle, then, being given, the rules of Article 109, page 98, are to be applied to the designing of the pitch-lines of the wheels. The greatest and least radii of those wheels are approximately 0·634 and 0·366 of the line of centres respectively.

The following are the comparative velocities, at different instants, of a revolving point in A at a given distance from its axis, of a revolving point in B at the same distance from its axis, and of a point in C connected by a very long link with the point in B *:—

* Mr. Willis, in his *Treatise on Mechanism*, investigates the figures of a pair of wheels on A and B for giving exact uniformity to the ratio C ÷ A. The equations are as follows:—Let c be the line of centres; r, a radius of the wheel on B, making the angle θ with the shortest radius; r', the corresponding radius of the wheel on A, making the angle θ' with the longest radius of this wheel; then we have

$$r = c \cdot \frac{\pi \sin \theta}{\pi \sin \theta + 2}; \; r' = c - r; \text{ and } \theta' = \frac{\pi}{2} \cdot \text{versin } \theta.$$

Mr. Willis points out that the forms of the pitch-lines given by the equations must in practice be slightly modified at the points which gear together when the crank is at its dead-points.

Angles moved through by A,	0°	45°	90°	135°	180°
Velocity-ratio B ÷ A,	1·732	0·866	0·577	0·866	1·732
Velocity-ratio C ÷ B,	0	0·866	1·000	0·866	0
Velocity-ratio C ÷ A,	0	0·750	0·577	0·750	0

Mean value of each of the velocity-ratios C ÷ B and C ÷ A, 0·637.

A similar adjustment may be made by connecting the shafts A and B by means of an universal joint (Article 191, page 203); the fork on the shaft B being so placed as to have its plane perpendicular to the plane of the axes when the crank is at its dead-points; the angle made by those axes with each other should be that whose cosine is 0·577, viz., 54¾°.

The Double Hooke's Joint (Article 192, page 205) is an example of a train in which the fluctuation of the velocity-ratio is corrected exactly.

SECTION X.—*References to Combinations arranged in Classes.*

215. **Object of this Section.**—In the preceding sections the various elementary combinations in mechanism have been arranged according to the mode of connection. The object of the present section is to give a list of such combinations, arranged according to Mr. Willis's system—that is, according to the comparative motion—with references to the previous Articles and pages of this treatise, where the several combinations are described. Two deviations from or modifications of Mr. Willis's system are used; first, the addition, at the commencement of each Class, of references to places where the comparative motions of two points in one primary piece are treated of; and secondly, the placing of combinations in which the connection is intermittent, in a class by themselves, entitled Class D.

216. CLASS A. **Directional-Relation Constant—Velocity-Ratio Constant.**

COMBINATIONS.

Velocity-Ratio that of Equality alone.

	ARTICLES.	PAGES.
Pair of Points in one straight-sliding Primary Piece,	43	22
Sliding Contact, Oldham's Coupling,....................	158	166
Bands, equal and similar Non-circular Pulleys,........	167	182
Linkwork, Coupled Parallel Shafts,.....................	181	194
„ Drag-link: Shafts in one straight line,	182	194
„ Double Hooke's Joint,........................	192	205
„ Double Hooke-and-Oldham Coupling,....	193	206

Any Constant Velocity-Ratio.

	Articles.	Pages.
Pair of Points in one Rotating Primary Piece,........	53	31
Pair of Points in one Screw,...........................	60	37
Rolling Contact: Circular Toothless Wheels and Sectors, and Straight Racks,......................	102 to 106	84 to 92
Rolling Contact: Frictional Gearing,..................	111	102
Sliding Contact; Circular Toothed Wheels and Sectors, and Straight Racks,........................	112 to 141 144 to 151	103 to 139 143 to 157
Sliding Contact: Screw Gearing,........................	152 to 157	157 to 166
Bands and Pulleys,..	165 to 177	179 to 192
Blocks and Tackle,..	200 to 205	214 to 218
Hydraulic Connection: Pistons and Cylinders,.......	207 to 210	221 to 224

217. Class B. **Directional-Relation Constant; Velocity-Ratio Variable.**

Mean Velocity-Ratio that of Equality alone.

	Articles.	Pages.
Rolling Contact: Smooth Elliptic and Lobed Wheels,	108 109	95 to 99
Sliding Contact: Toothed Elliptic and Lobed Wheels,	143	141
,, ,, Pin and Slot Coupling,...............	159	167
Crossed Cord and Elliptic Pulleys,......................	175	189
Linkwork: Drag-Link,....................................	182	194
,, Link for Contrary Rotations,.............	183	196
,, Single Hooke's Joint,........................	191	203
,, Single Hooke-and-Oldham Coupling,....	193	206

Any Mean Velocity-Ratio.

	Articles.	Pages.
Rolling Contact: Non-Circular Wheels and Sectors,	107 to 110	92 to 102

Any Mean Velocity-Ratio—Continued.

218. Class C. **Directional-Relation Variable.**

219. Class D. **Intermittent Connection.**

Section XI.—*Comparative Motion in the "Mechanical Powers."*

220. **Classification of the Mechanical Powers.**—"Mechanical Powers" is the name given to certain simple or elementary machines, all of which, with the single exception of the pulley, are more simple than even an elementary combination of a driver and follower; for, with that exception, a mechanical power consists essentially of only one primary moving piece; and the comparative motion taken into consideration is simply the velocity-ratio either of a pair of points in that piece, or of two components of the velocity of one point. There are two established classifications of

the mechanical powers; an older classification, which enumerates six; and a newer classification, which ranges the six mechanical powers of the older system under three heads. The following table shows both these classifications:—

Newer Classification.	Older Classification.
The Lever, ...	The Lever. The Wheel and Axle.
The Inclined Plane,........	The Inclined Plane. The Wedge. The Screw.
The Pulley,.....................	The Pulley.

In the present section the comparative motions in the mechanical powers are considered alone. The relations amongst the forces which act in those machines will be treated of in the dynamical division of this Treatise.

221. **Lever—Wheel and Axle.**—In the lever and the wheel and axle of the older classification, which are both comprehended under the lever of the newer classification, the primary moving piece turns about a fixed axis; and the comparative motion taken into consideration is the velocity-ratio of two points in that piece, which may be called respectively the *driving point* and the *following point.* The principle upon which that velocity-ratio depends has already been stated in Article 53, page 31—viz., that the velocity of each point is proportional to the radius of the circular path which it describes; that is, to its perpendicular distance from the axis of motion.

The distinction between the lever and the wheel and axle is this: that in the *lever*, the driving point, D, and the following point, F, are a pair of determinate points in the moving piece, as in figs. 161 to 164; whereas in the *wheel and axle* they may be any pair of points which are situated respectively in a pair of cylindrical pitch-surfaces, D and F, described about the axis A, fig. 160.

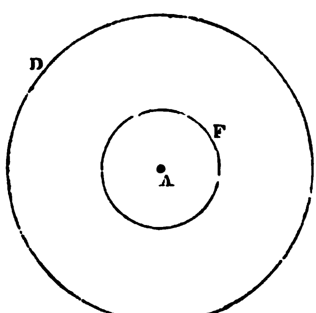

Fig. 160.

In each of these figures the plane of projection is normal to the axis, and A is the trace of the axis. In fig. 160, D and F are the traces of two cylindrical pitch-surfaces. In figs. 161 to 164, D and F are the projections of the driving and following points respectively.

The axis of a lever is often called the *fulcrum*.

A lever is said to be *straight*, when the driving point, D, and following point, F, are in one plane traversing the axis A, as in figs. 161, 162, and 163. In other cases the lever is said to be *bent*, as in fig. 164.

Fig. 161.

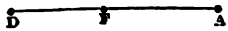

Fig 162.

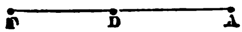

Fig. 163.

The straight lever is said to be of one or other of three kinds, according to the following classification:—

In a *lever of the first kind*, fig. 161, the driving and following points are at opposite sides of the fulcrum A.

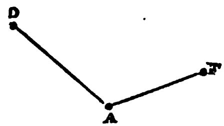

Fig. 164.

In a *lever of the second kind*, fig. 162, the driving and following points are at the same side of the fulcrum, and the driving point is the further from the fulcrum.

In a *lever of the third kind*, fig. 163, the driving and following points are at the same side of the fulcrum, and the following point is the further from the fulcrum.

222. **Inclined Plane—Wedge.**—In the inclined plane, and in the wedge, the comparative motion considered is the velocity-ratio of the entire motion of a straight-sliding primary piece and one of the components of that motion; the principles of which velocity-ratio have been stated in Article 43, pages 22, 23.

In the inclined plane, fig. 165, A A is the trace of a fixed plane; B, a block sliding on that plane in the direction B C; the plane of projection being perpendicular to the plane A A, and parallel to the direction of motion of B. B D is some direction oblique to B C. From any convenient point, C, in B C, let fall C D perpendicular to B D; then B D ÷ B C is the ratio of the component velocity in the direction B D to the entire velocity of B.

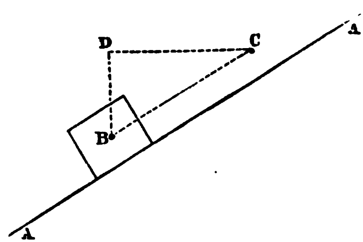

Fig. 165.

In fig. 166, A A is the trace of a fixed plane; B C D, the trace of a wedge which slides on that plane. While the wedge advances through the distance C *c*, its oblique face advances from the position C D to the position *c d*; and if C *e* be drawn normal to the plane C D, the ratio borne by the component velocity of the wedge

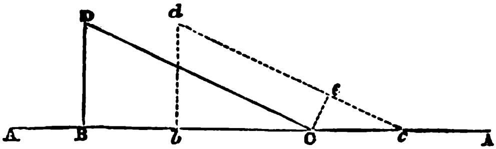

Fig. 166.

in a direction normal to its oblique face to its entire velocity will be expressed by C *e* : C *c*.

223. **Screw.**—In the screw the comparative motion considered is the ratio borne by the entire velocity of some point in, or rigidly connected with, the screw, to the velocity of advance of the screw.

The helical path of motion of a point in, or rigidly attached to, a screw may be developed (as has been already explained in Article 63, page 40) into a straight line: being the hypothenuse of a right angled triangle whose height is equal to the pitch of the screw, and its base to the circumference of a circle whose radius is the distance of the given point from the axis of the screw. Then if B D in fig. 165 be taken to represent the pitch of the screw, and D C, perpendicular to B D, the circumference of the circle described by the point in question about the axis, B C will be the development of one turn of the screw-line described by that point as it revolves and advances along with the screw; and B C ÷ B D will be the ratio of its entire velocity to the velocity of advance; just as in the case of a body sliding on an inclined plane, A A, parallel to B C. This shows why the screw is comprehended under the general head of the inclined plane, in the newer classification of the mechanical powers.

224. **Pulley.**—The term *pulley*, in treating of the mechanical powers, means any purchase or tackle of the class already described in Section VII. of this Chapter, pages 214 to 221.

CHAPTER V.

OF AGGREGATE COMBINATIONS IN MECHANISM.

SECTION I.—*General Explanations.*

225. **Aggregate Combination Defined.**—"Aggregate Combinations" is a term introduced by Professor Willis, to denote those assemblages of pieces in mechanism in which the motion of one follower is the resultant of motions impressed upon it by more than one driver. The number of independently-acting drivers which impress directly a compound motion on one follower cannot be greater than three; because each driver determines the motion of at least one point in the follower; and the determination of the motion of three points in a body determines the motion of the whole body. In most cases which occur in practice, the number of independent drivers which act directly on one follower is *two.*

226. **General Principle of their Action.**—The follower which has such a compound motion directly communicated to it by more than one primary piece must necessarily be a *secondary piece,* as defined in Article 37, page 17; its motion at any instant is the *resultant* of the motions impressed upon it separately by the pieces which act as its drivers; and the determination of that resultant motion depends upon the principles already explained in Chapter III. of this Division, pages 43 to 75. Several examples of the motion of secondary pieces have been given in the preceding Chapter, in treating of those secondary pieces, such as links and bands, and the sheaves of running blocks, which act as connectors in elementary combinations.

227. **Aggregate Combinations terminating in a Primary Piece.**—Very often an aggregate combination is of the nature of a train; and although a secondary piece receives in the first instance a compound motion from two or from three primary pieces, that secondary piece communicates motion in the end to a primary piece. In such cases the motion of that last primary follower may be determined, by finding the motions which would be communicated to it through the intermediate secondary piece or pieces by the several primary drivers acting separately, and taking the resultant of those motions.

228. **Shifting Trains.**—A secondary piece in an aggregate combination has very often a form like that of a primary piece, and

is distinguished from a primary piece only by the fact that its bearings, instead of being carried by the fixed frame, are carried by a moving frame; that moving frame being one of the primary pieces from which the secondary piece receives its motion. For example, a wheel may turn about an axis which is carried by an arm that turns about another axis. The compound motions of which such secondary pieces are capable have been treated of in Articles 72 to 79, pages 51 to 62, and Articles 81 to 86, pages 66 to 74. When such a secondary piece is to drive or to be driven by a primary piece, or another secondary piece not carried by the same

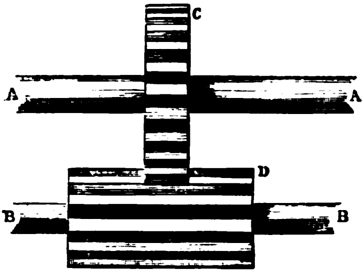

Fig. 167.

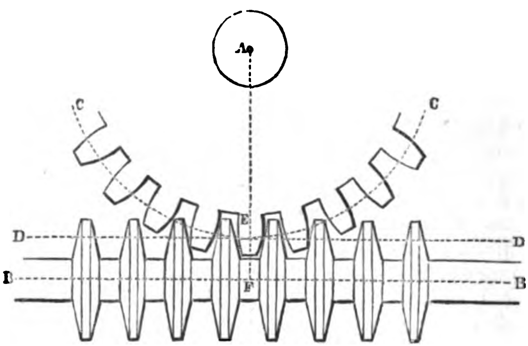

Fig. 168.

moving frame, special contrivances, which may be called *shifting trains*, have to be used in order to keep up the connection between the two pieces during their various changes of relative position. The following are examples:—

I. When two pieces turning about parallel axes are connected by toothed gearing, and one of them is free to shift its position along its axis relatively to the other, the Long or Broad Pinion may be used. In fig. 167 A A and B B are a pair of parallel axes; C, a spur-wheel on A A; D, a pinion on B B; and the breadth of the pitch-surface of D is made greater than that of C by a length equal to the distance through which D is capable of being shifted longitudinally.

II. When a toothed wheel, C C, fig. 168, gears with a rack, D D, and either the rack is to be capable of turning about an axis, B B, parallel to its pitch-line, or the axis A of the wheel is to be capable of being moved round the axis B B at the end of an arm, F A, the CIRCULAR RACK is to be used, being, as represented in the figure, a solid of revolution generated by the rotation of the trace of the rack-teeth about the axis B B. The pitch-line D D becomes the trace of an imaginary pitch-cylinder generated by its revolution about the axis B B; and the pitch-point E is the point of contact of that cylinder with the pitch-cylinder of the wheel.

It is easy to see that by fixing a broad pinion on one part of a shaft, and a circular rack on another, that shaft may receive at the same time two independent motions of rotation about its axis and translation along its axis respectively, from two different spur-wheels; the result being a helical motion; and this is one of the simplest of aggregate combinations.

III. TRAIN-ARM.—When rotation is to be transmitted from a fixed axis to a shifting axis, or from one shifting axis to another, and the relative motion of the two axes is such that their distance apart, and the angle which their directions make with each other, do not change,—in other words, when one of the two axes revolves round the other as if it were carried by a rotating arm,—the connection between those axes may be kept up by means of one rigid frame, which carries any combination or train of mechanism suitable for transmitting rotation from the one axis to the other: such a frame is called a *train-arm*.

The general principles of the velocity-ratios which are communicated by means of train-arms will be stated further on; but at present one particular case requires special mention,—it is that in which the train carried by the arm is such that the two axes connected by it are parallel, and the angular velocities of the pieces which turn about them equal and in the same direction. In fig. 169 the plane of projection is supposed to be normal to the two axes to be connected, A and B the traces of those two axes, and A B their common perpendicular. A moveable frame or train-arm connects the bearings of the axes with each other, so that the distance A B is invariable; and that frame carries a train of mechanism such as to transmit the angular velocity of the piece which turns about A unchanged in velocity and direction to the piece which turns about B. For example, those pieces may have pairs of parallel and equal cranks linked together by coupling-rods; or they may be equal and similar pulleys connected by a band; or equal and similar toothed wheels, with an intermediate wheel gearing with both. The result is, that while the train-arm turns

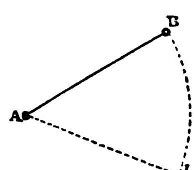

Fig. 169.

into any other position, such as A *b*, the angular velocities of the pieces which rotate about the axes A and B respectively continue to be equal in magnitude and identical in direction.

IV. When rotation is to be transmitted between a pair of axes whose common perpendicular alters in length as well as in direction, a COMPOUND TRAIN-ARM may be used, consisting of two or more train-arms jointed together at intermediate axes. For example, in fig. 170, A and C are the traces of two such axes. B is the trace of an intermediate axis, connected by means of two train-arms with A and with C respectively, so that the distances A B and B C are invariable; while A B can be turned into any angular position about A, such as A *b*, and B C into any angular position about B, such as *b c*. Then the relative position of A and C can be altered either in direction or in distance, so long as their distance apart does not exceed A B + B C; and the transmission of motion will still be kept up by means of the trains that are carried by the train-arms.

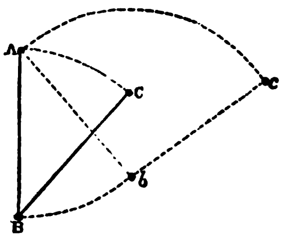

Fig. 170.

V. When motion is transmitted between two axes by means of a band, the connection may be maintained during changes of the relative position of those axes by means of STRAINING PULLIES and GUIDING PULLIES so arranged as to keep the band tight.

229. **Methods of Treating Problems respecting Aggregate Combinations.**—The methods by which problems respecting aggregate combinations are solved may be distinguished into two classes.

I. In one class a piece which may be regarded as a train-arm, or moving frame (and which may be designated by B), has a given motion relatively to the fixed frame, A, of the machine; and at the same time a secondary moving piece, C, has a given motion relatively to B. The resultant of those two given motions is the motion of C relatively to A; and the general rules for finding it in various cases have been stated in Articles 73 to 77, pages 52 to 56, and Articles 81 to 86, pages 66 to 74.

II. In the other class of methods the motions of three points in a secondary piece that is free to move in all directions, or, more frequently, the motions of two points in a secondary piece that is guided so as to move in one plane, or about one fixed point, are given; and the motion of the piece as a whole is to be deduced from them. The general rules for doing this have been given in Articles 69 to 71, pages 45 to 51.

There is no difference in principle between the kinds of problems that are treated by those two classes of methods respectively; the choice of methods is a matter of convenience only.

230. **Aggregate Combinations classed according to their Purposes —Aggregate Velocities—Aggregate Paths.**—The classification of aggregate combinations which will be adopted throughout the rest of this Chapter is that of Mr. Willis, and is founded on the purposes which the combinations are designed to effect. Those purposes are distinguished into (I.) *aggregate velocities*, and (II.) *aggregate paths*.

I. When an *aggregate velocity* is the object aimed at, the final piece of the train is usually a primary piece, whose comparative velocity, by the help of an aggregate combination, is made either to have a certain constant value or to vary according to a law which it might be difficult or impossible to realize by means of a train of elementary combinations only.

II. When an *aggregate path* is the object aimed at, a point in a secondary piece is made, by means of an aggregate combination, to move in a path of a figure which may be different from that which a point in a primary piece would describe.

The only paths which points in primary pieces can describe are straight lines, circles, and screw lines;* and paths of all other figures must be described by the help of aggregate combinations. Sometimes, indeed, it is found convenient to use aggregate combinations for describing, either exactly or approximately, even those elementary paths themselves—the straight line, the circle, and the screw-line. For example, there is a numerous class of aggregate combinations called parallel motions, whose object is to make a point move sensibly in a straight line.

* In other words, paths in which both the curvature and the tortuosity are either none or uniform. The curvature of a path is the reciprocal of the radius of curvature. The tortuosity is the reciprocal of the length, measured along the path, in the course of which the radius of curvature rotates round a tangent to the path as an axis, through the angle which subtends an arc equal to radius. In the case of a helix, or screw-line, let r be the radius of the cylinder on which the screw-line is described, and p the pitch of that line; let $q = \frac{p}{2\pi}$ be the radius of a circle whose circumference is equal to the pitch; let θ be the obliquity of the screw-line to a plane normal to its axis; let ρ be its radius of curvature; and let σ be the reciprocal of the tortuosity. Then $q = r \tan \theta$; and according to Article 64, page 41, the radius of curvature is

$$\rho = r + \frac{q^2}{r} = r \sec^2 \theta.$$

Also, it can be shown that the reciprocal of the tortuosity is

$$\sigma = q + \frac{r^2}{q} = q \operatorname{cosec}^2 \theta = 2 r \operatorname{cosec} 2 \theta = \rho \operatorname{cotan} \theta.$$

A further subdivision of the purposes of aggregate combinations leads to the following classification:—

AGGREGATE VELOCITIES.
Production of Uniform Velocity-Ratios (as in Willis's Class A).
Production of Varying Velocity-Ratios (as in Willis's Classes B and C).

AGGREGATE PATHS.
Description of Curved Paths, (Ellipses, Epicycloids, &c.)
Description of Sensibly Straight Paths (Parallel Motions).

231. **Converging Aggregate Combinations.**—This term may be applied to denote those trains in which the drivers in an aggregate combination are themselves the followers in aggregate combinations. By means of trains of that kind, any number of component motions may be combined. Suppose, for example, that a piece, A, is driven jointly by B and C, and that B is driven jointly by D and E, and C by F and G; then the motion of A is the resultant of four component motions, due respectively to the actions of D, E, F, and G.

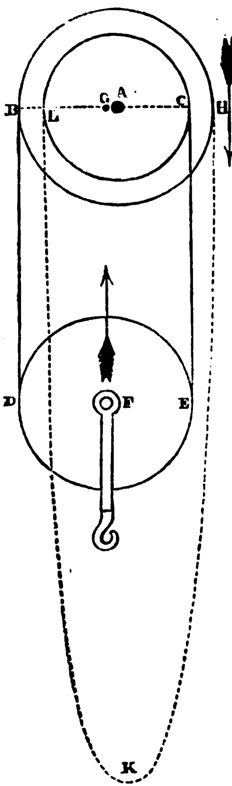

Fig. 171.

SECTION II.—*Production of Uniform Aggregate Velocity-Ratios.*

232. **Differential Pulley and Windlass.**—In this combination, two pulleys, B and C (fig. 171), of different radii, rotate as one piece about a fixed axis, A. An endless chain, B D E C L K H, passes over both pulleys. The rims of the pulleys are shaped so as to hold the chain, and prevent it from slipping. The lines in the figure represent the pitch-lines of the pulleys and the centre line of the chain respectively. As to the relation between those lines and the actual figures of the pieces, see Articles 166, 176, pages 180, 190. One of the *bights* or loops in which the chain hangs, D E, passes under and supports the running block F. The other loop or bight, H K L, hangs freely; and very often the combination is driven by hauling upon the part H K; which therefore may be called the *hauling part.* It is evident

that the velocity of the hauling part is equal to that of the pitch-circle B. Sometimes the compound pulley is driven by other means; as by a second endless chain acting on a sprocket-wheel.

In order that the velocity-ratio may be exactly uniform, the radius of the sheave F should be an exact mean between the radii of B and C; but it is not necessary to follow this rule strictly in practice. In stating the velocity-ratio, however, it will be assumed that the rule has been observed.

Let the velocities of the pitch-circles of B and C be denoted by B and C respectively. Then the proportion of those velocities to each other is

$$\frac{C}{B} = \frac{AC}{AB}$$

Let F denote the velocity of the running block. Then, if C were a fixed point, and consequently C E a "standing part" of the chain, the value of F would be $\frac{1}{2}$ B, and the direction of its motion would be upward (agreeably to the principles of Article 201, page 215). Also, if B were a fixed point, and B D a standing part, the value of F would be $-\frac{1}{2}$ C; the negative sign being used to denote downward motion. The actual value of F is the resultant of those two components; that is to say,

$$F = \frac{B - C}{2};$$

whence we have the comparative motion of the larger pitch-circle B, and the running block F, expressed by the following velocity-ratio:—

$$\frac{F}{B} = \frac{1}{2}\left\{1 - \frac{C}{B}\right\} = \frac{AB - AC}{2\,AB}.$$

The velocity of the running block is the same with that of the pitch-circle of a pulley of the radius $AG = \frac{AB - AC}{2}$, turning with the same angular velocity with the actual differential or compound pulley.

To calculate the *length of chain* required for a differential pulley, take the following sum: half the circumference of A + half the circumference of B + half the circumference of F + twice the greatest distance of F from A + the least length of the loop H K L. This last quantity is fixed according to convenience.

The *differential windlass* or *differential barrel* (fig. 172) is identical in principle with the differential pulley; the difference in construction being, that in the differential windlass the running block hangs in the bight of a rope whose two parts are wound round, and have their ends respectively made fast to, two barrels of different radii, which rotate as one piece about the axis A. The differential windlass is little used in practice, because of the great length of rope which it requires. That length is expressed by the following sum:—Twice the least distance of the running block from A + half circumference of running block + $\frac{B}{F}$ × total distance through which F is lifted; and the last term is often an inconveniently great quantity.

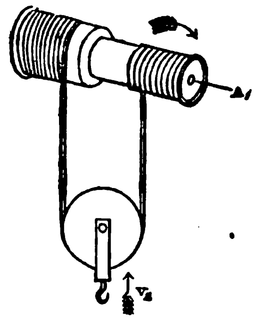

Fig. 172.

233. **Compound Screws.** (*A. M.*, 505.)—A compound screw consists of two screws cut upon the same spindle, and each having a nut fitted upon it. The screw turns: one of the nuts is usually fixed, so that the screw in turning in that nut is made to advance; the other nut slides, but does not turn; and the sliding motion of the second nut relatively to the first nut is the resultant of the advance of the screw relatively to the first nut, and of a motion equal and opposite to the advance of the screw relatively to the second nut; that is to say, the second nut moves relatively to the first nut as if it were acted upon by a single screw of a pitch equal to the *difference between the pitches* of the two screw-threads that are cut on the spindle; supposing those threads to wind the same way. But if the threads are *contrary-handed,* for the difference of their pitches is to be substituted the *sum.*

Fig. 173 represents a *differential screw:* that is, a compound screw

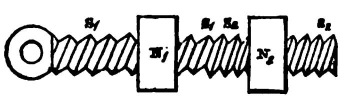

Fig. 173.

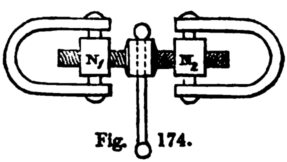

Fig. 174.

in which the threads wind the same way. N_1 and N_2 are the two nuts; S_1 S_1, the longer-pitched thread; S_2 S_2, the shorter-pitched thread: in the figure both those threads are left-handed. At each turn of the screw the nut N_2 advances relatively to N_1 through a distance equal to the difference of the pitches. The use of the differential screw is to combine the slowness of advance due to a fine pitch with the strength of thread which can be obtained by means of a coarse pitch only.

Fig. 174 represents a compound screw in which the two threads are contrary-handed; and the effect of each turn of the screw is to alter the distance between the nuts N_1 and N_2 by an amount equal to the sum of the pitches of the threads, which are usually equal to each other. This combination is used to tighten the couplings of railway carriages.

234. **Epicyclic Trains with Uniform Action.**—An epicyclic train for producing an uniform aggregate velocity-ratio consists essentially of four parts, whose general arrangement may be held to be represented by the diagram in Fig. 175—viz., the primary wheels B and C, turning about the same axis, O, with different uniform velocities; the *train-arm* A, being a moveable frame, turning with an uniform velocity about the same axis; and the *shifting train* of secondary pieces, carried by the train-arm A, and transmitting an uniform velocity-ratio from B to C, in the manner of an ordinary train. The shifting train may consist of any kind of mechanism belonging to Class A; such as circular toothed wheels, whether spur, bevel, or skew-bevel; screw-gearing; circular pulleys and bands; links with equal parallel cranks; and double universal joints.

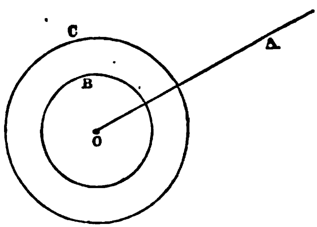

Fig. 175.

The comparative motions of the three primary pieces, A, B, and C, are determined in the following manner:—Let a, b, and c represent numbers proportional to the respective angular velocities of those pieces; it being understood that rotations in one direction are to be considered as positive, and those in the contrary direction as negative.

First, suppose that B is fixed relatively to A; that is to say, that it simply turns along with A, having the same angular velocity; or, in symbols, that $b = a$; then it is evident that C must turn along with A also, with the same angular velocity; that is to say, on this supposition, we have $c = a$.

Next, let B have a different angular velocity from A; then $b - a$ will represent the angular velocity of B relatively to A.

Determine, from the construction of the shifting train, the ratio of the velocity of C to that of B, as if the train-arm A were fixed; and denote that ratio by n; taking care to mark the value of n as positive or negative, according as the rotations of B and C are in similar or contrary directions. That ratio will also be the ratio which the angular velocity of C relatively to A bears to the angular velocity of B relatively to A, when the train-arm A is in motion; that is to say, in symbols,

$$\frac{c-a}{b-a} = n; \qquad \text{(1.)}$$

and this is *the general equation of the action of an epicyclic train.*

Two particular cases may be distinguished, according as the wheel C or the train-arm A is the follower in the combination.

CASE I.—The wheel B and the train-arm A are driven by means of diverging trains, with angular velocities proportional to given numbers, b and a; then the proportionate angular velocity of C is given by the following formula:—

$$c = n(b-a) + a = n\,b + (1-n)\,a \qquad \text{(2.)}$$

CASE II.—The primary wheels B and C are driven by means of diverging trains with angular velocities proportional to given numbers, b and c; then the proportionate angular velocity of the train-arm a is given by the following formula:—

$$a = \frac{c - n\,b}{1-n} = \frac{b}{1-\frac{1}{n}} + \frac{c}{1-n} \qquad \text{(3.)}$$

In some examples of both cases one of the primary wheels is fixed. Let B be that wheel; then $b = 0$; and we have

$$\frac{c}{a} = 1 - n. \qquad \text{(4.)}$$

One of the uses of epicyclic trains is to obtain with precision velocity-ratios in toothed wheel-work which are expressed by numbers whose factors are too large to be suitable for the teeth of wheels. For example, $\frac{c}{b}$ may be such a ratio; and it may be possible to divide $\frac{c}{b}$ into two parts, as expressed by the following formula:—

$$\frac{c}{b} = n + (1-n)\frac{a}{b};$$

such that each of those parts is expressed by numbers whose factors are not too large; and then, by using a train-arm with the velocity-ratio $\frac{a}{b}$, and a shifting train with the velocity-ratio n, the required velocity-ratio may be obtained with precision by means of wheels of moderate size.*

Another use of epicyclic trains is to make the train-arm move, for purposes of regulation (as in certain governors), with a velocity proportional to the difference between the velocities of the primary wheels B and C. This is best effected by causing the primary wheels B and C to rotate in contrary directions, and to connect them by means of a shifting train such that, when the train-arm is at rest, the angular velocities of those wheels are equal and opposite. This amounts to making $n = -1$ in equation 3, and c = a negative quantity, say $-k$; and then the expression for the angular velocity of the train-arm becomes

$$a = \frac{b - k}{2}.$$

For example, in Fig. 176, O is a vertical spindle, about which the equal and similar bevel wheels B and C turn in opposite directions. A is the train-arm, being a horizontal spindle carried by a collar which turns about the vertical spindle. The shifting train consists of a bevel wheel turning about the spindle A, and gearing with the wheels B and C. In order to produce a balance of forces, two, and sometimes three or four, equal and similar horizontal spindles like A project from the collar, and carry equal and similar bevel wheels. In the figure two are shown. The result is, that when the wheels B and C turn in opposite directions with equal speed, the train-arm stands still; but when the velocities of those wheels become unequal, the train-arm turns in the direction

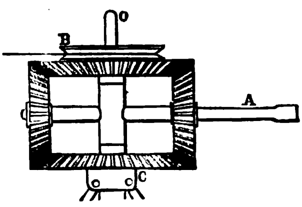

Fig. 176.

* The solution of this problem is to be obtained in any particular case by a series of trials conducted generally in the following manner:—Let n be an approximation to the ratio $\frac{c}{b}$, not containing factors exceeding what is considered a convenient limit (values of n may be found by the method of continued fractions, Article 117, page 107). Then make $a = \frac{c - bn}{1 - n}$; and try whether the ratio $\frac{a}{b}$ contains inconveniently large factors. The trial is to be repeated with the various different values of n, until a satisfactory result is arrived at. This method cannot fail, provided it is c only, and not b, which contains inconveniently large factors.

of the greater of the two velocities, with a speed equal to half their difference. Other applications of epicyclic trains, where the last follower is a secondary piece, will be mentioned under the head of aggregate paths.

SECTION III.—*Production of varying Aggregate Velocity-Ratios.*

235. The **Reciprocating Endless Screw** may be used where it is desired that there shall be periodic fluctuations in the ratio of the speed of the follower to that of the driver. In this combination a wheel is driven by a rotating screw, as in fig. 112, page 164, which screw has at the same time a reciprocating motion along its axis.

236. **Epicyclic Trains with Periodic Action** are used for the same purpose. This is effected by communicating, by means of suitable mechanism, such as a cam, or a crank and link, the required reciprocating motion to the train-arm A, fig. 175, page 243. The angular velocity of the follower, C, is expressed, as in Article 234, by

$$c = n\,b + (1 - n)\,a;$$

in which $n\,b$ is a constant term, and $(1 - n)\,a$ a periodically varying term; the factor $1 - n$ being constant, and the factor a periodic.

236 A.—**The Sun-and-Planet Motion** is a sort of epicyclic train with periodic action. In fig. 177, C is a shaft which overhangs

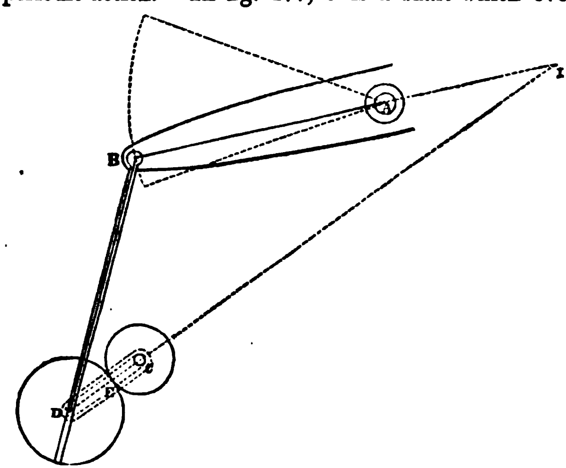

Fig. 177.

its bearing, and carries on its overhanging end a toothed wheel, C E, called the *sun-wheel*. This gears with another toothed wheel, D E, called the *planet-wheel*, which is made fast to the connecting-rod D B, which hangs from one end of the lever or walking-beam, A B. At the centre, D, of the wheel D E is a pin which is connected with the shaft C by a link or bridle, C D (shown by dotted lines); so that it revolves round the axis of C like a crank-pin, making one revolution for each double-stroke of the beam A B.

In the first place, to determine the *mean ratio of the linear velocity* of the pin D to that of the pitch-circle of the sun-wheel, C E, it is to be observed that the latter velocity is at every instant equal to that of the pitch-point E in the planet-wheel. Now, the motion of the planet-wheel is one of translation in a circle along with the pin D, compounded with an angular oscillation to and fro along with the rod D B. Hence the *mean linear velocity* of D is equal to that of the pitch-circle of C E.

Secondly, as to the *mean ratio of the angular velocity* of the bridle C D to that of the sun-wheel C E, it is obvious that as the mean linear velocities of D, and of the pitch-circle of C E, are equal, their mean angular velocities are inversely as the radii C E and C D; or in symbols—

$$\frac{\text{mean angular velocity of C D}}{\text{mean angular velocity of C E}} = \frac{\text{C E}}{\text{C D}}.$$

In the sun-and-planet motion, as originally contrived and constructed by Watt, the sun-wheel and planet-wheel were made of equal radii; so that C D was = 2 C E; and the sun-wheel made two turns for each revolution of the planet-wheel round it.

Thirdly, as to the ratio of the linear velocities of the points D and E at any instant; this is to be found by producing D C till it cuts A B in I, which will be the instantaneous axis of the planet-wheel; and then taking the proportion,

$$\frac{\text{velocity of D}}{\text{velocity of E}} = \frac{\text{I D}}{\text{I E}}.$$

The mean value of this ratio is unity, as already stated. It attains its greatest and least values in the two positions of the combination when B D and C D are in one straight line, so that I coincides with B; and then its values are respectively

$$\frac{\text{B D}}{\text{B D} - \text{D E}} \text{ and } \frac{\text{B D}}{\text{B D} + \text{D E}}.$$

237. **Eccentric Gearing.**—This is a combination for producing a periodically varying velocity-ratio by means of a train of circular wheels, one of which turns eccentrically about an axis. It is

nearly, but not exactly, equivalent in its action to a pair of elliptic toothed wheels (Article 100, page 95). In fig. 178, A is the axis of a shaft, which carries an eccentric circular toothed wheel. This

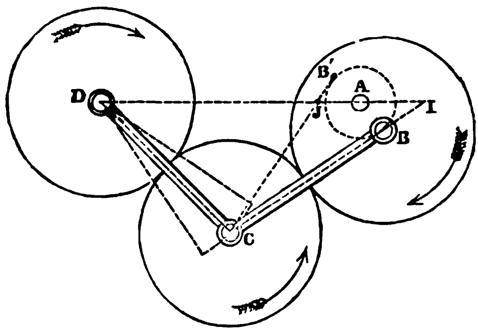

Fig. 178.

gears with a second toothed wheel, centred on a moveable axis, C, which gears with a third toothed wheel, centred on a fixed axis, D. The centres of the three wheels are linked together by the two train-arms B C, C D; so that the wheels are kept always in gearing, while the centre pin B revolves round the axis A. Suppose the wheels on B and D to be of equal size. Then, if the train-arms were fixed, the rotation of the first wheel about B would produce a rotation of the third wheel about D, with equal speed and in the same direction. The effect of the revolving of B about A is to combine that rotation of D with an alternate increase and diminution of speed, corresponding to the alternate diminution and increase of the angle B C D. The greatest and least values of the velocity-ratio take place when the line of connection, C B, touches the two sides of the circle described by B about A; that is to say, when that line is in the two positions marked C B I and C J B′ respectively. Let I and J be the points where C B cuts the line of centres, D A, when in those positions; then the two corresponding values of the velocity-ratio of D to A are respectively

$$1 + \frac{A\,I}{D\,I} \text{ and } 1 - \frac{A\,J}{D\,J}$$

238. **Aggregate Linkwork in General.**—A combination in aggregate linkwork is usually of the following kind:—A bar, or other rigid body, capable of moving parallel to a given plane, has two of its points connected by means of rods with two drivers:—A third point is connected by means of a rod with a follower. The motions

of the first two points, as compared with those of their drivers, are determined by the principles of elementary combinations in link-work, and so also is the motion of the follower, as compared with that of the third point; but the determination of the motion of the third point from that of the first two is a problem to be solved by the principles of the motion of secondary pieces, Article 69, pages 45 to 48; that is, by the process of finding the instantaneous axis, or by some equivalent process.

In most of the particular cases of aggregate velocities obtained by linkwork which occur in practice, the three points in the bar are either situated in one straight line to which their motions are perpendicular, or are so nearly in that position that their comparative motions, as determined on the supposition of their being in it exactly, are sufficiently near to the truth for practical purposes. In such cases let A and B, figs. 179 and 180, be the

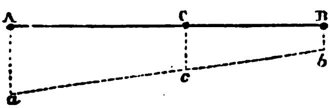

Fig. 179.

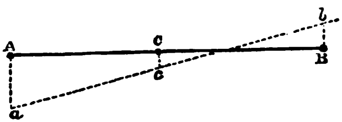

Fig. 180.

two points whose velocities at a given instant are given, and C the third point. Draw A a and B b perpendicular to A B, and of lengths proportional to the given velocities, and in the proper direction; join a b; draw C c also perpendicular to A B, cutting a b in c; C c will represent the velocity of C. The following formula is the symbolical expression of the same rule, in which a, b, and c denote the velocities of A, B, and C respectively:—

$$c = \frac{a \cdot \mathrm{B\,C} + b \cdot \mathrm{C\,A}}{\mathrm{A\,B}}.$$

The formula, as it stands, is applicable to the case in which C lies between A and B, the velocities a, b, and c being treated as positive or negative according to their directions. When C lies beyond B, B C is to be treated as negative, and C A as positive; when beyond A, C A is to be treated as negative, and B C as positive.

The velocity of C may be regarded as the resultant of two components, $\frac{a \;\mathrm{B\,C}}{\mathrm{A\,B}}$, which would be its velocity if B were fixed; and $\frac{b \cdot \mathrm{C\,A}}{\mathrm{A\,B}}$, which would be its velocity if A were fixed.

Trains of aggregate linkwork may be used to combine any number of component motions. For example, in fig. 181, A, B, D, and E receive motion from four different drivers: C has a motion whose components depend on the motions of A and B, and F a motion whose components depend on the motions of D and E; and G has a motion whose components depend on the motions of C and F, and therefore on the motions of A, B, D, and E, jointly.

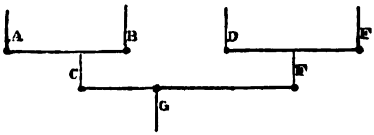

Fig. 181.

239. **Harmonic Motion in Aggregate Linkwork.**—By *harmonic motion* is to be understood the motion of a point which moves to and fro in a straight line in such a manner that its velocity at every instant is equal to the component, parallel to that straight line, of another point which revolves uniformly in a circle. The length of the straight line is called the *travel* of the reciprocating point, and is equal to the diameter of the circle. (As to the component velocities of a revolving point, see Article 55, pages 34, 35.)

Harmonic motion is exactly realized by any point in a slot-headed sliding rod, driven by an uniformly rotating crank, as explained in Article 159, page 169. The angle which the crank makes with its dead points is called, in mathematical language, the *phase* of the motion. The velocity of the reciprocating point varies proportionally to the *sine of the phase;* and the distance of that point from its middle position varies as the *cosine of the phase.*

Harmonic motion is approximately realized by any point in a piece, such as a piston, which is driven by means of a connecting-rod and an uniformly rotating crank. The extent of error in that approximation may be expressed either in the form of *greatest error in position* or of *greatest error in velocity.* The greatest error in position is the distance of the reciprocating point from the middle of its travel, when the crank is midway between its dead points; and when the line of stroke passes through the axis of the crank, its value may be found either by constructing a figure, or by the following formula:—

$$l - \sqrt{l^2 - c^2};$$

in which l denotes the length of the line of connection, and c that of the crank-arm. The *comparative error in position* is the ratio of this error to the half-travel c; that is to say,

$$\frac{l}{c} - \sqrt{\frac{l^2}{c^2} - 1};$$

which, when l is many times greater than c, is nearly equal to

$\frac{c}{2\,l}$. The greatest error in velocity is the proportionate excess of the greatest velocity of the reciprocating piece above that of the crank-pin, as found by the rules of Article 188, pages 199 to 201. When l is not less than $2\,c$, the value of the error in velocity is given approximately by the expression

$$\sqrt{1 + \frac{c^2}{l^2}} - 1,$$

or $\frac{c^2}{2\,l^2}$, nearly.

When the line of stroke does not pass through the axis of rotation of the crank, there are other errors arising from the two dead points not being diametrically opposite. Those errors may be found by applying the rules of Article 196, page 198.

The present and the following Article relate to cases in which two points in a bar receive given transverse movements, which are either exactly harmonic, or so nearly so that they may be treated as harmonic for practical purposes, and are also of equal period, and have a given constant difference of phase; and it is required to find the extent of travel and the relative phase of the motion of a third point, situated either exactly or nearly in one straight line with the first two.

The following is the general rule for the solution of all such cases. Some of its applications will be given in the next Article:—

RULE.—In fig. 182 draw the straight line A B to represent the bar in question, and let A and B represent the points whose motions are given, and C the point whose motion is to be found. Perpendicular to A B, draw A a to represent the half-travel of A, and B b to represent the half-travel of B. These distances may be laid off in both directions, so that a a shall represent the whole travel of A, and b b that of B. The *difference of phase* of A and B is supposed to be given; that is to say, A moves as if driven by a crank A′ A″ (= A a), and B as if driven by a crank B′ B″ (= B b), which cranks rotate with the same angular velocity, and make a given constant angle with each other.

At A and B lay off the angles B A D = A B D, each equal to half the difference of phase; and about the triangle A D B describe a circle. Join a b, a b, and through the point of intersection, E, draw the straight line D E, cutting the circle in F. Join F A, F B; then the angle A F B will be equal to the given difference of phase. Lay off F a' = A a, and F b' = B b; then F a' and F b' will represent the two cranks which actually or virtually drive A and B, in their angular position relatively to each other. Join a' b'; this will be parallel to A B (because it can be shown by plane geometry that F A B and F a' b' are similar triangles). Finally, draw the straight line F C, cutting a' b' in c'; then the point C will

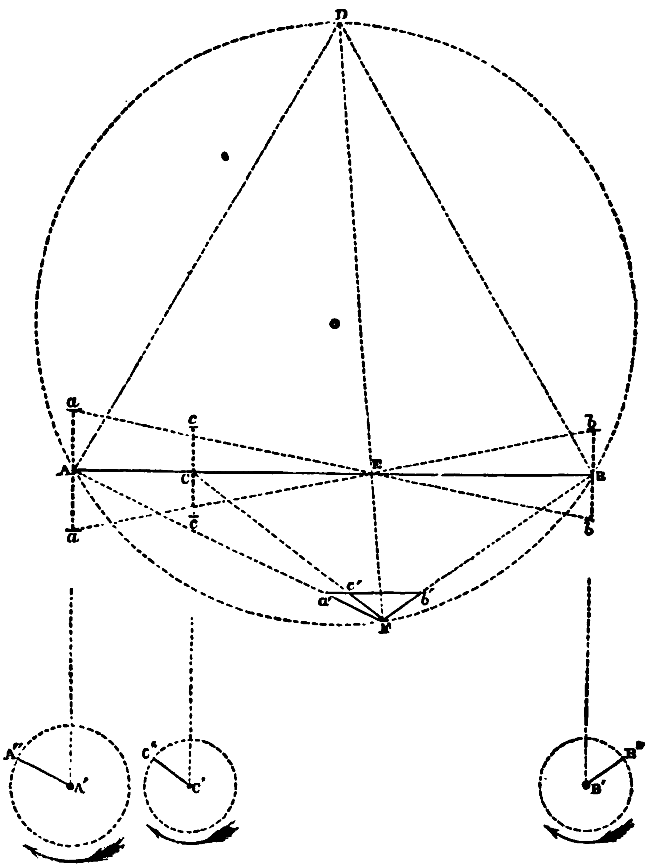

Fig. 182.

move almost exactly as if it were driven by a crank-arm, C′ C″, equal in length to F c', and having the angular position relatively to the cranks that drive A and B which F c' has relatively to F a' and F b'; that is to say, being in advance of the crank which drives A by the angle a' F c', and behind the crank which drives B by the angle b' F c'.

The travel of C may be represented in the figure by drawing, perpendicular to A B, the straight line $c\ c = 2\ C\ c = 2\ F\ c'$.

When the extent of travel of A and B is the same, part of the trouble of the construction is saved; for the point F is found simply by laying off the angles B A F = A B F, each equal to half the supplement of the difference of phase.

The construction which has been described solves the problem by drawing alone. Sometimes it may be convenient to use calculation combined with drawing; and then the whole process consists in drawing the triangle F a' b' in any convenient position, with its legs, F a' and F b', equal to the half-travel of the points A and B respectively, and its angle, a' F b', equal to the difference of phase of their motions, and dividing, by calculation, the base a' b' at c' in the same proportion in which A B is divided at C.

240. **Link-Motions for Slide-Valves** belong to the kind of combinations mentioned in the preceding Article. The bar which receives harmonic motion is called the *link;* it is in general slightly curved, and only sometimes straight. Two points in it, marked A and B in figs. 183 to 186, receive approximately-harmonic motions from two eccentrics, E and F, on the engine-shaft, O, called respectively the forward and the backward eccentrics. The link carries a slider, C. That slider is attached to the head of the slide-valve spindle either directly (as shown at C in figs. 183, 184, and 185), or by means of an intermediate rod, C X (as in figs. 186, 187). The slider is capable of being adjusted to different positions in the link, either by shifting the link (as in figs. 183, 184, and 185, which represent Stephenson's link-motion) or by shifting the slider (as in fig. 186, which represents Gooch's link-motion), or by shifting the link and the slider at the same time in opposite directions (as in Allan's link-motion, represented in fig. 187). In Stephenson's link-motion the form of the link is an arc of a circle, concave towards the shaft, and of a radius equal to the length of the eccentric rods E A, F B. In Gooch's link-motion the figure of the link is an arc of a circle described about the head, X, of the valve-spindle. In Allan's link-motion the link is straight, and the adjustment of the proportions of the mechanism for shifting it will be described presently. In each case the object is, that the shifting of the position of the slider, C, relatively to the link, A B, shall not cause any sensible alteration of the middle position of the slide-valve. In each of the figures, O D represents the crank of the engine to which the link-motion belongs, the positions of the parts being those which they take when that crank is at a dead-point.

In each of the figures, also, the eccentrics are represented simply by points, E, F, which mark the centres of the eccentric discs. It has already been explained, in Article 195, page 197, that an eccentric is equivalent to a crank whose arm coincides with the eccentricity; that is, the distance from the axis of rotation to the centre of figure of the disc.

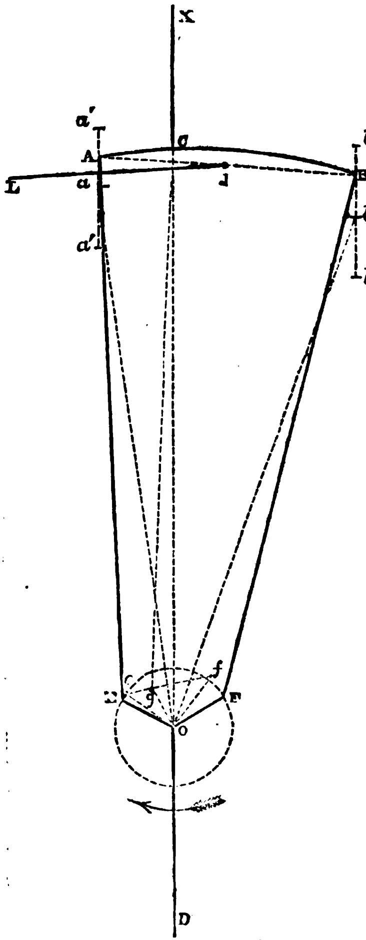

Fig. 183.

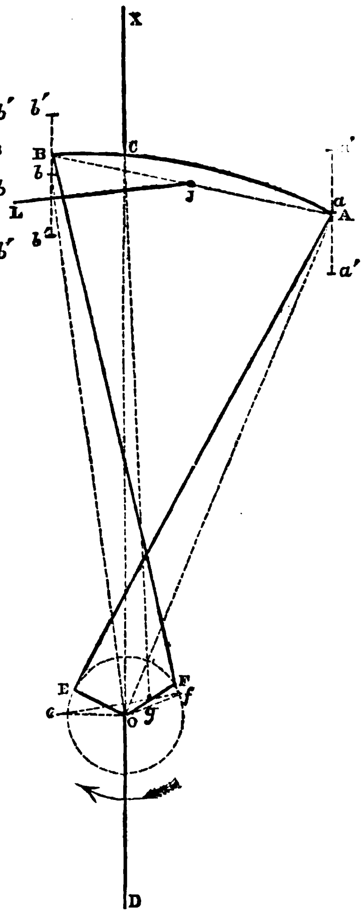

Fig. 184.

The general problem in questions as to the action of link-motions is this: the dimensions of the parts being given, and the angles made by the eccentric-arms O E, O F, with the crank O D; also the position of the slider C in the link A B; to find the length and positions of what may be called the *single virtual eccentric-arm* O g; that is to say, the arm of a single eccentric, which, if connected with the valve-spindle by a rod marked g C in figs. 183 and 184, and $g\ c$ in figs. 186 and 187, would produce, *approximately*, the same motion of the valve-spindle that the actual mechanism produces.

The solution of that problem consists generally of two steps; the first being to find the two virtual eccentric arms, O e, O f, which, on the supposition of the eccentric rods, E A, F B, being indefinitely long, or of slotted cross-heads being used instead of eccentric rods, would be equivalent in their action to the actual eccentrics with their oblique rods; and the second step being to find the single virtual eccentric arm, O g, whose action is equivalent to the combined action of those two, on the same supposition of an indefinite length of rod or a slotted cross-head being used.

There are two different arrangements of the eccentric rods, which are said to be *crossed* or *open* according as they cross each other or not when the crank O D is pointing away from the cylinder. In figs. 183 and 186 the rods are open; in figs. 184 and 187 they are crossed. The two following rules apply to either arrangement:—

I. *To find the virtual forward and backward eccentric arms.* Through A and B draw straight lines parallel to the line of stroke, O X, of the valve-spindle, and mark on those straight lines the ends of the travel of the points A and B respectively, a', a', and b', b'. These points are to be found by Rule I. of Article 186, page 198. Bisect $a'\ a'$ in a, and $b'\ b'$ in b. Join O a and O b; these straight lines will pass *nearly*, though not exactly, through the dead-points of the eccentrics E and F respectively. Lay off the angles X O e = a O E and X O f = b O F, and make O e = $a\ a'$ and O f = $b\ b'$. Then O e and O f will be the required virtual forward and backward eccentric arms.

II. *To find the single virtual eccentric arm.* Draw the straight line $e\ f$, and in it take the point g, dividing $e\ f$ in the same proportion in which the slider C divides A B. O g will be the required virtual single eccentric arm.

The preceding rules are applicable to all the three constructions of link-motion. But for each particular construction there are special rules by which the process may be simplified, to the extent of dispensing with the whole or part of the detailed process, except for certain principal positions of the slider in the link.

III. In *Gooch's Link-Motion* (fig. 186) the link is hung or attached to a fixed pin by means of the rod L J, and the alteration of the position of the slider C in it is effected by shifting a lever (not shown), one end of which is connected, by means of the rod N M,

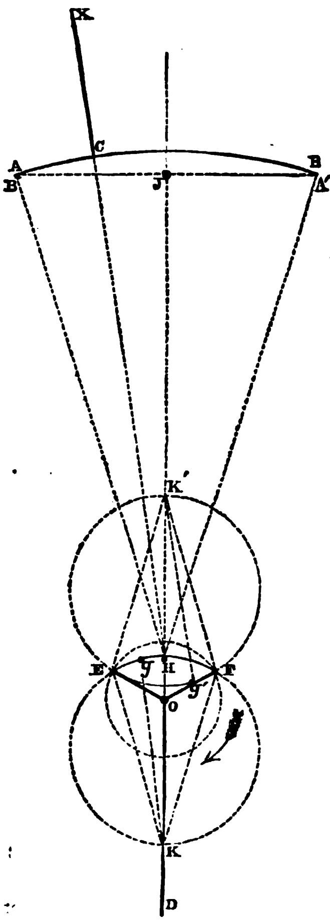

Fig. 185.

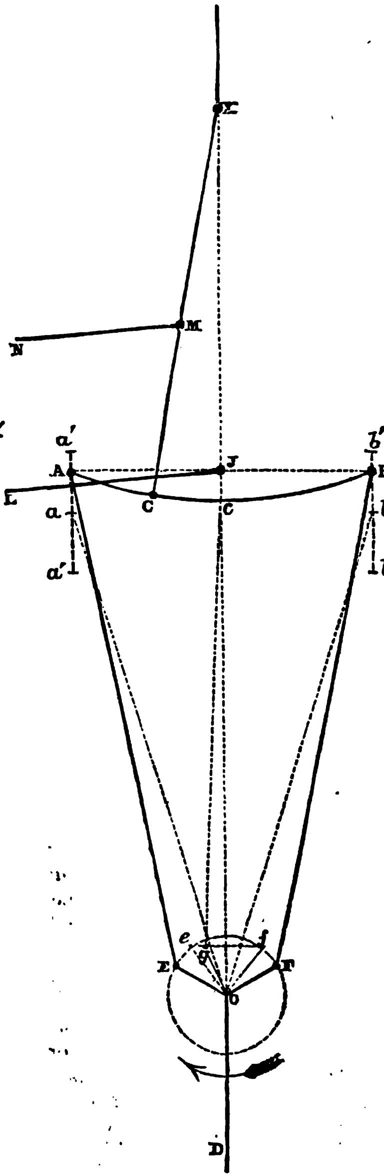

Fig. 186.

with the valve-rod C X. The result is that the virtual forward and backward eccentric arms O e and O f are the same for all positions of the slider in the link, and have only to be found once for all. The figure shows this motion with open eccentric-rods; and the angle e O f = E O F − a O b; but it is also made with crossed eccentric-rods, and then the angle e O f = E O F + a O b.

IV. In *Stephenson's Link-Motion* (figs. 183, 184) the link is attached by the rod J L to the end of a lever (not shown); and by shifting that lever the position of the link relatively to the slider is changed when required. The consequence is that the virtual eccentric arms O e and O f are different in length and position for every different position of the slider in the link; and the application of the general Rules I. and II. to a variety of such positions becomes a tedious process. The time and labour, however, required for that process are to a great extent saved by using the following approximate method, which is sufficiently accurate for practical purposes:—When the link-motion is in full forward gear—that is, when C coincides with A—the actual forward eccentric arm O E is itself the virtual eccentric arm. When the link-motion is in full backward gear—that is, when C coincides with B—the actual backward eccentric arm O F is itself the virtual eccentric arm. For intermediate positions proceed as follows (see fig. 185):—Draw the link A B in that position in which the straight line A B is parallel to the straight line E F, and let H be the centre of curvature of the link when in that position; its radius, H A = H B, being equal to the length of each of the eccentric-rods E A, F B. Then, *if the eccentric-rods are open,* draw E K parallel to A H, and F K parallel to B H, cutting each other in K; and through the three points E, F, K describe a circle; the arc E g F of that circle will be a very close approximation to the curve that contains the ends of all the virtual single eccentric radii. For a given position, C, of the slider, take the point g, dividing the arc E F in the same proportion in which C divides A B; and O g will be the required virtual eccentric radius. *If the eccentric-rods are crossed,* draw E K′ parallel to A′ H, and F K′ parallel to B′ H, cutting each other in K′; and through the three points E, F, K′ describe a circle. The arc E g' F of that circle will be a very close approximation to the curve containing the ends of all the virtual single eccentric radii. In this arc take the point g', dividing it in the same proportion in which C divides the link; and O g' will be the required virtual eccentric radius.

V. *Allan's Link-Motion—Centre of the Shifting-Lever.*—In Allan's link-motion (fig. 187) the middle point, J, of the link, and any convenient point, M, in the valve-rod C X, are attached by rods to the two ends of a lever, N L, which turns about an axis at P; and the position of the slider C in the link A B is adjusted by moving the handle P Q. In order that the motion may work correctly, the centre of the shifting-lever must be properly placed; and the following is the rule for finding its position:—In fig. 188,

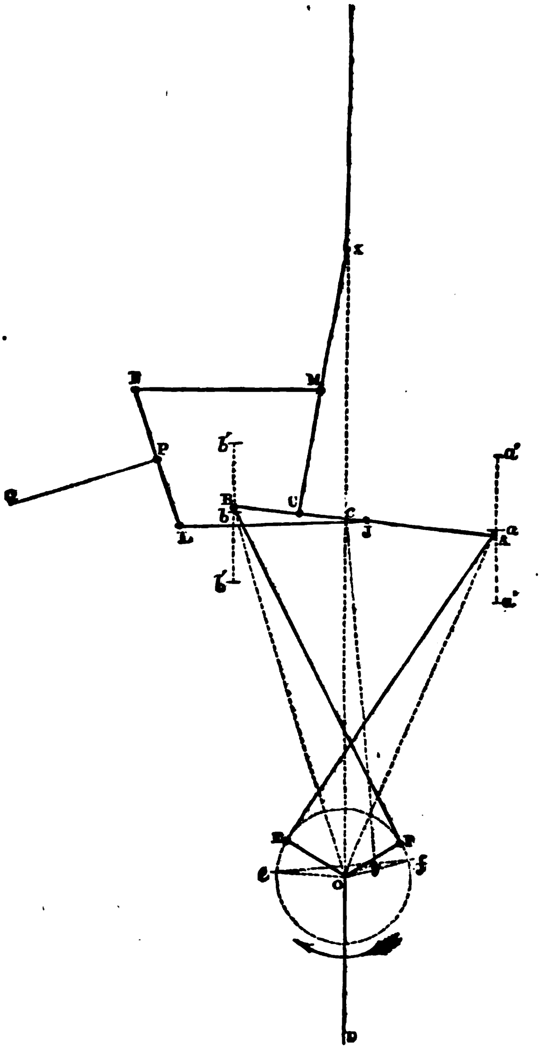

Fig. 187.

draw the isosceles triangle O A B, in which A B is the length of the link, and O A and O B are each equal to the length of an eccentric-rod. About O draw a circular arc through A and B. Bisect the straight line A B in J; join O J, and produce it, making J X equal to the length of the valve-rod; and mark, in J X, the position of the point M to which the rod M N is to be attached. About X, with the radius X J, draw a circular arc, J B′, cutting the arc through A and B in B′. Make B′ A′ = B A, and O A′ = O A; join B′ A′, and bisect it in J′. Also join X B′; and in it take X M′ = X M. Then M M′ and J J′ will be the distances through which M and J are respectively to be shifted in order to shift the slider C from mid-gear to full gear—that is, from J to B, or to A, as the case may be. Draw the straight line M′ J′, cutting M J in R, and through R draw R P perpendicular to O X; then the axis of the shifting lever is to be placed at a convenient point in the line R P.

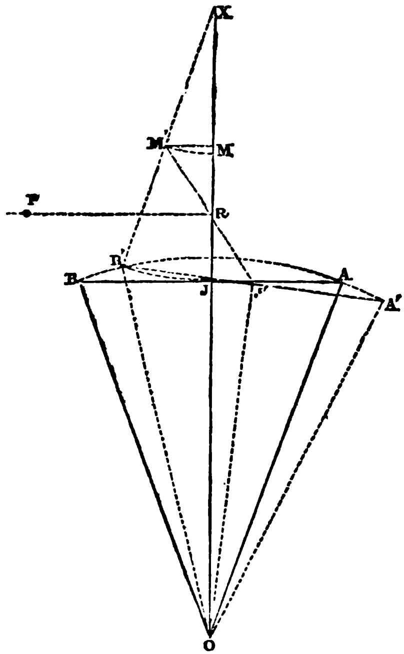

Fig. 188.

VI. *Allan's Link-Motion—Virtual Eccentric Arm.*—By the application of the general Rules I. and II., find the virtual eccentric arms in full forward gear, O *e*; in full backward gear, O *f*; and in mid-gear, O *g*; and draw them in a diagram, as in fig. 189. Then through the points *e*, *g*, and *f* draw a circular arc; this will be approximately the curve containing the ends of all the virtual eccentric arms for different positions of the slider; and it is to be used like the corresponding curve for Stephenson's link-motion. If the rods are open, this arc will be concave towards O, as *e g f*; if crossed, convex, as *e g′ f*.*

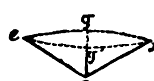

Fig. 189.

* On the subject of Link-Motions, see Zeuner's *Schiebersteuerungen*, and M'Farlane Gray's *Geometry of the Slide-Valve.*

It is to be remarked that in order to show distinctly the principles of the construction of the figures illustrating this article, all those dimensions which give rise to errors—that is, to deviations from the exact law of harmonic motion—are exaggerated; such as the lengths of the link A B, and of the eccentric-arms O E and O F, as compared with that of the eccentric-rods E A and F B. In ordinary practice the link is from one-third to one-fifth, and each of the eccentric-arms about one-twentyfifth of the length of an eccentric-rod; and the effect of these proportions is to make the deviation of the resultant motion of the slide-valve from true harmonic motion practically inappreciable.

241. **Differential Harmonic Motions.**—Two primary pieces, having different motions, may be regarded as constituting an aggregate combination with respect to the motion of one of them relatively to the other; because that motion is the resultant of two components: for example, if A be taken to denote the frame, and B and C the two primary pieces, the motion of C relatively to B is the resultant of the motion of B relatively to A, and of a motion equal and contrary to the motion of B relatively to A. This principle has been already stated in Article 42, page 21. The following is one of its most frequent applications:—In fig. 190, let *b* and *c* be two pieces which have approximately harmonic motions in parallel directions and of equal periodic time, but differing in phase, given to them respectively by two cranks or eccentric-arms, A B and A C, which turn as one piece with the same angular velocity about axis A, the angle B A C being the difference of phase. Then the motion of the slide *c* relatively to the slide *b* is approximately the same with that which would be produced by a crank or eccentric-arm, B C, turning with the same angular velocity; that is to say, it is an approximately harmonic motion of the same periodic time with the two elementary motions of B and C; its half-travel is equal to B C, and its phase at any instant is that corresponding to the direction of B C at that instant. This is what may be called a differential harmonic motion; and upon such motions depends the action of double slide-valves and moveable slide-valve seats in steam engines.

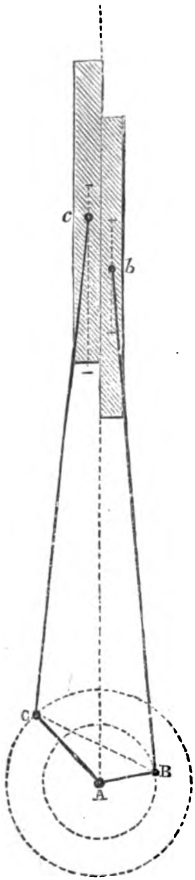

Fig. 190.

SECTION IV.—*Production of Curved Aggregate Paths.*

242. **Circular Aggregate Paths.**—Some circular aggregate paths are traced by means of mechanical combinations, which are capable also of tracing ellipses, if required; and these will be described further on. The present Article relates to combinations in which circular paths alone are traced.

Amongst such combinations may be classed the coupling-rod shown in fig. 32, Article 68, page 44; for every point in or rigidly attached to that rod traces a circle of a radius equal in length to the crank-arms by which the rod is carried; and the same takes place in every case in which a secondary piece has a motion of circular translation without rotation. For example, in fig. 191, A is a centre pin, carrying a fixed spur-wheel—in other words, a spur-wheel without rotation. About the axis of that wheel there turns a disc, carrying a set of diverging epicyclic trains. Each epicyclic train consists of a spur-wheel, B, gearing with the fixed wheel A, and another spur-wheel, C, gearing with B. The last spur-wheel, C, is exactly equal in radius and in number of teeth to the fixed wheel A; and the consequence is, that each of the wheels marked C has an angular velocity equal to that of A—that is to say, equal to nothing: in other words, when the disc rotates, the wheels marked C have a motion of circular translation without rotation. Let E be any point in one of the wheels C; and draw A D equal and parallel to C E; then E traces a circle round D, exactly equal to the circle which the centre C of the wheel to which E belongs traces round A at the same time. This combination is used in spinning wire ropes. Each of the wheels C carries a bobbin from which a wire or a strand is paid out as the spinning goes on; and the effect of the absence of rotation in the wheels C is, that the wires or strands are spun together without being twisted, which would overstrain the material.

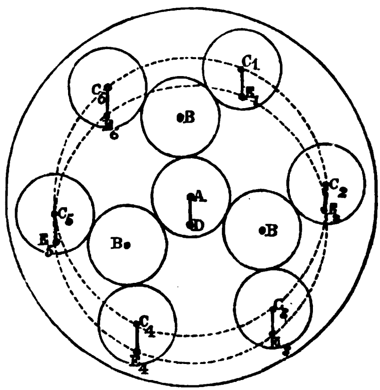

Fig. 191.

The combination shown in fig. 192 serves to guide a point C along an arc, B C A, of a circle of a radius so great that it would be inconvenient to guide the point C by connecting it directly with the centre of the circle. It is based upon this well-known geometrical principle:—Let A and B be any two fixed points in the circular arc to be traced; then the two chords C A, C B make with each other a constant angle at C—viz., the supplement of one-half of the angle which the arc A B subtends at the centre of the circle. Two rods are fastened together at C, so as to make with each other the proper constant angle; and they are guided by passing through sockets at A and B, which sockets are free to turn about A and B respectively, but not to move otherwise. Then, when the rods are made to slide through the sockets, the point C traces the required circular arc. The angle made by the rods with each other may be made adjustable by means of a screw or otherwise, so as to vary the curvature of the arc when required.

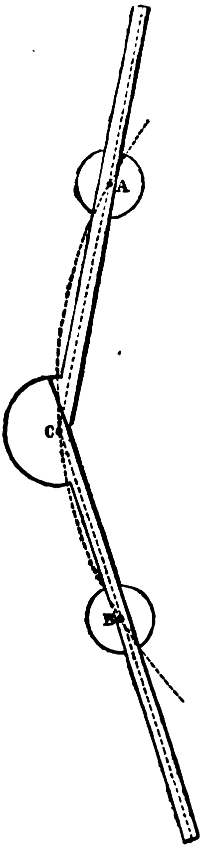

Fig. 192.

243. **Epitrochoidal Paths.**—An epitrochoid is the curve traced by a point in or rigidly attached to a circle which rolls either inside or outside of another circle (called the *base-circle*); also, if two circles (as the pitch-circles of two spur-wheels) turn in rolling contact with each other about fixed axes, a point rigidly attached to one of those circles traces an epitrochoid upon a disc rigidly attached to the other.

When the tracing-point is *in the circumference* of the rolling-circle, the curve traced becomes that particular kind of epitrochoid that is called an *epicycloid*. The properties of this curve have already been explained in Article 78, page 56, with a view to its adaptation to the figures of the teeth of wheels.

If the circumferences of the rolling-circle and of the base-circle are commensurable with each other, the epitrochoid returns into itself, and has a finite number of lobes or coils—viz., the denominator of the fraction which, being in its least terms, expresses the ratio borne by the circumference of the rolling-circle to that of the base-circle. If those circumferences are incommensurable, the epitrochoid does not return into itself, so that the number of its lobes or coils is indefinite.

When the rolling-circle rolls outside a base-circle of equal-radius, the epitrochoid is one-lobed, and is called a *cardioid*.

In the examples shown in figs. 193, 194, and 195 the ratio of

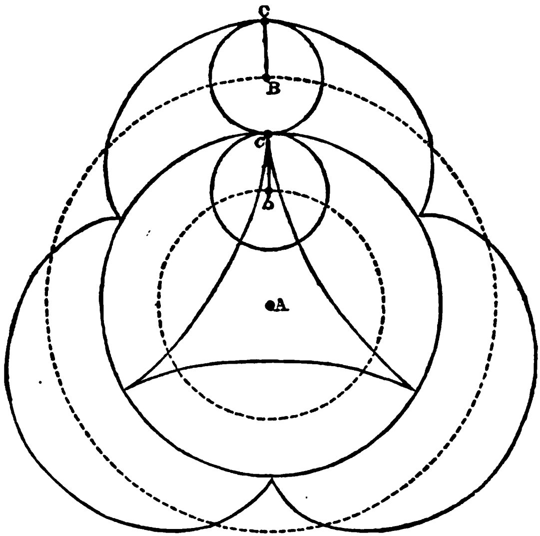

Fig. 193.

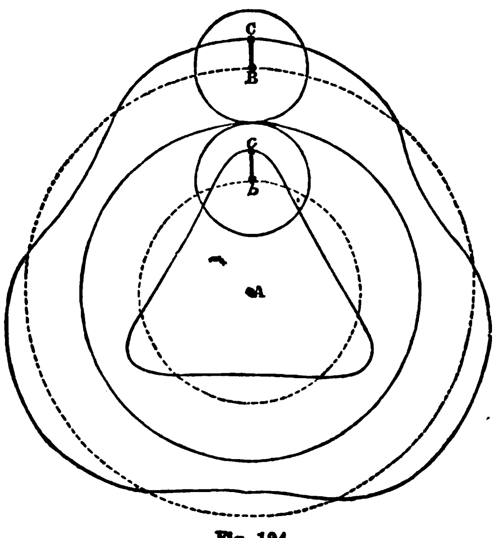

Fig. 194.

the rolling-circle to the base-circle is $\frac{1}{3}$, so that the epitrochoids are three-lobed. Each figure shows an external and an internal epitrochoid, traced by rolling the rolling-circle outside and inside the base-circle respectively. The centres of the base-circles are marked A; those of the external rolling-circles, B; those of the internal rolling-circles, *b*; and the tracing-points of the external and internal rolling-circles are marked C and *c* respectively.

In fig. 193 the tracing-points are in the circumferences of the rolling-circles; and the curves traced are epicycloids, distinguished by having *cusps* at the points where the tracing-point coincides with the base-circle. In fig. 194 the tracing-points are inside the rolling-circles; and the curves traced are *prolate epitrochoids*, distinguished by their wave-like form. In fig. 195 the tracing-

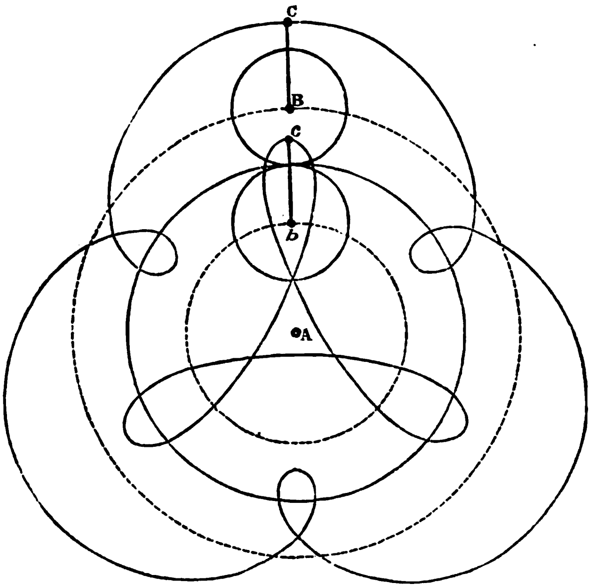

Fig. 195.

points are outside the rolling-circles; and the curves traced are *curtate epitrochoids*, distinguished by their looped form.

An important property of curves traced by rolling has already been mentioned—viz., that at every instant the straight line joining the tracing-point and the pitch-point, or point of contact of the rolling-curve and base-curve, is normal to the traced curve at the tracing-point.

The distance B C or *b c* may in each case be called the *tracing-arm.*

In mechanism for the tracing of epitrochoids (used chiefly in ornamental turning), the rolling and base-circles are the pitch-circles of a pair of spur-wheels, made with great accuracy. (See page 290.)

244. **Rolled Paths in General.**—An infinite variety of curves may be traced by rolling different pairs of non-circular curves upon each other. In practice it is most convenient to limit this process to pairs of non-circular curves which are capable of turning in rolling contact about fixed parallel axes; that is to say, which are suitable for the pitch-lines of wheels, according to the principles explained in Article 107, page 92. Suppose any such pair of pitch-lines to turn in rolling contact with each other; then a point rigidly attached to one of them (the rolling-line) will trace upon a disc rigidly attached to the other (the base-line) a third curve; at every point in that third or traced curve the normal will be the straight line traversing the tracing-point and the pitch-point, or point of contact of the rolling-line and base-line.*

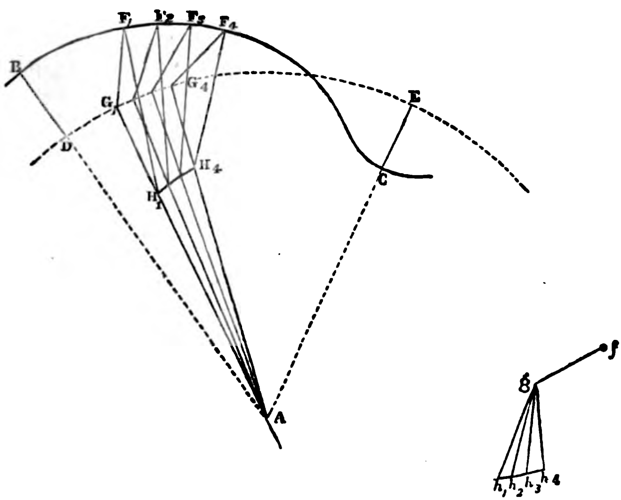

Fig. 195 A.

* The following rule gives an approximate method of determining the figures of a rolling-curve and base-curve suited for tracing one lobe of a given curve, B C, about a given pole, A (see fig. 195 A); the term *pole* being used to

245. **Elliptic Paths traced by Rolling** form a particular case of internal epitrochoids. In fig. 196 is represented a rolling-circle, which rolls inside a base-circle of exactly twice its radius. Then

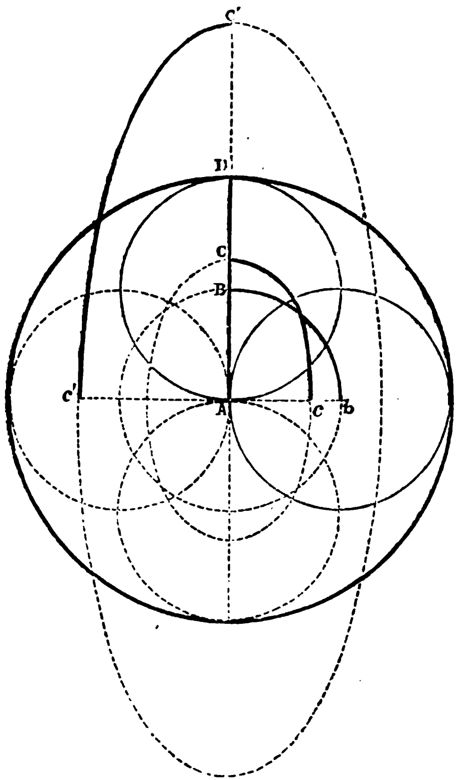

Fig. 196.

denote the trace of the axis of the base-wheel upon the plane of the given curve:—

Let A B and A C be the greatest and least distances of the given curve from the pole. Make the line of centres $A D = A E = \frac{A B + A C}{2}$; and the tracing-arm $D B = C E = \frac{A B - A C}{2}$. With the line of centres A D as radius, draw a circle, D E; this will be the path of the axis of the rolling-curve. Take a series of points, F_1, F_2, F_3, &c., in the curve to be traced, and from each of them lay off, to the circumference of the circle, D E, a distance, F G, equal to the tracing-arm; thus finding a corresponding (considering a quarter of a revolution at a time), while the centre of the rolling-circle traces a quadrant, B b, of an equal circle about A, a point D in the circumference of the rolling-circle traces a straight line traversing A, and a point C, inside the rolling-circle, traces a quadrant, C c, of an ellipse whose semiaxes are A C = A B + B C, and A c = C D = A B − B C; also a point C′ outside the rolling-circle, but rigidly attached to it, traces a quadrant, C′ c', of an ellipse whose semiaxes are A C′ = B C′ + A B, and A c' = C′ D = B C′ − A B. The former may be called an *internal*, and the latter an *external*, ellipse. The proportions of the axes of either of them may be indefinitely varied by adjusting the position of the tracing-point; but in every internal ellipse the sum, and in every external ellipse the difference, of the semiaxes is equal to the diameter of the rolling-circle—that is, to the radius of the base-circle.

This is the principle of the mechanism commonly used for turning ellipses. (See Addendum, page 290.)

It is evident that by having a number of tracing-points carried by one rolling-circle, several ellipses differently proportioned and in different positions may be traced at the same time.

246. A **Trammel** is a substitute for a pair of rolling-circles suited for tracing ellipses; but it is less used in mechanism than in drawing instruments. It depends on the following principles:—That every point in the circumference of the rolling-circle in fig. 196 traces a straight line through A; and that consequently, if two points in a rigid body, so chosen as to be in the circumference of one circle through A, be so guided as to move in straight lines traversing A, the whole body will move as if it were carried by that circle, rolling inside a circle of twice the radius.

In fig. 197, let X X and Y Y be the centre lines of two straight grooves, cutting each other in A; and let B and C be the centres of two pins which slide along those grooves respectively at an invariable distance, B C, from each other. Through B, perpendicular to X X, and through C, perpendicular to Y Y, draw the straight lines B G, C G, cutting each other in G; this point will evidently be the trace of the *instantaneous axis* of a rigid body attached to B and C. Join A G, bisect it in D, and about D draw a circle

series of points, G_1, &c., in that circle. (Two only of these points are lettered, to prevent confusion.)

Draw the radii A G_1, &c. Then, through the points F_1, &c., draw a series of normals, F_1 H_1, &c., cutting their corresponding radii in a series of points, H_1, &c. These will be points in the *base-curve*.

To construct the *rolling-curve*, draw, in a separate diagram, the tracing-arm $g f$ = G F; draw the radii g h, &c., equal to the corresponding lines G_1 H_1, &c., in the original diagram, and making the angles f g h_1, &c., equal to the corresponding angles F_1 G_1 H_1, &c., in the original diagram; then h_1, &c., will be points in the required rolling-curve; g being the trace of its axis, which is at the invariable distance A D from the axis of the base-curve; and f the tracing-point.

through A and G. That circle will also traverse B and C, because the angles A B G and A C G are right angles. Now, the length of

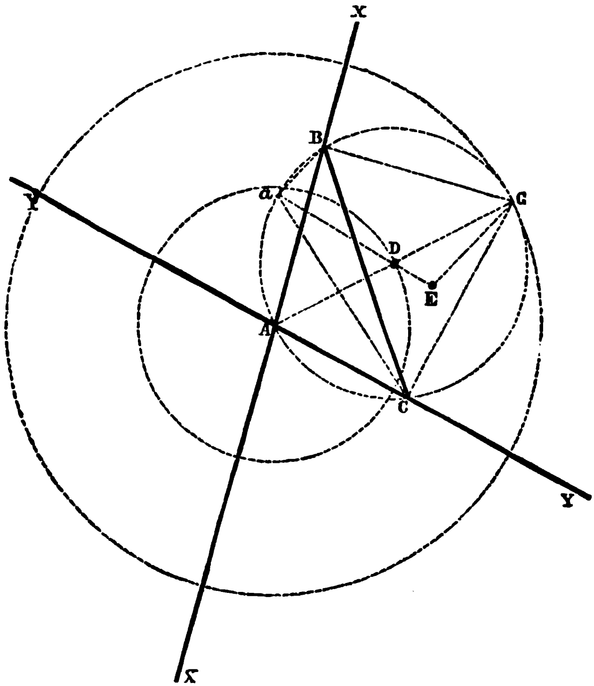

Fig. 197.

the chord B C in that circle is constant, and it subtends at the circumference the constant angles Y A X and B G C; therefore the diameter A G of that circle is constant in all positions of the pins B and C as they slide along the grooves; therefore the several positions of the instantaneous centre, G, are all in one circle described about A; therefore the motion of a rigid body attached to the pins B and C is the same as if it were carried by the circle A B G C rolling inside a circle of twice the radius described about A. Hence the point D in such a rigid body traces a circle about A; and any other point, such as E, traces an ellipse whose semi-axes are respectively equal to A D + D E and A D − D E. The straight line G E is at every instant a normal to the ellipse traced by the point E.

In the ordinary form of the trammel, represented in fig. 198, the grooves X X and Y Y are at right angles to each other; and the moving rigid body is a straight rod, B C F, carried by two pins at B and C, which are fixed in blocks that slide along the grooves.

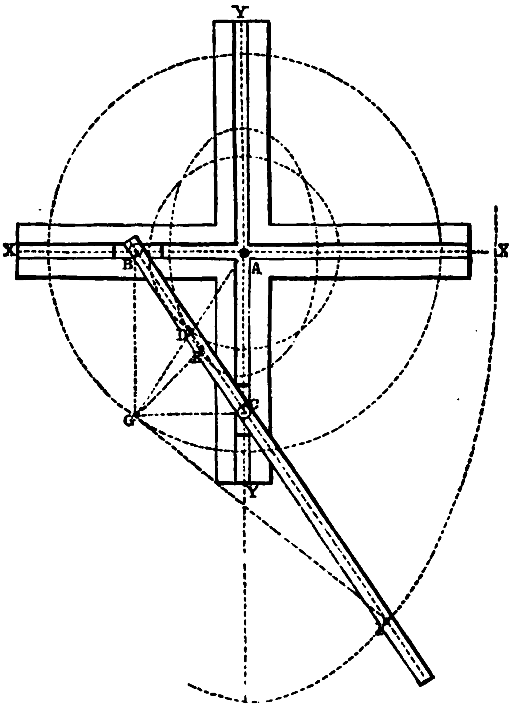

Fig. 198.

The tracing-point is moveable, and can be adjusted to any required position along the rod. When midway between B and C, at D, it traces a circle of the radius $\frac{B\,C}{2}$; when at any other intermediate point, such as B, it traces an ellipse whose semiaxes are equal respectively to E B and E C; and if the rod be prolonged, a point F in the prolongation traces an ellipse whose semiaxes are equal respectively to F B and F C. In each case the axes of the ellipse coincide with the centre lines of the grooves X X and Y Y.

When the trammel is oblique-angled at A, the positions of the axes of the ellipse described by a given tracing-point, E (fig. 197), are found as follows:—Produce E D till it cuts the virtual rolling-circle in a; join B a, C a; then E a B and E a C will be equal to the angles made by the longer axis with X X and Y Y respectively; and the shorter axis will of course be perpendicular to the longer.*

247. **Feathering Paddle-Wheels** exemplify a class of aggregate combinations in which linkwork is the means of producing the aggregate motion. Each of the paddles is supported by a pair of journals, so as to be capable of turning about a moving axis parallel to the axis of the paddle-wheel, while its position relatively to that moving axis is regulated by means of a lever and rod connecting it with another fixed axis. Thus, in fig. 199, A is the axis of the paddle-wheel; K the other fixed axis, or eccentric-axis; B, E, N, C, P, M, D the axis of a paddle at various points of its

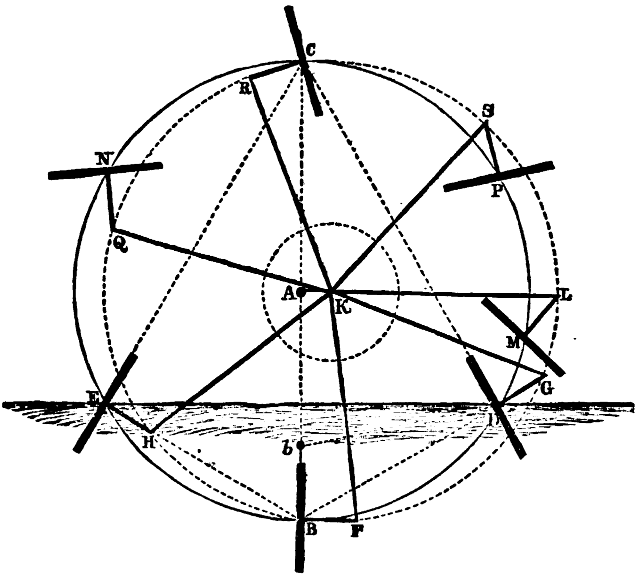

Fig. 199.

revolution round the axis A of the wheel; B F, E H, N Q, C R, P S, M L, D G, the *stem-lever* of the paddle in various positions; K F, K H, K Q, K R, K S, K L, K G, various positions of the guide-rod which connects the stem-lever with the eccentric-axis.

When the end of the paddle-shaft *overhangs*, and has no outside

* The oblique-angled trammel is believed to be the invention of Mr. Edmund Hunt.

bearing, the eccentric-axis may be occupied by a pin fixed to the paddle-box framing; but if the paddle-shaft has an outside as well as an inside bearing, the inner ends of the guide-rods are attached to an *eccentric collar*, large enough to contain the paddle-shaft and its bearing within it, and represented by the small dotted circle that is described about K. One of the rods, called the *driving-rod*, is rigidly fixed to the collar, in order to make it rotate about the axis K; the remainder of the rods are jointed to the collar with pins.

The object of the combination is to make the paddles, so long as they are immersed, move as nearly as possible edgewise relatively to the water in the paddle-race. The paddle-race is assumed to be a uniform current moving horizontally, relatively to the axis A, with a velocity equal to that with which the axes B, &c., of the paddle-journals revolve round A. Let E be the position of a paddle-journal axis at any given instant; conceive the velocity of the point E in its revolution round A to be resolved into two components,—a normal component perpendicular, and a tangential component parallel, to the face of the paddle. Conceive the velocity of the particles of water in the paddle-race to be resolved in the same way. Then, in order that the paddle may move as nearly as possible edgewise relatively to the water, the normal components of the velocities of the journal E and of the particles of water should be identical.

Let B be the lowest point of the circle described by the paddle-journal axes; that is, let A B be vertical. Draw the chord E B. Then it is evident that the component velocities of the points B and E along E B are identical. But the velocity of B is identical in amount and direction with that of the water in the paddle-race. Therefore the face of a paddle at E should be normal to the chord E B, or as nearly so as possible. Another way of stating the same principle is to say that a tangent, E C, to the face of the paddle should pass through the *highest point*, C, of the circle described by the paddle-journal axes, C A B being the vertical diameter of that circle.

It is impossible to fulfil this condition exactly by means of the combination shown in the figure; but it is fulfilled with an approximation sufficient for practical purposes, so long as the paddles are in the water, by means of the following construction:—Let D and E be the two points where the circle described by the paddle-journals cuts the surface of the water. From the uppermost point, C, of that circle draw the straight lines C E, C D, to represent tangents to the face of a paddle at the instant when its journals are entering and leaving the water. Draw also the vertical diameter C A B, to represent a tangent to the face of a paddle at the instant when it is most deeply immersed. Then draw the stem-lever projecting from the paddle in its three positions, D G, B F, E H. In the figure, that lever is drawn at right angles to the face of the paddle; but the angle at which it is placed is to a certain extent arbitrary, though it seldom deviates much from a right angle. The length of the stem-lever is a matter of convenience: it is usually about $\frac{3}{5}$ of the depth of the face of a paddle. Then, by plane geometry, find the centre, K, of the circle traversing the three points, G, F, and H; K will mark the proper position for the eccentric-axis; and a circle described about K, with the radius K F, will traverse all the positions of the joints of the stem-levers.

From the time of entering to the time of leaving the water, paddles fitted with this feathering gear move almost exactly as required by the theory; but their motion when above the surface of the water is very different, as the figure indicates.

To find whether, and to what extent, it may be necessary to notch the edges of the paddles, in order to prevent them from touching the guide-rods, produce A K till it cuts the circle G F H in L; from the point L lay off the length, L M, of the stem-lever to the circle D B E, and draw a transverse section of a paddle with the axis of its journals at M, its stem-lever in the position M L, and its guide-rod in the position L K. This will show the position of the parts when the guide-rod approaches most closely to the paddle.

Some engineers prefer to treat the paddle-race as undergoing a gradual acceleration from the point where the paddle enters the water to the point of deepest immersion. The following is the consequent modification in the process of designing the gear:—Let the final velocity of the paddle-race be, as before, equal to that of the point B in the wheel, and let the initial velocity be equal to that of the point *b*, at the end of a shorter vertical radius, A *b*. Let D be the axis of a paddle-journal in the act of entering the water, and E the same axis in the act of leaving the water. Join *b* D and B E; draw the face of the paddle at D normal to D *b*, the face of the paddle at B vertical, as before, and the face of the paddle at E normal to E B. Then draw the stem-lever in its three positions, making a convenient constant angle with the paddle-face; and find the centre of a circle traversing the three positions of the end of the stem-lever; that centre will, as before, mark the proper position for the eccentric-axis.

248. **Spherical Epitrochoidal Paths—Z-Crank.**—A point rigidly attached to a cone which rolls on another cone describes a *spherical epitrochoid*, situated in a spherical surface whose centre is at the common apex of the two cones. This sort of aggregate motion is illustrated by Mr. Edmund Hunt's Z-crank.

In fig. 200, A A is a rotating shaft, carrying at B, B, two crank-arms, which project in opposite directions, and are connected with each other by means of a cylindrical crank-pin, B B. The shaft, crank-arms, and crank-pin, are all rigidly fastened together; and they rotate as one piece about the axis A A.

The crank-pin is contained within a hollow cylindrical sleeve or

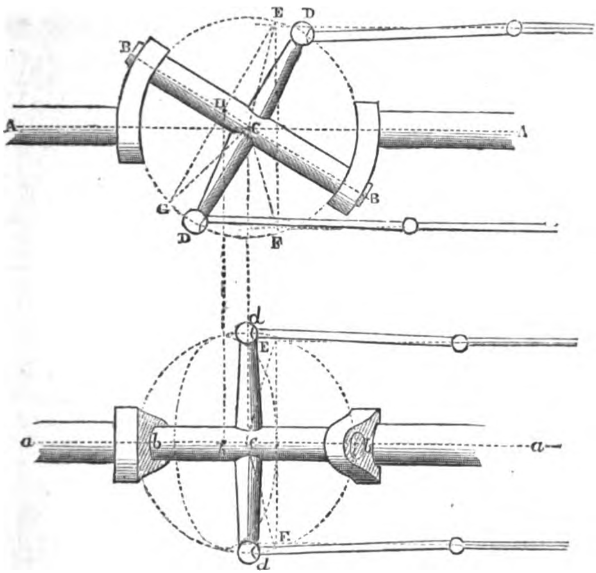

Fig. 200.

tube, fitting it accurately, but not tightly, and free to rotate, relatively to the pin, about the axis B B, but not to shift longitudinally. From that tube, and in a plane normal to B B, and traversing C, the intersection of B B and A A, there project any required number of arms, such as C D, C D. Those arms move as one piece with the tube; and each of them, at its end D, is connected, by means of a ball-and-socket joint, with a link, and through the link with a piston-rod, which has a reciprocating sliding movement parallel to A A.

The lower part of the figure shows the position of the mechanism after a quarter of a revolution has been made from the position represented in the upper part of the figure.

The motion of the sleeve, with its arms, is compounded of a rotation about the fixed axis A A, and of a rotation with equal speed in the contrary direction about the revolving axis B B.

Therefore, in the plane of those axes at any instant draw C E, bisecting the obtuse angle B C A, and C E will be the instantaneous axis of the sleeve and arms.

Draw C F, making the angle A C F = A C E; and C G, making the angle B C G = B C E. Then E C F will be the trace of a fixed cone, having A C A for its axis; and E C G will be the trace of a rolling cone, having B C B for its axis; and the motion of the sleeve with its arms will be the same as if it were rigidly attached to the rolling cone.

Each of the points marked D describes a spherical epitrochoid, shaped like a long and slender figure of 8, and situated in the surface of a sphere of the radius C D, whose trace in the figure is marked by a dotted circle. The trace of the fixed cone on that sphere is projected in the figure by the straight line E F; that of the rolling cone, in the upper part of the figure, by the straight line E G, and in the lower part by a dotted ellipse. The centre of the base of the rolling cone is marked H and *h* in the two parts of the figure respectively.

SECTION V.—*Parallel Motions.*

249. **Parallel Motions in General.**—A *parallel motion* is a combination of turning pieces in mechanism, usually links and levers, designed to guide the motion of a reciprocating piece either exactly or approximately in a straight line, so as to avoid the friction which arises from the use of straight guides for that purpose. Its most common application is to the heads of piston-rods.

Some parallel motions are exact; that is, they guide the piston-rod head or other reciprocating piece in an exact straight line; but these parallel motions cannot always be conveniently made use of. Other parallel motions are only approximate; that is, the path of the piece which they guide is near enough to a straight line for the practical object in view; and these are the most frequent. They are usually designed upon the principle, that the two extreme positions and the middle position of the guided point shall be exactly in one straight line; care being taken, at the same time, that the deviations of the intermediate parts of the path of that point from that straight line shall be as small as possible.

There are purposes for which no merely approximate parallel motion is sufficiently accurate; such as the guiding of the tool in a planing machine, whose motion ought to be absolutely straight.

250. **Exact Parallel Motions.**—When a wheel rolls round inside a ring of exactly twice its radius, any point in the pitch-circle of the wheel traces a straight line, being a diameter of the pitch-circle of the ring (Article 245, page 266). This combination, then, has sometimes, though seldom, been used as an exact parallel motion for a piston-rod; the head of the piston-rod being jointed to a pin at the pitch-circle of the rolling wheel, and the crank to another pin at the centre of that wheel.

Fig. 201 represents another exact parallel motion, first proposed, it is believed, by Mr. Scott Russell.

The arm C D turns on the axis C, and is jointed at D to the middle of the bar A D B, whose length is double of that of C D, and one of whose ends, B, is jointed to a slider, sliding in straight guides along the line C B. Draw B E perpendicular to C B, cutting C D produced in E; then E is the instantaneous axis of the bar A D B; and the direction of motion of A is at every instant perpendicular to E A; that is, A moves along the straight line A C *a*. While the stroke of A is A C *a*, extending to equal distances on either side of C, and equal to twice the chord of the arc D *d*, the stroke of B is only equal to twice the deflection of that arc; and thus A is guided through a comparatively long stroke by the sliding of B through a comparatively short stroke, and by rotatory motions at the joints C, D, B. This may be called the *grasshopper* parallel motion.

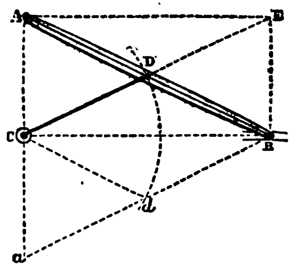

Fig. 201.

251. **Approximate Grasshopper Parallel Motion.**—The point B, instead of sliding between straight guides, may be carried by the end of a lever which, in its middle position, is parallel to *a* C A, and which is so long that the deviation of the arc described by B from a straight line is insensible. (See also page 292.)

252. **Watt's Parallel Motion—General Description.**—The general construction of the ordinary form of Watt's approximate parallel motion is shown in fig. 202. A *c* is one arm of the walking-beam of the engine, turning about an axis at the *main-centre c*. A B is the *main-link*, connecting the end A of the beam with the piston-rod B D. T *t* is the *back-link*, equal and parallel to the main-link; and B T is the *parallel-bar*, equal and parallel to the part A *t* of the walking-beam, and completing the *parallelogram* A *t* T B. The piston-rod head, B, is to be guided so that its highest, middle, and lowest positions shall be in one straight line; and this is effected by guiding in the same manner the point P, where the straight line B *c* cuts the back-link. The guiding of the point P is effected by means of the *radius-bar*, or *bridle*, C T, which is a lever that turns about an axis at C, and is

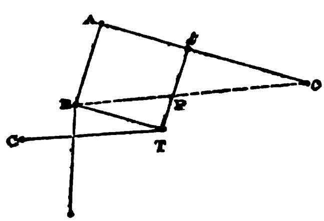

Fig. 202.

jointed at T to the back-link. From the point P a pump-rod is often hung. Some variations in detail will be explained further on. The links, bridle, and parallel-bar usually consist each of a pair of equal and similar pieces, connected respectively with the two sides of the piston-rod head.

Fig. 203 shows on a larger scale the principle according to which the point P is guided. S *s* is the line of stroke of that point; P_1, P_2, and P_3, its upper, middle, and lower positions. T P *t* (shown in three positions, numbered 1, 2, and 3 respectively)

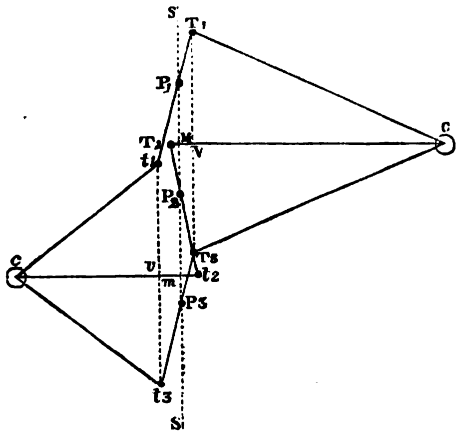

Fig. 203.

is the link; C T and *c t*, the two levers which guide that link, each shown in three positions. In designing a parallel motion the principal problem is to adjust the relative positions and proportions of the levers and link, so that the three points P_1, P_2, and P_3 shall lie in the straight line S *s*. There are also subordinate problems, of which the first has for its object to make the deviations of the point P from the straight line of stroke, in positions intermediate between P_1, P_2, and P_3, as small as possible. It will be shown further on that those deviations arise from the varying obliquity of the link T *t* to the line of stroke S *s*; therefore each lever should be so placed relatively to the link that the greatest obliquity of the link shall be as small as possible; and for that purpose the link, in its positions of greatest obliquity to the right and left of the line of stroke respectively, should make *equal angles* at opposite sides of that line; which condition is to be fulfilled by making the middle position of each lever perpendicular to the line of stroke, and making that line bisect, in the points M and *m*, the deflections T_2V and t_2v of the arcs described by the points T and t respectively. This adjustment is to be made for the beam and link before proceeding to design the parallel motion strictly so called; that is, the position and dimensions of the bridle relatively to the beam and link.

Another subordinate problem is to design the parallelogram, when a second point, such as B in fig. 202, is to be guided.

253. **Watt's Parallel Motion—Rules for Designing.**—I. Given (in fig. 204) the line of stroke, G D, of a piston-rod, the middle position of its head, B, and the centre, A, of a walking-beam or lever, which, in its middle position, A D, is perpendicular to G D: to find the radius of the lever, so that the link connecting it with B shall deviate equally to the two sides of G D during the motion; also, the length of the link.

Make $DE = \frac{1}{4}$ stroke; join A E; and perpendicular to it draw E F, cutting A D produced in F; A F will be the required radius. Join F B; this will be the link.

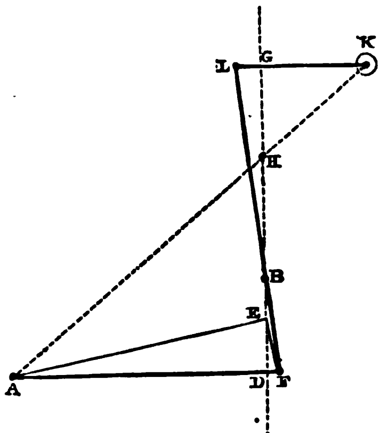

Fig. 204.

II. Given the data and results of Rule I.; also the point, G, where the middle position of a bridle or second lever, connected with the same link, cuts G D: to find the second lever, so that the two extreme positions of B shall lie in the same straight line, G B D, with the middle position.

Through G draw a straight line, L G K, perpendicular to G D; produce F B till it cuts that line in L; this point will be one end of the required second lever at mid-stroke, and F L will be the entire link. Then, in D G lay off D H = G B; join A H, and produce it till it cuts L G K in K; this will be the centre for the second lever.

III. Given (in fig. 205) the middle positions, A C and B E, (parallel to each other) and the extreme positions, A D, A D′, B F, B F′, of two levers centred at A and B respectively: required the radii of those levers, so that a link connecting them shall deviate in direction by equal angles alternately to the two sides of a perpendicular to A C and B E.

From B let fall B G perpendicular to A C (produced if necessary).

Draw A H bisecting the angle C A D. Also, at the point G, lay off the angle A G H = $\frac{1}{2}$ E B F; and let G H cut A H in H. Draw H K perpendicular to H A, and H L perpendicular to H G, cutting A G in K and L respectively. In B E lay off B M = G L. Join K M. Then will A K and B M be the required radii of the levers, and K M the link connecting them.

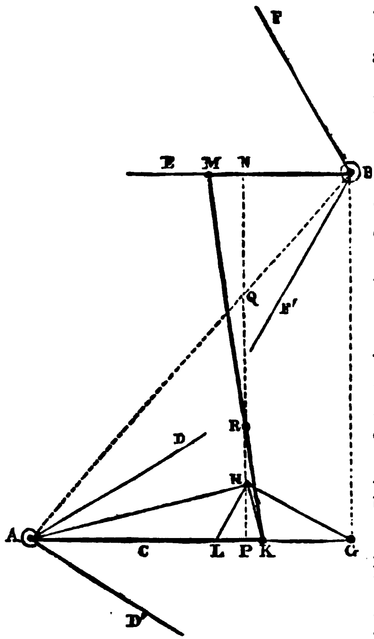

Fig. 205.

IV. In the link K M, found by the preceding rule, to find the point whose middle and two extreme positions lie in one straight line; also the length of stroke of that point.

Through H draw P H N perpendicular to A K and B M. The required length of stroke is 4 P H. The required point is R, where P H N cuts K M; but as the acuteness of the angle of intersection causes this method of finding R to be wanting in precision, it is better to proceed as follows:—Draw the straight line A B, cutting N P in Q; then lay off P R = N Q: R will be the required point.

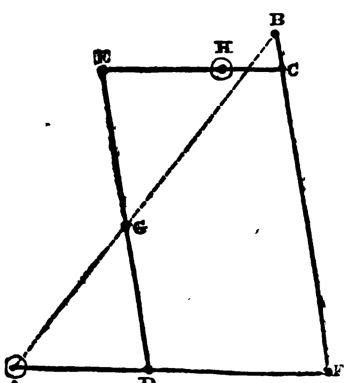

Fig. 206.

V. Given (in fig. 206 or in fig. 206 A) the *main-centre*, A, the middle position of the main-lever, A F, the piston-rod head, B, and its length of stroke; the radius, A F, of the lever, and the *main-link*, F B, having been found by Rule I. Let the figure represent those parts at mid-stroke; and let it be required to construct a parallel motion consisting of a parallelogram, C E D F (in which C E = F D is called the *parallel-bar*, and D E = F C the *back-link*), and a radius-lever, or bridle, H E, jointed to the angle E of the parallelogram.

Draw the straight line A B, cutting the back-link D E in G; then by Rule II. find the lever H E, such that the middle and extreme positions of G shall lie in one straight line.

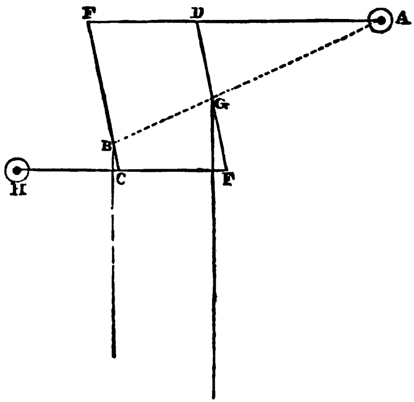

Fig. 206 A.

(The point G shows where a pump-rod may, if convenient, be jointed to the back-link).

Fig. 206 B represents that case of the application of this rule in which the points B and C coincide with each other; and it requires no separate description except the substitution of B for C in the Rule. This construction is the same with that which is roughly represented in fig. 202.

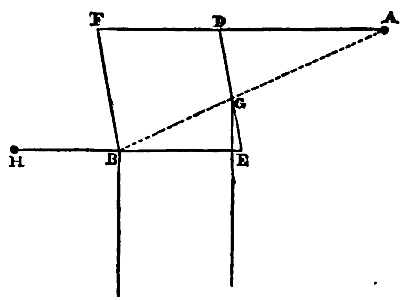

Fig. 206 B.

VI. Given (in fig. 207) the main-centre, A; also the main-lever or walking-beam, A F, and main-link, F B, found by Rule I. Let B be the middle position, and C and C′ the two extreme positions, of the point where the parallel-bar is jointed to the main-link, whether at the piston-rod head or at some other point. Let A G and A G′ be the two extreme positions of the main-lever. Let H be a convenient point in the straight line, B H, parallel to F A, chosen as the centre for the bridle; and let H J be a convenient length chosen for the radius of that lever; J being the middle position of the point where it is jointed to the parallel-bar.

Required, the points where the back-link ought to be jointed to the main-lever, A F, and the bridle, H J, respectively.

About H, with the radius H J, describe a circular arc; then take the length, B J, of the parallel-bar in the compasses, and lay off the same length from C and C′ respectively to K and K′ in that arc: C K and C′ K′ will be the extreme positions of the parallel-bar.

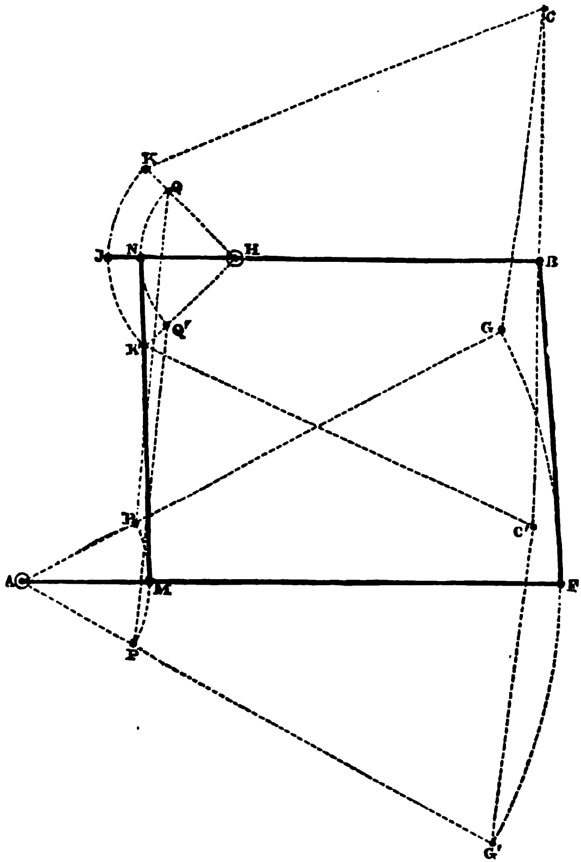

Fig. 207

Join H K, H K′: these will be the extreme positions of the bridle. Then, by Rule III., find the ends, M and N, of the back-link, M N, which is to connect the bridle with the main-lever.

The two extreme positions of the back-link are marked P Q, P′ Q′.

If it be desired to joint a pump-rod to the back-link, the proper point for that purpose may be found by Rule IV.

254. **Watt's Parallel Motion—Extent of Deviation.**—The following rule serves to determine the extent of the greatest deviation of the guided point in Watt's parallel motion from an exactly straight path. In fig. 208, let D G be the line of stroke; A F and K L, the two levers; F L, the link; and B, the guided point; the moving parts being in their middle positions. About A and K respectively, draw the circular arcs F *f* and L *l*, being the paths of the points F and L, cutting the line of stroke in *f* and *l*. Take the length F L of the link in the compasses; lay it of from one of those arcs to the other, by trial, or by the eye, so as to be parallel to D G; and mark its ends, *f′* and *l′*.

Draw the straight line *f′ l′*, cutting K L in G′; this will be the position of the link when its deviation towards K is greatest; and G G′ will be that deviation. The greatest deviation towards A will be of equal extent, and will take place when the levers make nearly equal angles with their middle position in the contrary direction.*

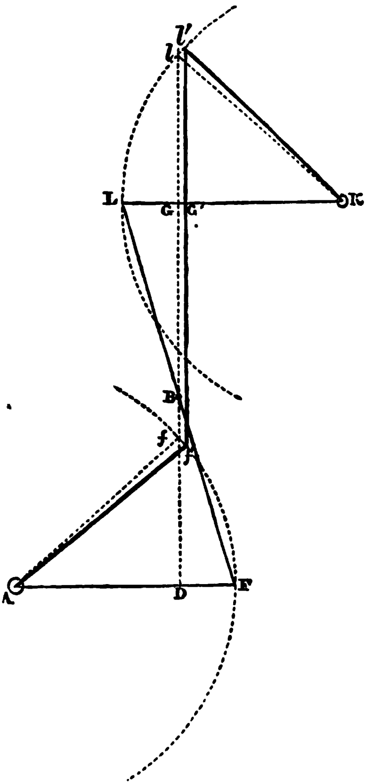

Fig. 208.

* *To calculate the greatest deviation approximately*, let the radii A F and K L be denoted respectively by a and a'; the distances, A D and K G, of the centres from the line of stroke, by b and b'; the distance D G by c; the length of the link F L by l; and the greatest deviation G G′ by d. Also make

$$Df = \sqrt{a^2 - b^2} = f; \text{ and}$$

$$G\,l = \sqrt{a'^2 - b'^2} = f'$$

Then

$$d = \frac{l - c}{\frac{b}{f} + \frac{b'}{f'}}, \text{ very nearly. } \ldots\ldots\ldots\ldots \text{(1.)}$$

[Note continued on next page.

As is shown in the note, the fraction expressing the proportion borne by the deviation to the length of stroke varies, to a rough approximation, inversely as the length of the link, inversely as the cube of the mean of the lengths of the levers, and directly as the fourth power of the stroke; and when the angular stroke is 42°, and the link about equal in length to the levers, that fraction is about $\frac{1}{5000}$th. In order to make the deviation visible in the figure, the dimensions which tend to increase it are much exaggerated beyond those which are admissible in practice.

The rules of Article 253 give the following values for some of the data in the preceding formula in terms of the stroke s:-

$$a = b + \frac{s^2}{16\,b};$$

$$a' = b' + \frac{s^2}{16\,b'};$$

$$l = \sqrt{\left\{c^2 + \left(a + a' - b - b'\right)^2\right\}}$$

$$= \sqrt{\left\{c^2 + \frac{s^4}{256}\left(\frac{1}{b} + \frac{1}{b'}\right)^2\right\}}$$

$$= \text{(nearly) } c + \frac{s^4}{512\,c}\left(\frac{1}{b} + \frac{1}{b'}\right)^2$$

When the stroke is not too great, compared with the lengths of the levers, we may also put

$$f \text{ nearly} = f' \text{ nearly} = \frac{s}{3};$$

whence we obtain the following approximate formula, as being near enough to the truth in practice, for the proportion borne by the greatest deviation to the length of stroke:—

$$\frac{d}{s} \text{ nearly} = \frac{s^4\,(b + b')}{1536\,c\,b^2\,b'^2} \quad \text{.........................(2.)}$$

For example, let $b = b' = c$; and let $s = b$; which gives an angular stroke of about 56°; then

$$\frac{d}{s} \text{ nearly} = \frac{1}{1536} = 0{\cdot}00065.$$

Next, let $b = b' = c$; and let $s = \frac{3}{4}\,b$; which gives an angular stroke of about 42°; then

$$\frac{d}{s} \text{ nearly} = \frac{3^4}{4^4 \times 1536} = \text{about } 0{\cdot}0002.$$

The *Radius of Mean Curvature* of the path of the guided point is found by the following formula:—

$$\varrho = \frac{s^2}{32\,d} \quad \text{.................................(3.)}$$

255. **Tracing Approximate Circular Arcs by the Parallel Motion.**—An intermediate point in the link of a parallel motion, not coinciding with the point whose path approximates to a straight line, moves nearly in an arc of a circle; and this fact may be made useful in the approximate tracing of arcs of long radius. The approximation is not sufficient for purposes which require the utmost precision, such as cutting the slots in curved links for slide-valve gear; but where less precision is necessary, as in shaping barrel staves, it might prove useful. The following are rules relating to this process:—

I. To find the radius of a circular arc traversing the extreme and middle positions of a given point in the link of a parallel motion. In fig. 209, let A F and K L be the levers, in their middle position; F L, the link; B, the point which is guided nearly in a straight

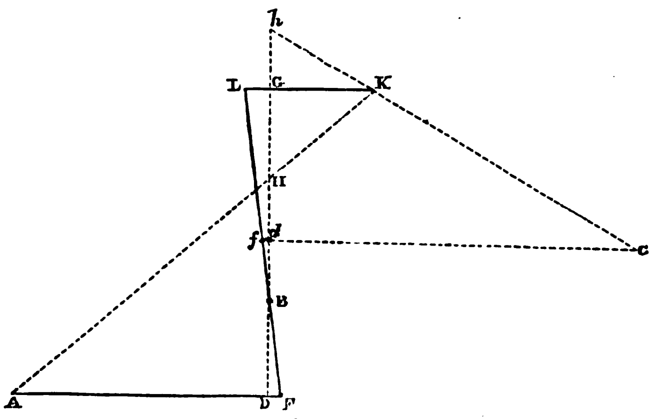

Fig. 209.

line; D G, its line of stroke; *f*, another point in the link. Perpendicular to G D, draw *f d a*, cutting it in *d*. In D G produced, lay off G *h* = B *d*; join *h* K, and produce it till it cuts *f d a* in *a*; this will be the centre of the circular arc traversing the middle and extreme positions of the point *f*; and *a f* will be the radius of that arc.

II. To design a parallel motion which shall guide a point approximately along a given circular arc, B D C, fig. 210, of which D is the middle point and A the centre.

Draw the straight chord B C, cutting A D in E; bisect the deflection D E in F; through F draw F G parallel to B C (and therefore perpendicular to A D). Choose any convenient position, I, for the centre of motion of one of the levers, and draw I G parallel to D A. Lay off G H = $\frac{1}{2}$ E B = $\frac{1}{4}$ B C; join I H; and perpendicular to this line draw H K, cutting I G produced in K; I K will be one of the levers. Join K D; this will be part of the link; and M, where K D cuts F G, will be the point in the link which moves nearly in a straight line; but that point is found with more precision as follows:—Join I A, cutting G D in L, and lay off F M = G L. Produce K M D; and make M N = M K; then

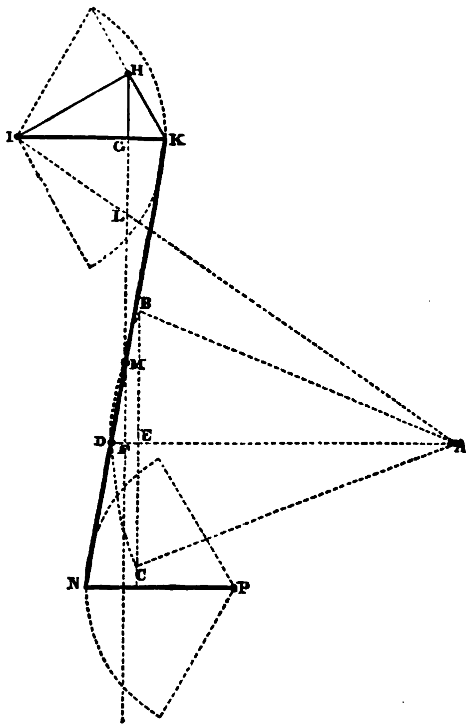

Fig. 210.

draw N P parallel and equal to I K; K N will be the link, and N P the other lever, centred at P.

III. To find approximately the greatest deviation of the point which is guided nearly in a circular arc. Find by the rule of the preceding Article the greatest deviation of the point which moves nearly in a straight line. The greatest deviation of any other point in the link, from its approximate circular path, is proportional very nearly to the product of the lengths of the segments into which it divides the link. For example, in fig. 209 we have

$$\frac{\text{greatest deviation of } f}{\text{greatest deviation of B}} = \frac{\text{L}f \cdot f\text{F}}{\text{L B} \cdot \text{B F}}, \text{ nearly;}$$

and in fig. 210, $\frac{\text{greatest deviation of D}}{\text{greatest deviation of M}} = \frac{\text{K D} \cdot \text{D N}}{\text{K M} \cdot \text{M N}}$, nearly.

256. **Roberts's Parallel Motion.**—This might also be called the W parallel motion, from its shape. It is considered to be a convenient form for horizontal engines. Two cases of this combination are shown in the figures: a simpler case in fig. 211; a more complex case in fig. 212.

I. In Fig. 211, let D E be the straight line of stroke which the guided point is to follow approximately, and A the middle of that line. Draw two isosceles triangles, D B A, A C E, with equal legs, D B = A B = A C = C E. The length of each of these legs should not be less than D A × 0·843; but it may be as much greater as the available space will permit; and the greater it is the more accurate will be the motion. Join B C, which of course is = D A = A E. Then A B C represents a rigid triangular frame, of which the apex A is the guided point; while the angles of the base, B and C, are jointed to two levers or bridles, B D, C E, which turn about axes at D and E respectively. The two extreme positions of the triangle A B C are marked respectively D B′ C′ and E B″ C″, the points B″ and C′ coinciding.

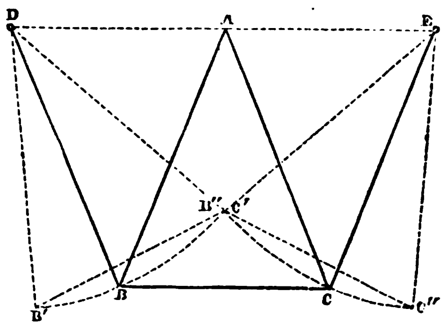

Fig. 211.

The reason for prescribing that the length of each leg of the triangles shall not be less than the base × 0·843 is, that this proportion between these lengths makes the points B′, C′, and E lie in one straight line, and the points C″, B″, and D in another straight line, in the two extreme positions of the combination respectively.

II. The second arrangement, fig. 212, is to be used in those cases in which it may be inconvenient to have the axes of motion D and E of the bridle-levers traversing the line of stroke. Let A′ A A″ be the line of stroke, and A its middle point. Draw an isosceles triangle, A B C, of a convenient size and figure, with its apex, A, at the middle point of the stroke, and its base, B C, parallel to the line of stroke. Draw the same triangle in its two extreme positions, A′ B′ C′ and A″ B″ C″, leaning over equally in opposite

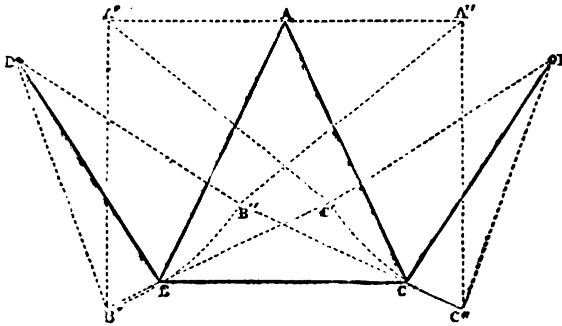

Fig. 212.

directions; the sides A′ B′ and A″ C″ may conveniently be made vertical; but this is not essential. Find, by plane geometry, the centre D of a circular arc traversing the three points B′, B, B″; also the centre E of a circular arc traversing the three points C′, C, C″. D B and E C will be the two bridle-levers, and D and E the bases of their axes of motion.

Addendum to Article 142, Page 141.

Intermittent Gearing—Counter-Wheels—Geneva Stop.—In the intermittent toothed wheelwork described at pages 139 to 141, the wheels and their teeth are so designed that, during the transmission of motion, the velocity-ratio has a constant value. In some cases, however, it is not necessary that the velocity-ratio should be constant, provided only that the follower performs a certain part of a revolution for each revolution of the driver, as in mechanism for *counting revolutions.* The simplest mechanism of that sort consists of a toothed wheel of the ratchet form (Article 194, page 207, fig. 145), driven by a wiper or single tooth (Article 164, page 175), which projects from a rotating cylinder, and has its length adjusted so that the arc of contact is equal to the pitch of the ratchet wheel. This requires no special explanation. But there are cases in which the abruptness of the action of the wiper would be disadvantageous, and in which it is desirable, in order to prevent shocks, that the follower should be set in motion and stopped by insensible degrees.

The following is the most precise method of designing a pair of wheels turning about parallel axes, in which the follower is to count the revolutions of the driver by turning through a certain aliquot part of a revolution for each revolution of the driver; the action being absolutely without shock, and capable of taking place in either direction.

In fig. 100 A, let A B be the line of centres, A the trace of the axis of the driver, and B that of the follower. In the example shown in the figure, the follower is to make *one-fifth* of a revolution for each revolution of the driver; but the same rules are applicable to any given number of aliquot parts.

Draw straight lines radiating from B (marked by dots in the figure), so as to divide the angular space round B into twice the given number of equal

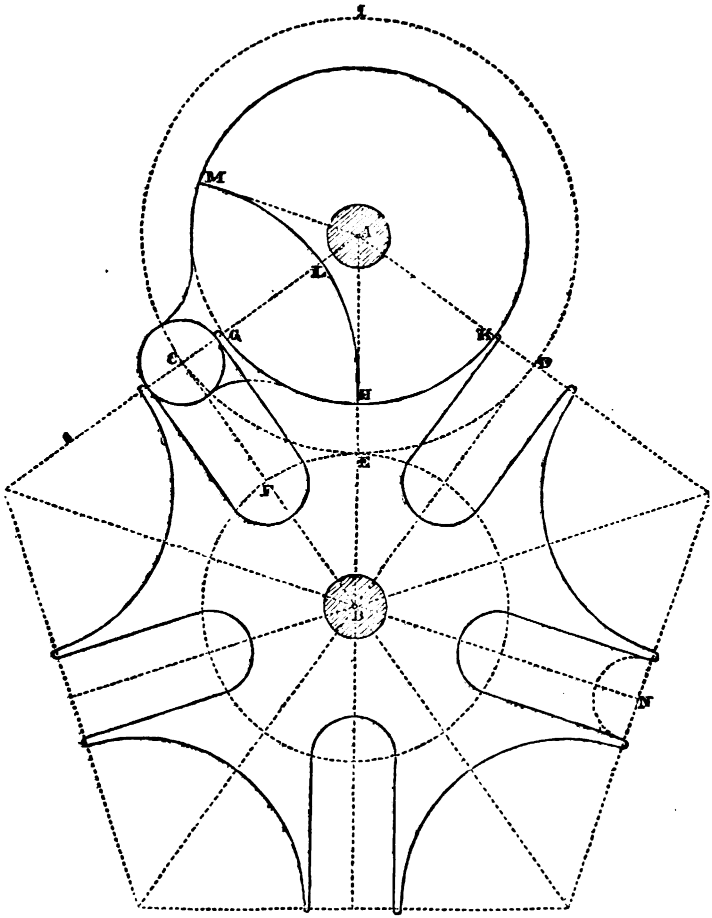

Fig. 100 A.

aliquot parts; the line of centres, B A, being one of these radiating lines. Then from A let fall perpendiculars A C and A D on the two radiating lines which lie nearest to the line of centres, and complete the regular polygon of which B is the centre, and A C and A D are two of the half-sides. (In the figure this is a regular pentagon.) On the line of centres lay off A E = A C = A D; and about B, with the radius B E, describe a circle; this circle will cut the alternate radiating lines in a set of points, such as F, which will be the centres of the semicircular bottoms of a set of notches that will gear in succession with a pin carried by the driver. Having assumed a convenient radius for that pin, draw a circle to represent it about the point C as a centre. The pin will be carried by a plate or arm projecting from the axis A, in a different plane from that of the driven wheel. For the bottom of the notch C F draw about F, as a centre, a semicircle of a radius equal to that of the pin C, increased by an allowance sufficient for clearance; and for the two sides of that notch draw two straight lines touching that semicircle, and parallel and equal to F C. Draw all the other notches of the driven wheel of the same figure and dimensions.

In the intervals between the notches the rim of the driven wheel is to consist of a series of equal hollow circular arcs, described respectively about the angles of the polygon with a radius such as to leave a thickness of material sufficient for strength and durability at each side of each of the notches; for example, the arc G H K is described about the centre A, so as to leave a sufficient thickness of material at G and K.

The periphery of the driver is to consist of a cylindrical surface extending round the *dead arc*, H K M, and fitting smoothly, but not tightly, into each of the hollows, such as G H K, in the rim of the follower; and of a hollow, H L M, of a depth and figure such as to clear the *horns*, such as G and K, of the notches in the follower. The angular extent of that hollow, H A M, is to be equal to the angle C A D, that is, to the supplement of C B D, and is to lie so that the radius A C shall bisect it.

The effect of this construction is as follows:—While the pin moves through the arc C E D, the follower is driven through the angle C B D; and as the pitch-point evidently moves from A to E, and then back to A again, the angular velocity of the follower gradually increases from nothing to a maximum, and then gradually diminishes to nothing again. At the point D the pin leaves the notch, and while it moves through the arc D I C, the follower remains at rest, and is kept steady by the dead arc, M K H, fitting into one of the hollows in its periphery. When the pin arrives again at C, it enters and drives a second notch, and so on. The combination evidently works with rotation in either direction.

The Geneva Stop is the name given to the form of this combination that is employed when the object is that the follower shall stop the driver after it has turned through a certain number of revolutions and fractions of a revolution. For that purpose one of the notches is to be filled up, as shown by the dotted semicircle at N, so as to leave only a recess fitting the pin in the position C or D. The extent of rotation to which the driver will then be limited is expressed by as many revolutions as there are intervals in the circumference of the follower, *less* the angle C A D; and as the angle C A D is the supplement of C B D, this is expressed in algebraical symbols as follows: Let n be the number of intervals in the circumference of the follower; then the driver is limited to the following number of turns:—

$$n - \frac{1}{2} + \frac{1}{n}.$$

For example, in the figure, we have $n = 5$; therefore, if one of the notches is stopped, the rotation of the driver will be limited to

$$5 - \frac{1}{2} + \frac{1}{5} = 4{\cdot}7 \text{ turns.}$$

This contrivance is used in watches, to prevent their being overwound.

Very often a hammer-headed tooth is used instead of the cylindrical pin C.

This enables the horns of the notches to be shortened; but it gives more backlash, and less precision and smoothness of action.

Addendum to Article 154, Page 161.

Racks in Screw Gearing.—When a straight rack gears with a screw, the pitch-surface of the rack is a plane, touching the pitch-cylinder of the screw. The traces of the teeth of the rack on its pitch-plane are parallel straight lines, and are the development on that plane of the traces of the threads of the screw upon its pitch-cylinder. The principal rule to be used in designing a rack and screw is a modification of Rule II. of Article 154, and is as follows:—

In fig. 110 A, let the plane of projection be the pitch-plane of the rack; and let A I P be the projection of the axis of the screw: being also the straight line in which the pitch-surfaces touch each other. Let I c represent the direction in which the rack is to slide; and let the length I c represent *the surface-velocity of the rack.* Having assumed a proper transverse obliquity for the teeth of the rack, draw the straight line c P, to represent the trace of a tooth.*

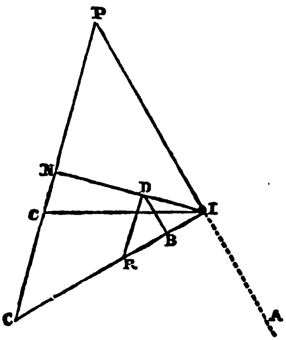

Fig. 110 A.

Draw I C perpendicular to I A, cutting P c in C; then I C will represent *the surface-velocity of the screw;* and c C will represent *the velocity of transverse sliding* of the threads of the screw on the teeth of the rack. On C c P let fall the perpendicular I N; this will represent *the common component* of the surface velocities. Also, the proportions borne to each other by the divided pitches are as follows:—as

I c : longitudinal pitch of rack
: : I N : divided normal pitch of rack and screw
: : I C : circular pitch of screw
: : I P : divided axial pitch of screw.

Having assumed a convenient absolute value for the longitudinal pitch of the rack, find, by the help of the diagram, the circular pitch of the screw; multiply that circular pitch by a convenient number of threads, for the circumference of the pitch-cylinder, and find its radius by construction or by calculation.

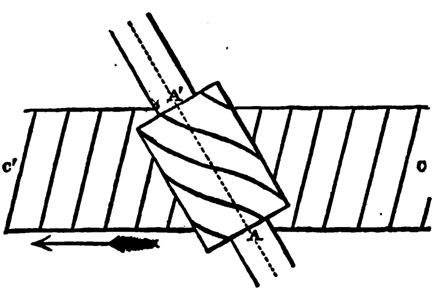

Fig. 110 B.

Fig. 110 B is a projection of a rack, C C′, and screw, A A′, showing the traces of the teeth and threads on the pitch-surfaces.

During the action of a tooth and a thread, the

* The proper obliquity for the rack-teeth depends on friction, a subject belonging to the Dynamics of Machinery; but it may here be stated, that when the screw is to drive the rack

pitch-point travels along the whole length of the line of contact of the pitch-surfaces, whose projection is A A′. In the figure the breadths of those surfaces are such, that at any given instant there are at least three pitch-points.

The teeth and threads are to be designed by drawing their *normal sections*, as described in Article 155, page 163. The normal section of a tooth of the rack will be that adapted to its normal pitch. The normal section of a thread of the screw is to be found by the help of the osculating circle of the normal helix, as found by Rule III. of Article 154, page 161, which is repeated here for convenience. On I C in fig. 110 A lay off I B equal to the radius of the pitch-cylinder; then, perpendicular to I C, draw B D, cutting I N in D; then, perpendicular to I N, draw D R, cutting I C in R; I R will be the radius of curvature of the normal helix.

A screw and rack are used in one form of Sellers's planing machine.

Addendum to Article 243, Page 265, and Article 245, Page 267.

Epitrochoidal Paths.—In tracing epitrochoidal paths (including ellipses) it is obviously not essential that there should actually be two wheels having the fixed and rolling circles respectively for their pitch-circles, provided only that the wheel which carries the tracing-point is carried by a train-arm and driven by an epicyclic train, so as to have the same motion with the imaginary rolling circle. As to the principles which regulate the motion of the rolling circle, see Articles 76, 77, pages 54 to 56. As to the action of the epicyclic train, see Article 234, pages 243, 244. The wheel-work of the epicyclic train may be varied in detail according to convenience, so long as it gives the required velocity-ratio.

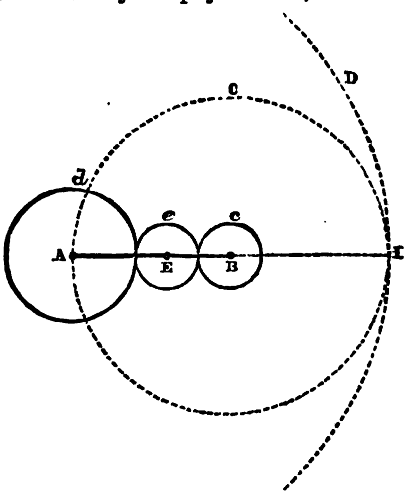

Fig. 196 A.

To exemplify this, let it be required to design an instrument for tracing ellipses by rolling motion; and in Fig. 196 A, let A be the centre of the fixed circle D, and B the centre of the rolling circle C, whose diameter, A I (as already stated in Article 245), is the radius of the fixed circle. The simplest gearing obviously consists of an internally toothed fixed ring whose pitch-circle is D, and a spur-wheel whose pitch-circle is C; but this combination may be inconvenient in practice, and it may be desirable to use an epicyclic train in outside gearing. The train-arm will of course be represented by A B; and the absolute angular velocity of the wheel centred on B, which is to carry the tracing-arm, must be made equal and contrary to that

(as is almost always the case in practice), the angle I c P should be acute, and equal to the complement of the angle of repose of the teeth,—say about 85°, if the teeth are smooth and greasy.

of the train-arm. To effect this, let d be a fixed spur-wheel centred on A; let c be a spur-wheel centred on B, and of half the radius of the fixed wheel; and let e be an idle wheel of any convenient radius, gearing with both the wheels before mentioned. Then, if a be taken to denote the angular velocity of the train-arm, the angular velocity of the fixed wheel d relatively to the train-arm will be $= -a$; and the angular velocity of the wheel c, and of the tracing-arm which it carries, relatively to the train-arm will be $= -2a$; therefore the absolute angular velocity of the tracing-arm will be $a - 2a = -a$; that is to say, equal and contrary to that of the train-arm, as required.

As represented in the figure, this instrument is capable of tracing various ellipses in which either the half-sum or the half-difference of the semiaxes is = A B; and it is easy to see that by jointing the train-arm at E, the centre of the idle wheel, and providing the means of fixing its two divisions, A E and E B, so as to make different angles with each other, the distance A B may be varied within certain limits.

Addendum to Article 244, Page 265.

Aggregate Paths traced by Cam-Motions.—An endless variety of aggregate paths may be traced by means of combinations in which rolling action is combined with the action of a cam.

Suppose two axes, not intersecting, to be connected by means of a pair, or of a train, of suitable wheels, so as to have any required velocity-ratio; and suppose that along with one axis there rotates a disc on which an aggregate path is to be traced; and that along with the other axis there rotates a cam of any required figure, which cam, by acting on a sliding bar, causes the tracing-point to move along the line of centres, or common perpendicular of the axes, towards and from the axis that carries the disc, according to a law determined by the figure of the cam; then the tracing-point will trace on the disc a curve whose figure will depend on that of the cam, and on the velocity-ratio of the cam and disc.

(As to the action of cams, see Article 161, pages 170 to 173.)

Combinations of this sort are used in lathes for ornamental turning. The tracing-point is the point of a suitable cutting tool; and the cam which regulates its motion is commonly called the *copy-plate*, or *shaper-plate*. (See Northcott *On Turning*, 1868, Part III.)*

* To express by algebraical symbols the action of this combination, let the polar equation of the pitch-line of the cam be

$$r = f(\theta), \quad \text{(1.)}$$

θ being the angle made by the radius r with a fixed radius. Also, let a be the length of the line of centres, and b the constant distance along that line from the pitch-line of the cam to the tracing-point.

Let the polar equation of the curve traced by that point on the disc be

$$r' = \phi(\theta'); \quad \text{(2.)}$$

and let n be the ratio borne by the angular velocity of the cam to that of the disc. Then the quantities in the equations 1 and 2 are connected with each other by the following pair of equations:—

$$r + r' = a - b; \quad \text{(3.)}$$

$$\theta = n\theta'. \quad \text{(4.)}$$

Hence, if the figure of the cam is given—that is, if the function f in equation 1 is given—the figure of the traced curve is determined by the following equation:—

$$r' = a - b - f(n\theta'); \quad \text{(5.)}$$

and if the figure of the traced curve is given—that is, if the function ϕ in equation 2 is given—the figure of the cam is determined by the following equation:—

$$r = a - b - \phi\left(\frac{\theta}{n}\right). \quad \text{(6.)}$$

Addendum to Article 251, Page 275.

Approximate Grasshopper Parallel Motion.—Another form of approximate parallel motion of the grasshopper kind is designed as shown in fig. 201 A. Let A, A′, A″ be the extreme and middle positions of the guided point, lying in one straight line. Draw the straight line A′ B B′ perpendicular to A A′ A″; and lay off the intended length of the guiding-bar, A B = A′ B′ = A″ B, so as to find the extreme positions, B and B′, of its further end. This end may be guided either by straight guides, or by a lever centred at a point, C, equidistant from B and B′; that lever being so long as to make the point B describe a very flat circular arc, deviating very little from a straight line.

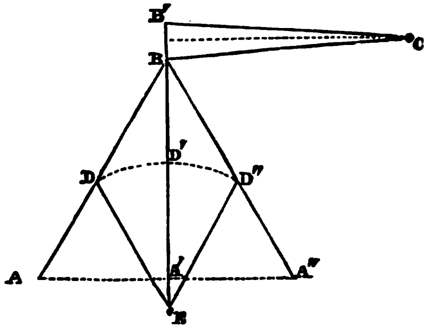

Fig. 201 A.

Choose a convenient point, D, for the attachment of the bridle to the bar A B, and lay off A″ D″ = A′ D′ = A D, so as to find the extreme and middle positions of that point. Then, by plane geometry, find the centre E of a circular arc traversing the points D, D′, D″; E will be the trace of the axis of motion of the bridle E D. The error of this parallel motion is the less the nearer D is to the middle of A B.

Addendum to Article 143, Page 143.

Involute Teeth for Elliptic Wheels are designed by drawing an ellipse confocal with the elliptic pitch-line, and having its major axis smaller in a fixed proportion, and then drawing, for the traces of the fronts and backs of the teeth, involutes of the smaller ellipse. The proportion in which this ellipse is smaller than the pitch-ellipse should be such, that every tangent to the smaller ellipse shall cut the fronts of two teeth at least between that ellipse and the pitch-line. The pair of smaller ellipses in a pair of elliptic-toothed wheels are analogous in their motion to a pair of elliptic pulleys; as to which, see Article 175, page 189.

CHAPTER VI.

OF ADJUSTMENTS.

257. **Adjustments Defined and Classed.**—The word "adjustments" was introduced by Professor Willis, in order to comprehend under one general term all contrivances for varying at will the comparative motions in a machine. Every adjustment may be regarded as an aggregate combination in which the action is temporary or intermittent; and the various kinds of adjustments might have been classed under the head of "Aggregate Combinations," in the preceding chapter; but it is more convenient to treat of them by themselves. Various contrivances which belong to the class of adjustments have already been described under the head of "Elementary Combinations," as well as of aggregate combinations: these will be specified in their order further on. Other contrivances belonging to the class of adjustments involve the application of the principles of dynamics and of the strength of materials, to such an extent that their description, at all events in detail, must be reserved for later divisions of this book.

When adjustments are classed according to the purposes to which they are applied, they may be arranged as follows:—

Traversing-Gear and Feed-Motions;
Engaging, Disengaging, and Reversing-Gear;
Gear for varying Speed or Stroke.

258. **Traversing-Gear and Feed-Motions in General.**—By *traversing-gear* is meant the mechanism by means of which a machine, consisting of framework and moving pieces, is shifted from place to place without being thrown out of connection with the driver from which it receives its motion; such, for example, as the mechanism by which the truck in a travelling crane, that carries the hoisting machinery, is made to move to different positions on a travelling platform, which itself is capable of being moved to different positions on a fixed framework; or the mechanism by which the arm in a drilling machine is made to move to various positions, carrying with it the boring-tool and the machinery by which that tool is driven; or that by which the tool-holder in a shaping machine is turned into various positions, according to the varying directions in which the strokes of the tool are to be made. By a *feed-motion* is meant the mechanism in a machine-tool by

means of which, after a stroke has been made, either the cutting-tool or the *work* (that is, the piece of material operated upon) is shifted into a new position, preparatory to making the next cut;—for example, in a lathe for turning axles, the feed-motion causes the tool to shift, at each revolution of the axle that is being turned, through a certain distance in a direction parallel to the axis of rotation; and in a sawing machine, the feed-motion causes the log of wood that is being sawn to advance through a certain distance either during or after each cut of the saw. Some feed-motions are continuous in their action; others are intermittent.

It is obvious that the general principles of traversing-gear, and of those feed-motions in which the tool is shifted, are those of *shifting-trains*, already stated in Article 228, pages 235 to 238. The consideration of traversing-gear and feed-motions in detail belongs to the subject of the construction of machinery, and must therefore be deferred.

SECTION I.—*Of Engaging, Disengaging, and Reversing-Gear.*

259. **General Explanations.**—Engaging and Disengaging-Gear, or sometimes Disengaging and Re-engaging-Gear, is the name given to those contrivances by means of which the connection between a follower and its driver can be begun and stopped at will;—in other words, by means of which the combination can be thrown *into gear* and *out of gear* when required. For brevity's sake, such contrivances may be called simply *Disengagements.* Disengagements may be classed in different ways. According to one mode of classification, they are distinguished into those which, in the communication of motion, act by *pressure*, and those which act by *friction*. Disengagements which act by pressure are precise and definite in their action; that is, the connection between the pieces that are thrown into gear at a given instant is established at once, in a certain definite position of the pieces, and with a certain definite velocity-ratio. Disengagements which act by friction are to a certain extent indefinite in their action; that is, the velocity-ratio corresponding to the complete establishment of the connection is produced by degrees; and the relative position of the pieces when the connection is completely established is uncertain. In certain cases the definite action of the former class of disengagements is necessary: in other cases it is unnecessary; and in these the frictional class of disengagements have a great advantage, because of their avoiding the shocks and straining actions which accompany sudden changes of velocity. The principles upon which such straining actions depend belong to the dynamics of machinery.

By another mode of classification, disengagements are arranged according to the kind of mechanism of which they consist, as follows:—

I. *Disengagements by means of Couplings;* where two pieces that turn about one axis are coupled or uncoupled at pleasure; so that when coupled, they turn as one piece. These may transmit motion either by pressure or by friction.

II. *Disengagements with Rolling Contact.*—These always transmit motion by friction.

III. *Disengagements with Sliding Contact.*—These transmit motion by pressure; and in most cases they act by throwing toothed wheels or screws into and out of gear.

IV. *Disengagements by Bands* transmit motion by friction.

V. *Disengagements by Linkwork* transmit motion by pressure.

VI. *Disengagements with Hydraulic Connection* transmit motion by the pressure of a fluid; and they are made to act by the opening and shutting of valves.

Reversing-Gear usually consists simply of a double set of engaging and disengaging-gear; that is to say, an arrangement of mechanism by means of which the follower can, when required, be thrown into gearing with one or other of two drivers that drive it in opposite directions, or may be disengaged from both.

It is obvious that all the combinations in which the connection is intermittent (enumerated in Article 219, page 231) are examples of self-acting disengagements; and that some of them (such as the escapements described in Article 164, pages 176 to 179) are examples of self-acting reversing-gear.

260. **Clutch.**—A clutch is a sort of coupling, in which one rotating piece drives another piece that turns about the same axis, by means of two or more projecting claws or horns, that fit into corresponding recesses, or lay hold of corresponding horns, on the second piece. In a disengaging clutch the driving piece is a cylindrical box or collar with suitable horns, which is capable of easily sliding lengthwise upon a rotating shaft, and is made to rotate constantly along with the shaft, by having in its internal cylindrical surface a slot or longitudinal groove, fitting a longitudinal key or feather that projects from the shaft. In the outer cylindrical surface of the clutch is a circular groove, into which there fit easily the rounded ends of the prongs of a forked hand-lever, by means of which the clutch can be shifted lengthwise on the shaft through a distance sufficient to engage its horns with or disengage them from those of the following piece. The following piece may be another length of shaft, turning about the same axis; or it may be a wheel or a pulley, loose upon the same shaft with the clutch.

Sometimes the acting faces of the clutch, instead of being planes traversing the axis of rotation, are inclined backwards as regards the direction of motion at an angle of 15°, or thereabouts. The effects of this are, that a certain forward pressure must be continually exerted by the lever on the clutch when in gear, in order to make it keep its hold; and that any sudden acceleration of one of the parts of the coupling causes the clutch to lose its hold, and thus prevents the transmission of a shock to the machinery which is driven by means of it.

261. **Friction-Clutch—Friction-Cones—Friction-Sectors—Friction-Discs.**—In the *friction-clutch* the following piece is a circular disc, having a hoop which grasps it, and which can be tightened or slackened by means of screws until the friction between the hoop and the disc is just sufficient to transmit the required power. The hoop has two projecting horns, corresponding to those of the clutch. When this combination is thrown into gear, the clutch instantly communicates its own velocity to the hoop; but the hoop at first slips on the disc, which is set in motion by degrees; and thus dangerous shocks are avoided.

In the *friction-cones* the driver, as in the case of the clutch, is a cylindrical box, turning along with the shaft, and capable of being shifted lengthwise by means of a hand-lever; but instead of horns, it has a disc with a rim turned to a very accurate and smooth convex conical surface. The follower is a disc whose rim is turned to a concave conical surface, exactly fitting that of the driver. When the driver is pushed forward by means of the lever, so as to press the two conical surfaces together, it gradually imparts its rotation to the follower by means of the friction of those surfaces. On drawing back the driver by means of the lever, the connection immediately ceases.

The angle of obliquity of the conical surfaces should be just great enough to prevent any risk of their becoming jammed against each other, so as to prevent disengagement; and for that purpose an angle of 10° or thereabouts is sufficient.

In the *frictional sector* coupling (invented by Mr. Bodmer) the follower is a cylindrical box, loose on the shaft, and carrying a circular disc-plate with a hoop-shaped rim. The inner cylindrical surface of that rim is turned true and smooth. The driver consists of a boss fixed on and turning with the shaft, and carrying an expanding disc composed of two sectors, with true and smooth cylindrical rims, fitting the inner surface of the rim of the followers. Those sectors can be simultaneously moved from or towards the shaft by means of right and left-handed screws, turned by levers and links; the links lie parallel to the shaft, and are jointed to a collar which is shifted by means of a forked lever, as in the ordinary clutch. When the sectors are moved outwards, they fit tightly to the inside of the hoop-shaped rim of the follower, and by their friction communicate to it the rotation of the shaft.

When moved inwards, they cease to touch that rim, and the connection ceases. (See Fairbairn *On Millwork*, Part II, Edition of 1863, pp, 91, 92.)

In Mr. R. D. Napier's disengaging-gear the pushing forward in the usual way of a cylindrical clutch-box causes two segmental pieces to grasp between them a drum that rotates with the shaft, and so to communicate rotation to a disc to which they are attached.

In the *friction-disc* disengagement (Mr. Weston's invention) a set of flat discs are made to turn along with the shaft by means of grooves and feathers, and are capable of shifting longitudinally to a small extent. Between each pair of that first set of discs is placed a disc belonging to a second set, which are loose on the shaft, but are made to turn along with it when required, by pressing them between the discs of the first set. The second set of discs, by means of grooves and feathers at their outer edges, carry along with them in their rotation a wheel or a pulley concentric with the shaft.

262. **Disengagements acting by Rolling Contact.**—A pair of wheels acting on each other by rolling contact may be engaged and disengaged when required by pressing them together and drawing them asunder, the axis of one of them being made moveable for that purpose; and this is practised in grooved frictional gearing of the kind described in Article 111, page 150.

The principle of another method of effecting engagement and disengagement by wheels in rolling contact is shown in fig. 213.

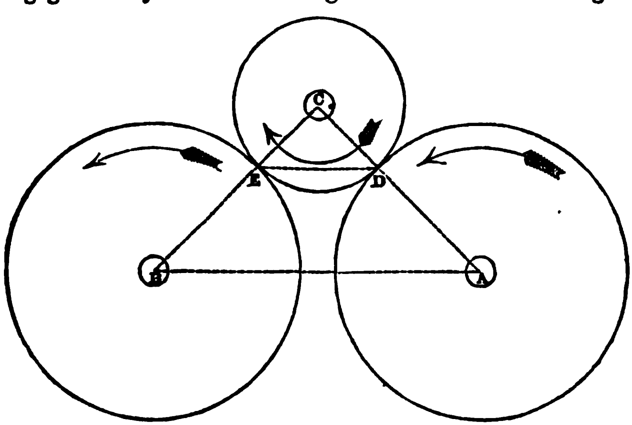

Fig. 213.

A and B are the traces of the fixed axes of a pair of smooth wheels, whose surfaces do not touch each other: A being the driver and B the follower. C is the trace of the moveable axis of an intermediate idle wheel, which drives B, and is driven by A; D being the pitch-point of A and C, and E the pitch-point of C and B. The straight line D E is the common line of connection of the three wheels; and as pressure only, and not tension, can be transmitted along that line from the first wheel to the third wheel, the connection ceases if the motion is reversed. To disengage the wheels while in motion forwards, the axis C is shifted so as to put an end to the contact at D or at E, or at both those points.

The angles of obliquity, C D E = C E D, which the line of connection D E makes with the two lines of centres, A C and B C, ought to be a little greater than the "angle of repose" of the surfaces of the wheels, in order that the wheel C may not become jammed between the wheels A and B; but it ought not to be greater than is just sufficient to prevent the risk of jamming; in order that the force with which C must be pressed towards A and B may not become unnecessarily great. The value of that force and of the angles of obliquity will be considered under the head of the "Dynamics of Machinery;" meanwhile, in anticipation of that division of this treatise, the following values are given of the angle of repose for different surfaces:—

Cylindrical surfaces without grooves.
Metal on Metal; dry, 10°; slimy, 8°; greasy, 4°.
Metal on Oak; dry, 28°; wet, 14°.
Metal on Elm; dry, 13°.
Leather on Metal; dry, 29½°; wet, 20°; greasy, 13°.
Leather on Oak, 17°.
Grooved metal surfaces, as in frictional gearing; about 28°.

The construction, therefore, for designing this disengagement is as follows:—Construct an isosceles triangle C D E, with the angles at D and E each a little greater than the angle of repose; produce C D and C E, laying off upon them D A and E B proportional to the radii of the wheels to be connected; join A B. Then the proportions borne respectively by A D, B E, and C D, to A B, will be those which the radii of the wheels are to bear to the line of centres.

263. **Disengagements and Reversing-Gear acting by Sliding Contact.**—A pair of toothed wheels, whether spur, bevel, or skew-bevel, may be thrown into or out of gear by shifting one of them along its axis. This sort of disengagement belongs to the class in which motion is transmitted by pressure; so that the velocity-ratio and the relative position of the pieces are definite, and the communication of motion abrupt. Another way of making it act is to have the wheels always in gearing with each other, and to effect the engagement and disengagement of one of them by

means of a clutch upon the shaft that carries it; as in Article 260, page 295.

The most common kind of reversing-gear which acts by means of toothed wheels is shown in fig. 214; A is the driving-shaft, carrying a bevel-wheel which drives in contrary directions a pair of bevel-wheels, B, C, that turn loose on the driven shaft, D D. A double clutch, E, sliding along a feather on the latter shaft, is made, by means of a collar and lever, to lay hold of the one or the other of the bevel-wheels B, C, according to the direction in which the shaft D D is to rotate.

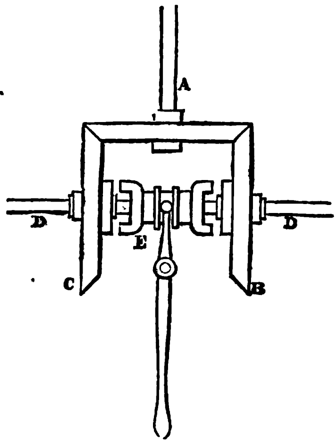

Fig. 214.

264. **Disengagements and Reversing-Gear by Bands.**—When rotation is transmitted from one shaft to another by means of a belt and a pair of pulleys, the form of engaging and disengaging-gear employed is the *"fast and loose pulley"* already described in Article 170, pages 184 and 185. The fork mentioned there is called a *belt-guide*, or *belt-shifter*. It is evident that the contrivance of the fast and loose pulley is applicable to belts alone, and not to cords and chains.

Reversing-gear by means of belts with fast and loose pulleys is arranged in the following way: on the driven shaft is one fast pulley, between two loose pulleys, one for each of the two belts, which run in opposite directions. In the act of reversing the motion, care should be taken that the belt which has been driving the fast pulley is shifted completely on to its own loose pulley before any part of the other belt is shifted on to the fast pulley.

A method of engaging and disengaging connection by bands, applicable to cords as well as to belts, is to tighten and slacken the band when required, by means of a *straining pulley*, as already described in Article 174, page 188.

265. **Disengagements and Reversing-Gear acting by Linkwork.**—Amongst disengagements acting by linkwork are all the examples of intermittent linkwork described in Article 194 to 197, pages 206 to 213; and in most of those examples, besides the periodical disengagement which takes place at each return stroke, there exists also the means of making a permanent disengagement, by fixing the click or catch so as to prevent it from taking hold of the

teeth. In fig. 146, page 207 (described at page 208), is an example of reversing-gear in linkwork.

In ordinary linkwork (as distinguished from click-and-ratchet-work) the means of disengagement consist in connecting the link with the pin at one of the connected points by means of a *gab* (as at A, fig. 215); that is, a deep notch with plane sides and a semi-cylindrical bottom, fitting the pin accurately but easily. The link is thrown out of gear when required, by moving the gab clear of the pin, either by hand or by suitable mechanism. Sometimes the gab is provided with spreading jaws, to enable it the more easily to lay hold of the pin when the connection is to be re-engaged.

Fig. 215.

Another case of disengagement by linkwork is that of the *hooks in a Jacquard loom*. At each shot or stroke of the loom there are certain threads of the warp that have to be raised and lowered again, while other threads remain at rest; the order and arrangement of threads so treated being varied at each shot, in a manner depending on the pattern to be woven. In fig. 216, B C is a hook, of which the lower end is connected with a thread: the hook is kept in a nearly vertical position by passing easily through a hole at D in a horizontal sliding-bar, D E, called a *needle;* and the hooked upper end, at B, overhangs a horizontal bar or rail, A, which is carried by a frame having a vertical reciprocating motion, of the extent represented by A A′. In the position shown by full lines and capital letters the hook stands ready to be lifted by the rail A; but when the needle is drawn back to the position *d e*, the hook is made to assume the position C *d b*, shown in dotted lines, in which it stands disengaged from the rail, and remains at rest during the next stroke. The needles are usually drawn back by means of springs, and pushed forward by the forward stroke of a drum, F, which turns about a horizontal axis, and has also, along with that axis, a reciprocating motion in the direction of the length of the needles. The drum is

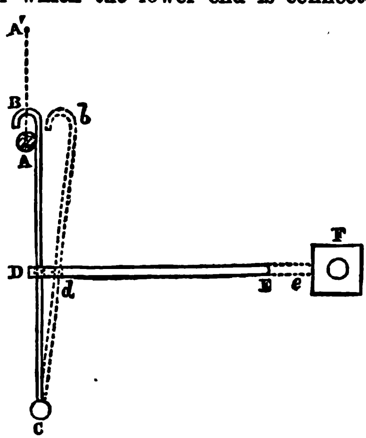

Fig. 216.

of the form of a polygonal prism, usually square, as in the figure; its acting face is covered with an oblong card (of pasteboard or sheet metal), having holes in it opposite the ends of those needles which are *not* to be pushed forward. The drum does not rotate during its forward stroke, when it is pushing the needles; but during the return stroke a catch pulls it round so as to bring a new face opposite the needles, with a new card upon it, having a proper arrangement of holes for the next stroke. The cards, in sufficient number to produce the entire pattern, are linked together at their longer edges, so as to form, as it were, a flat chain, which hangs over the drum, by whose rotation they are brought round one by one to act on the ends, E, of the needles.

The Jacquard Apparatus, of drum, cards, needles, and hooks, may be applied to many purposes besides that of lifting the threads of a warp.

266. **Disengagements acting by Hydraulic Connection—Valves.**—When the driver and follower are two pistons, and the former transmits motion to the latter by means of an intervening mass of fluid (as in Articles 207 to 211 A, pages 221 to 227), the engagement and disengagement are effected by opening and closing a valve in the passage through which the displaced fluid flows: as has been already stated in Article 211, page 224. If the forward motion of the driving piston is to go on while the valve is closed, some other outlet must be opened for the fluid which it displaces.

A reversing action takes place in hydraulic connection, when a stream of fluid is admitted, by means of suitable valves, so as to act alternately on the two sides of a piston, as in a double-acting water-pressure engine.

In all cases in which the motion of a piston driven by a fluid is reversed, an outlet, with a suitable valve, must be provided for the escape from the cylinder, during the return stroke, of the mass of fluid by which the previous forward stroke was produced.

267. **Principles of the Action of Valves.**—It would be out of place here to describe in detail the various kinds of valves used in machinery; and therefore a summary only of the general principles of their construction and action, so far as those principles can be considered as forming part of the Geometry of Machines, will now be given, chiefly abridged from *A Manual of the Steam Engine and other Prime Movers.*

Valves in general, considered with reference to the means by which they are moved, may be divided into three principal classes:—Valves, sometimes called *clacks*, which are opened and shut by the pressure of the fluid that traverses their openings, and are usually intended for the purpose of permitting the passage of the fluid in one direction only, and stopping its return;—valves moved by hand;—and valves moved by mechanism. When a piston drives a fluid, as in ordinary pumps, the valves are usually moved by the fluid: when the fluid drives the piston, it is in general necessary that the valves should be moved by hand or by mechanism. In water-pressure engines that work occasionally and at irregular intervals, such as hydraulic hoists and cranes, the valves are usually opened and shut by hand; in those which work periodically and continuously, they are moved by mechanism connected with the engine. Valves, when considered with reference to the kind of motion by which they open and shut the *ports*, or orifices to which they are fitted, may be distinguished into *Drop-valves*, which are opened and shut by being lifted up and set down; *Flap-valves*, which turn on a hinge; and *Slide-valves*.

The *seat* of a valve is the fixed surface on which it rests, or against which it presses.

The *face* of a valve is that part of its surface which comes in contact with the seat.

When a valve occurs *in the course* of a pipe or passage, the valve-box or chamber, being that part of the passage in which the valve works, should always be of such a shape as to allow a free passage for the fluid when the valve is open, so that the fluid may pass the valve with as little change of area of the stream as possible; and if necessary for that purpose, the valve-chamber may be made of larger diameter than the rest of the passage.

A valve moved by mechanism has almost always a periodical reciprocating motion, by which it is alternately opened and shut. The simplest mode by which that motion can be given is by a crank, or an eccentric, carried by some continuously-rotating piece, and acting through a rod; as in Articles 184 to 186, pages 196 to 198; and such is the ordinary way of moving *slide-valves*. *Drop-valves* are sometimes worked by the same kind of mechanism, with the addition of a contrivance for setting them down very gently, of the kind described in Article 190, pages 202, 203; or by means of cams or wipers (Articles 160 to 164, pages 170 to 175).

The principal forms of valves are the following:—

I. The *Bonnet-Valve* or *Conical Valve* is the simplest form of drop-valve, and is a flat or slightly arched circular plate whose face, being formed by its rim, is sometimes a frustum of a cone, and sometimes a zone of a sphere, the latter figure being the best. Its *seat*, being the rim of the circular orifice which the valve closes, is of the same figure with the face or rim of the valve, and the valve-face and its seat are turned and ground to fit each other exactly, so that when the valve is closed no fluid can pass. The thickness of a valve of this form is usually from a fifth to a tenth of its diameter, and the mean inclination of its rim about 45°.

To ensure that the valve shall rise and fall vertically, and always return to its seat in closing, it is sometimes provided with a *spindle*, moving through a ring or cylindrical socket. A knob on the end of the spindle prevents the valve from rising too high. When the valve is to be moved by hand or by mechanism, the spindle may be continued through a stuffing-box, and connected with a handle or a lever, so as to be the means of transmitting motion to the valve.

II. The *Ball Clack* is a drop-valve of the form of an accurately-turned sphere. When of large size, it is in general hollow, in order to reduce its weight. Its face is its entire surface: its seat is a spherical zone.

III. The *Divided Conical Valve* is composed of a series of concentric rings. The largest ring may be considered as a bonnet-valve, in which there is a circular orifice, forming a seat for a smaller bonnet-valve, in which there is a smaller circular orifice, forming a seat for a still smaller bonnet-valve, and so on. This arrangement enables a large opening for the passage of fluid to be formed with a moderate upward motion of each division of the valve.

IV. The *Double-beat Valve* is a drop-valve so contrived as to enable a large passage for a fluid to be opened and shut easily under a high pressure. Fig. 217 represents a section of the valve, with its seats and chamber, and fig. 218 a plan of the valve alone.

The valve shown in the figure is for the purpose of opening and shutting the communication between the pipes A and B.

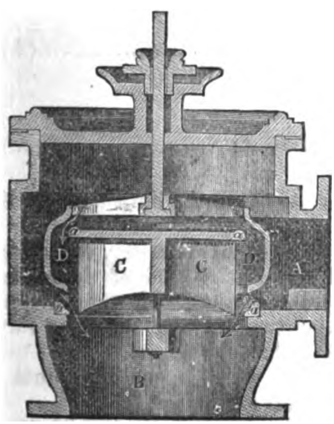

Fig. 217.

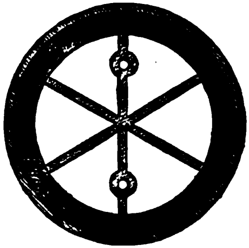

Fig. 218.

The pipe B is vertical, and its upper rim carries one of the two valve-seats, which are of the form of the frustum of a cone, and each marked *a*.

A frame C, composed of radiating partitions, fixed to and resting on the upper end of the pipe B, carries a fixed circular disc, whose rim forms the other conical valve-seat.

The valve D is of the form of a turban, and has two annular conical faces, which, when it is shut, rest at once on and fit equally close to the two seats, *a*, *a*. When the valve is raised, the fluid passes at once through the cylindrical opening between the lower edge of the valve and the upper edge of the pipe B, and through the similar opening between the upper edge of the valve and the rim of the circular disc.

The greatest possible opening of the valve is when its lower edge is midway between the disc and the rim of the pipe B, and is given by the following formula:—

Let

d_1 be the diameter of the pipe B;

d_2, that of the disc;

h, the clear height from the pipe to the disc, *less* the thickness of the valve;

A, the greatest area of opening of the valve; then

$$A = 3{\cdot}1416 \frac{d_1 + d_2}{2} \cdot h; \text{ .... (1.)}$$

and in order that this may be at least equal to the area of the pipe B, viz., $\cdot 7854\ d_1^2$, we should have

$$h \text{ at least} = \frac{d_1^2}{2(d_1 + d_2)}; \text{ .... (2.)}$$

which, if, as is usual, $d_1 = d_2$, gives

$$h \text{ at least} = \frac{d_1}{4}; \text{ .... (2 A.)}$$

but h is in general considerably greater than the limit fixed by this rule.

If the upper and lower seats are of equal diameter, the valve is called an *equilibrium-valve;* and this is the kind of double-beat valve most commonly used in steam engines. In water-pressure engines, pumps, and hydraulic apparatus generally, the lower valve-seat is generally made a little larger than the upper.

V. A common *Flap-Valve* is a lid which opens and shuts by turning on a hinge. The face and seat are planes.

A pair of flap-valves placed hinge to hinge constitute a "*butterfly clack.*" The chamber of a flap-valve should be of considerably greater diameter than the valve.

VI. A *Flexible Flap-Valve* consists of a piece of some flexible material, such as waterproof canvas or India rubber. It may be rectangular, so as to have one edge fixed to the seat, and the opposite edge attached to a bar, by moving which it is opened and shut; or it may be circular, and fixed to the seat at the centre; and this is the form usually adopted for self-acting flexible flap-valves in pumps. The seat of the flap consists of a flat horizontal grating, or a plate perforated with holes. To prevent a circular flap-valve from rising too high, it is usually provided with a guard, which is a thin metal cup formed like a segment of a sphere, grated or perforated like the valve-seat, to which it is bolted at the centre. When the valve is raised by a current from below, it applies itself to the bottom of the cup. When the current is reversed, the fluid from above, pressing on the valve through the holes in the cup, drives it down to its seat again.

VII. The *Disc-and-Pivot Valve*, or *Throttle-Valve*, consists of a thin flat metal plate or disc, which, when shut, fits closely the opening of a pipe or passage, generally circular in section, but sometimes rectangular. The valve turns upon two pivots or journals, placed at the extremities of a diameter traversing its centre.

When the valve is turned so as to lie edgeways along the passage, the current of fluid passes with very little obstruction: when it is turned transversely, the current is stopped, or nearly stopped. By placing the valve at various angles, various openings can be made. If the valve, when shut, is perpendicular to the axis of the pipe, the opening for any given inclination of the valve to that axis is proportional to the *coversed-sine of the inclination*. If the valve is oblique when shut, the opening at a given inclination is proportional to the *difference* between the *sine of that inclination* and the *sine of the inclination when shut*.

The *face* of this valve is its rim; its *seat* is that part of the internal surface of the passage which the rim touches when the valve is shut; and those surfaces ought to be made to fit very accurately, without being so tight as to cause any difficulty in opening the valve.

One of the journals of the valve usually passes through a bush or a stuffing-box in the pipe, so as to afford the means of communicating motion to the valve from the outside.

VIII. *Slide-Valves.*—The *seat* of a slide-valve consists of a plane metal surface, very accurately formed, part of which is a rim surrounding the orifice or *port*, which the valve is to close, and from $\frac{1}{4}$ to $\frac{1}{20}$ of the breadth of that orifice, while the remainder extends to a distance from the orifice equal to the diameter of the valve, in order that the valve, when in such a position as to leave the port completely open, shall still have every part of its face in contact with the seat.

The valve is of such dimensions as to cover the port together with that portion of the seat which forms a rim surrounding the port. The face of the valve must be a true plane, so as to slide smoothly on the seat. As to the periodical motion of slide-valves, see the next Article.

Rotating slide-valves are sometimes used, in which the valve and its seat are a pair of circular plates, having one or more equal and similar orifices in them. The passage is opened by turning the valve about its centre until its openings are opposite to those of the seat, and shut by turning it so that its openings are opposite solid portions of the seat. (See page 314.)

IX. *A Piston-Valve* is a piston moving to and fro in a cylinder, whose internal surface is the *valve-seat*. The *port* is formed by a ring or zone of openings in the cylinder, communicating with a passage which surrounds it; and by moving the piston to either side of those openings, that passage is put in communication with the opposite end of the valve-cylinder.

X. *Cocks.*—This term is sometimes applied to all valves which are opened and shut by hand; but its proper application is to those valves which are of the form of a frustum of a cone, or conoid, turning in a seat of the same figure.

In the most common form of cock, the seat is a hollow cone of slight taper, having its axis at right angles to the pipe in whose course it occurs. The valve is a cone fitting the seat accurately, and having a transverse passage through it of the same figure and size with the bore of the pipe, so that in one position it forms simply a continuation of the pipe, and offers no obstruction to the current, while, by turning it into different angular positions, the opening may be closed either partially or wholly.

268. **Periodical Motion of Slide-Valves.**—The motion of a slide-valve driven by a crank or an eccentric is a case of *approximate harmonic motion*, as already described in Article 239, page 250; and in most cases which occur in practice, it may be treated, without material error, as if it were exact harmonic motion: that is to say, the *travel* or *length of stroke of the slide* is twice the eccentric-arm; the slide is in its *middle position* when the eccentric-arm is sensibly at right angles to the line of its dead-points; in other words, when the *phase* of its revolution is sensibly 90°; and the *displacement* of the slide from its middle position at any given instant is sensibly equal to the eccentric-arm multiplied by the cosine of the phase. For example, in fig. 220 (page 308), the straight line F A L, bisected in A, represents twice the eccentric-arm; so that A F and A L respectively represent the displacements of the slide at the two dead-points of the revolution of the eccentric, when the phase is respectively 0° and 180°. On those two lines as diameters describe two equal circles, A H F G A, and A N L P A; then, when the phase is $= \angle$ F A D, the dis-

placement is = A G; and when the phase is = $\angle$ F A M, the displacement is = A N, in the contrary direction to that of the displacement A G.

Under the geometry of machinery are comprehended the rules by which the movement of the slide-valve of an engine is made to bear certain relations to that of the crank with which the piston is connected. The following are terms used in those rules:—

The two opposite sides of the port, or oblong opening in the seat of a slide-valve, are distinguished as the *induction-side* and the *eduction-side*;—the former being the side at which the fluid enters the port; the latter, the side at which it is discharged.

The *lap*, or *cover*, of a slide-valve at one of its edges is the extent to which that edge overlaps the adjoining edge of the port which it covers when the slide-valve is in its middle position. In fig. 219 is a section of part of a vertical slide-valve and its port; W is the lower port of a cylinder; X, the lower half of the slide-valve, in its middle position; U is the *induction-side*, and V the *eduction-side*, of the port; C is the *induction-edge*, and P the *eduction-edge* of the valve; U C is the *lap on the induction-side*, and V P the *lap on the eduction-side:* the hollow part of the valve opposite X is called the *exhaust-cavity*.

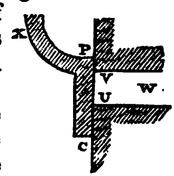

Fig. 219.

It is evident that the opening and closing of the port at either side take place at the instants when the displacement of the slide in a direction *away* from that side is equal to the lap at that side; and that the port remains open at that side so long as the displacement in the proper direction is greater than the lap. Thus, the port W remains open at the side U, so long as the displacement of the slide towards P is greater than U C; and at the side V, so long as the displacement of the slide towards C is greater than V P. If the lap at either side is nothing, the opening and closing at that side take place in the middle position of the slide; and the port remains open at that side during half a revolution of the eccentric.

The instant at which the port is first opened at the induction-side is called the instant of *admission;* that at which it is closed, of *suppression*, or *cut-off;* that at which it is first opened at the eduction-side, the instant of *release;* that at which it is closed at the eduction-side, the instant of *compression*.

By the *angular advance* of the eccentric is to be understood the angle at which the eccentric-arm stands in advance of that position, which would bring the slide-valve to mid-stroke when the crank is at its dead-points: in other words, the excess above 90° of the phase of the eccentric when the phase of the crank is 0°; or in symbols, phase of eccentric – phase of crank – 90°. When the slide is at its middle position at the same instant at which the crank is at a dead-point, the angular advance is nothing.

RULE I.—Given, the positions of the crank at the instants of admission and cut-off; to find the proper angular advance of the eccentric, and the proportion of the lap on the induction-side to the half-travel of the slide.*

In fig. 220, let A B and A C be the positions of the crank at the beginning and end of the forward stroke; let the arrow show the direction of rotation; let X x be perpendicular to B C; let A D be the position of the crank at the instant of cut-off, and

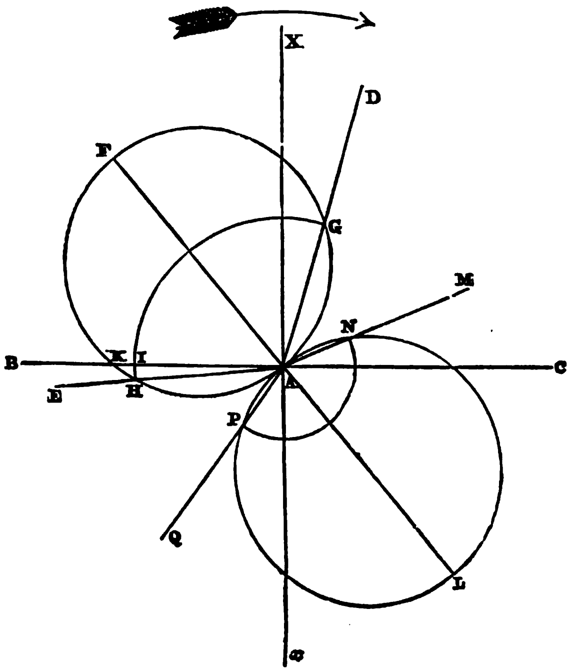

Fig. 220.

A E its position at the instant of admission. Draw A F, bisecting the angle E A D; A F will represent the position of the

* The method used in this and the following rules is that of Professor Dr. Zeuner, of the Swiss Federal Polytechnic School at Zürich, published in his treatise on Slide-valve Gearing, entitled, *Die Schiebersteuerungen.*

crank at the instant when the slide is at the *forward end* of its stroke; and F A X will be the *angular advance of the eccentric.*

Lay off the distance A F to represent the half-travel; and on A F as a diameter describe the circle A H F G, cutting A D in G and A E in H; then $\frac{AG}{AF} = \frac{AH}{AF}$ will be the *required ratio of lap at the induction-side to half-travel;* and A G = A H will represent that lap, on the same scale on which A F represents the half-travel.

On the same scale, I K represents the *width of opening of the valve at the beginning of the stroke,* sometimes called the "*lead of the slide.*" Strictly speaking, this is the lead of the induction-edge of the slide only; the lead of the centre of the slide being A K; that is, its distance from its middle position at the beginning of the forward stroke.

RULE II.—Given, the data and results of the preceding rule, and the position, A M, of the crank at the instant of release; to find the ratio of lap on the eduction-side to half-travel, and the position of the crank at the instant of compression. Produce F A to L, making A L = A F; on A L as a diameter draw a circle cutting A M in N; then $\frac{AN}{AL}$ will be the *required ratio of lap at eduction-side to half-travel.*

About A draw the circular arc N P, cutting the circle A L again in P; join A P; then A P will be the required *position of the crank at the instant of compression.*

RULE III.—Given, the data and results of Rule I., and the position, A Q, of the crank at the instant of compression; to find the ratio of lap at the eduction-side to half-travel, and the position of the crank at the instant of release. Produce F A as before; on A L = F A as a diameter draw a circle cutting A Q in P; $\frac{AP}{AL}$ will be the *required ratio of lap at the eduction-side to half-travel.*

About A draw the circular arc P N, cutting the circle A L again in N; join A N; A N will be the position of the crank at the instant of release.

RULE IV.—Given, the angular advance of the eccentric, the half-travel of the slide, and the lap at both sides; to find the positions of the crank at the instants of admission, cut-off, release, and compression. Draw the straight lines B A C and X A x perpendicular to each other; and take B and C to represent the dead-points. Let the arrow denote the direction of rotation. Draw F A L, making the angle F A X = the angular advance of the eccentric; and make A F = A L = half-travel. On A F and

A L as diameters, draw circles. About A, with a radius equal to the lap at the induction-side, draw an arc cutting the circle on A F in H and G; also, with a radius equal to the lap at the eduction-side, draw an arc cutting the circle on A L in N and P. Draw the straight lines A H E, A G D, A N M, A P Q. These will represent respectively the positions of the crank at the instants of *admission*, *cut-off*, *release*, and *compression*.

The eccentric may act on the slide, not directly, but through a train of levers and linkwork. The effect of this on the application of the rules is merely to substitute for the actual eccentric-arm a virtual eccentric-arm equal to the half-travel of the slide.

The effects of the link-motion, of double slides, and of moveable slide-valve seats, in modifying the length and position of the virtual eccentric-arm, have been already described in Articles 239 to 241, pages 250 to 260.

Section II.—*Of Adjustments for Changing Speed and Stroke.*

269. **General Explanations.**—All methods of changing the velocity-ratio of an elementary combination in a machine operate by changing the position of their line of connection; for on the position of that line the velocity-ratio depends, according to the principle already explained in Article 91, page 78. In some cases the combination contains two or more pairs of acting surfaces (such as wheels or pulleys), one or other of which can be thrown into gear according to the velocity-ratio required; and then it is in general necessary to stop the motion in order to change the velocity-ratio. In other cases there are contrivances for changing the velocity-ratio by degrees while the machine is in motion.

In the case of linkwork the change of velocity-ratio is often connected with a change of length of stroke.

Many of the most ordinary and useful adjustments for changing speed have already been described under the head of elementary or of aggregate combinations; and in such cases it will be sufficient in the present section to refer to the place where the detailed description is to be found.

Adjustments for changing speed, like engaging and disengaging-gear, may in most cases be distinguished into two classes, according as the connection is made by pressure or by friction. In the former case the change of velocity-ratio is definite, and in most instances sudden; in the latter case, gradual, and to a certain extent indefinite.

270. **Changing Speed by Friction-Wheels.**—To obtain changes of speed by means of friction-wheels, a pair of parallel shafts are to be provided with as many pairs of wheels as there are to be different velocity-ratios; each pair of wheels being connected with each other, not directly, but by means of an intermediate idle wheel,

which can be thrown into or out of gear at pleasure, as in the second method of disengagement described in Article 262, page 297; the only difference being that whereas in that Article the two principal wheels of the pair are described as being equal, in the present case they will in general be unequal. The rule as to the obliquity of the line of connection is the same. (See page 298.)

A combination of friction-wheels in which the velocity-ratio is changed by degrees during the motion, is shown in fig. 221. (It forms part of Morin's Integrating Dynamometer.) A is a plane circular disc, turning about an axis perpendicular to its own plane. B is a wheel driven by the friction of the disc against its edge; and it turns about an axis that cuts the axis of A at right angles. The angular velocity of B varies proportionally to its distance from the centre of A, and is varied by altering that distance.

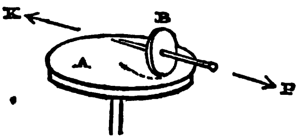

Fig. 221.

271. **Changing Speed by Toothed Wheels.**—The ordinary method of producing precise and definite changes of the angular velocity-ratio of two rotating shafts is by means of *change-wheels:* that is to say, there are several pairs or trains of wheels, suited to a certain series of velocity-ratios; and one or other of those pairs or trains of wheels is thrown into gear according to the comparative speed that is wanted at the time.

Sometimes the wheels are made so as to be put on the shafts and taken off at pleasure. If an intermediate idle wheel is not used, between two shafts connected by pairs of change-wheels, there must be as many pairs of change-wheels as there are different velocity-ratios; because the sum of the geometrical radii of each pair must be equal to the line of centres; but by the help of an intermediate idle wheel, any two wheels which are not so large as to touch each other may be put into connection; so that by a proper choice of numbers of teeth, the number of different ratios may be made equal to the product of the number of different wheels that can be fitted on one shaft into the number that can be fitted on the other after the first has been fitted.

Change-wheels are frequently arranged so as to be thrown into or out of gear by shifting the whole series longitudinally along with the shaft that carries them. For example, in fig. 222 A A and B B are a pair of parallel axes; and the transverse lines marked 1, 2, 3, &c., represent the radii of two series of change-wheels carried by shafts turning about those axes respectively. To each wheel of one series there corresponds a wheel in the other series, marked with the same figure; and any such pair can be thrown into gear when required, by shifting the shaft A longitudinally. To place the wheels on the shafts so as to occupy the least possible space, the following rules are to be observed:—Let b denote the breadth of the rim of a wheel, plus a small allowance for clearance. Range the radii of the wheels on A in such a manner that the greatest shall be in the middle, with a diminishing series on

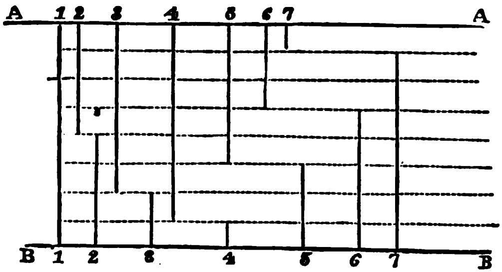

Fig. 222.

each side of it. Then, commencing at the two ends of the double series, make the two endmost intervals between the middle planes of the wheels on the axis A ($\overline{1\,2}$ and $\overline{7\,6}$ in the figure), each $= b$; the pair of intervals next them ($\overline{2\,3}$ and $\overline{6\,5}$ in the figure), each $= 2\,b$; the next pair ($\overline{3\,4}$ and $\overline{5\,4}$), each $= 3\,b$; and so for any number of intervals that may be required. Then make the interval between the middle planes of each pair of wheels on the axis B greater by *one breadth*, b, than the corresponding interval on the axis A.

272. **Changing Speed by Bands and Pulleys.**—The most convenient way of changing the velocity-ratio of rotation of a pair of shafts, where absolute precision in the ratio is not required, is by means of "*speed-cones*," which have already been described in Article 175, page 185. When a series of pulleys is used with radii changing step by step, the motion must be stopped in order to shift the band from one pair of pulleys to another; and this is applicable to cords as well as to belts. When tapering conoidal pulleys are used, the belt can be shifted, and the velocity-ratio gradually changed, while the machinery is in motion; and this is applicable to belts only.

273. **Changing Stroke in Linkwork.**—The principles upon which the length of stroke in linkwork depends have been explained in Article 186, page 197. When a piece receives a reciprocating motion from a lever, a crank, or an eccentric, the simplest way of changing the length of stroke is to change the distance of the connected point in the lever, crank, or eccentric, from its axis of motion. In the case of a continuously rotating crank or eccentric, this can be done by means of an adjusting screw, the motion being

stopped when an alteration is to be made; but in the case of a reciprocating lever, the pin to which the connecting-rod is jointed may be carried by a stud, capable of sliding in a slot in the lever, and having its position in that slot adjusted by means of a rod and a handle which can be shifted while the machinery is in motion. Sufficient examples of the latter kind of action have already been given under the head of link-motions, in Article 240, pages 253 to 260.

Fig. 223 represents a train of linkwork proposed by Willis, for adjusting the velocity-ratio and comparative length of stroke of two reciprocating points. The points to be connected are marked D and E; and D A and E A are their lines of stroke, intersecting each other in A. A B is a train-arm centred at A, and capable of being adjusted to any required angular position. At B, the other end of the train-arm, is centred the reciprocating lever B C, equal in length to B A, and connected with the points D and E by the links C D and C E.

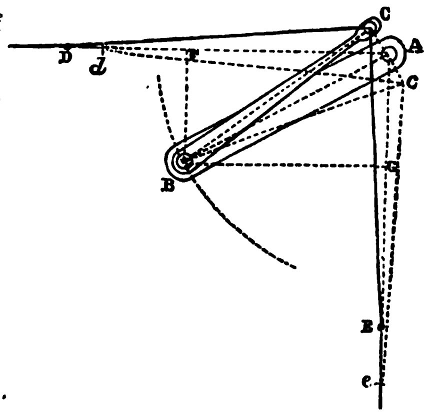

Fig. 223.

While the lever B C oscillates through a small angle to either side of B A, the motions of D and E are very nearly equal to the component motions of C along A D and A E respectively; that is to say, we have, at any given instant, the following proportion very nearly exact:—

$$\text{Velocity of C} : \text{velocity of D} : \text{velocity of E} \;::\; BA : BF : BG;$$

in which B F and B G denote the lengths of perpendiculars let fall from B on A D and A E respectively; and the same proportions hold very nearly for the lengths of stroke of those three points; hence those proportions can be made to assume any required value while the mechanism is in motion, by adjusting the position of the train-arm A B.

274. **Changing Speed with Hydraulic Connection.**—The comparative speed of a piston driven by a fluid may be altered by altering the number of driving-pistons which force the fluid into the cylinder of the driven piston at the same time. For example,

in some hydraulic presses it is desirable to diminish step by step the ratio which the velocity of the press-plunger bears to that of the pump-plungers; and that is done by forcing water into the press-cylinder at first by means of several pumps at once, and diminishing their number as the process goes on, until at last only one is kept at work.

Addendum to Article 267, Page 306.

Slide-Valves.—Another class of rotating slide-valves is that in which the seat of the valve forms part of a cylindrical surface, usually concave; the face of the valve forms an arc of a corresponding cylindrical surface, convex when the seat is concave; and the reciprocating motion of the valve takes place by rocking, or oscillating rotation, about the axis of the cylindrical surfaces. The "Corliss" valves are an example of this.

A straight-sliding slide-valve and its seat are also sometimes of a cylindrical form, the reciprocating motion taking place parallel to the axis of the cylinder.

There are instances of plane-faced slide-valves which have motions of curvilinear translation, produced by aggregate combinations of linkwork: for example, Hunt's slide-valves.

PART II.

DYNAMICS OF MACHINERY.

CHAPTER I.

SUMMARY OF GENERAL PRINCIPLES.

275. **Nature and Division of the Subject.**—In the present Part of this work, machines are to be considered not merely as modifying motion, but also as modifying force, and transmitting energy from one body to another. The theory of machines consists chiefly in the application of the principles of dynamics to trains of mechanism; and therefore much of the present part of this treatise will consist of references back to Part I.*

There are two fundamentally different ways of considering a machine, each of which must be employed in succession, in order to obtain a complete knowledge of its working.

I. In the first place is considered the action of the machine during a certain period of time, with a view to the determination of its EFFICIENCY; that is, the ratio which the *useful* part of its work bears to the whole expenditure of energy. The motion of every ordinary machine is either uniform or periodical; and therefore the principle of the equality of energy and work is fulfilled, either constantly, or periodically at the end of each period or cycle of changes in the motion of the machine.

II. In the second place is to be considered the action of the machine during intervals of time less than its period or cycle, if its motion is periodic, in order to determine the law of the periodic changes in the motions of the pieces of which the machine consists, and of the periodic or reciprocating forces by which such changes are produced.

The present Chapter contains a summary of the principles of dynamics—that word being taken in the comprehensive sense in which it is used in Thomson and Tait's *Natural Philosophy*, to

* A large portion of the present Part, and especially of the second Chapter, although originally written for this work, has already appeared as an Introduction to *A Manual of the Steam Engine and other Prime Movers;* for that book would have been incomplete without an explanation of the dynamical principles of the action of machines in general.

denote the science of forces, whether employed in balancing each other or in producing motion. The ensuing Chapters will contain the special application of those principles to machines.

276. **Forces—Action and Re-action.**—Every force is an action exerted between a pair of bodies, tending to alter their condition as to relative rest and motion; and it is exerted equally, and in contrary directions, upon each body of the pair. That is to say, if A and B be a pair of bodies acting mechanically on each other, the force exerted by A upon B is equal in magnitude and contrary in direction to the force exerted by B upon A. This principle is sometimes called *the equality of action and re-action.* It is analogous to that of relative motion, explained in Article 42, page 21.

The forces chiefly to be considered in machines are the following:—

I. *Gravity*, exerted between the parts of the machine, fixed and moving, and the whole mass of the earth. The action of the earth on the machine alone requires to be considered in practice; for although the re-action of the machine on the earth is equal and opposite, the enormous mass of the earth, as compared with the machine, causes the effects of that re-action to be inappreciable. This is the only case in which re-action may be disregarded.

II. Forces exerted *between parts of the machine* and *contiguous external bodies*, solid or fluid. Sometimes those bodies support the foundations of the machine: sometimes they drive the machinery; as when the impulse or the pressure of a fluid drives an engine: sometimes they are moved by it; as in the lifting of loads, the overcoming of friction against external bodies, the working of machine tools, &c.

III. Forces exerted *between a moving piece and the frame*, at their bearing surfaces. These forces may be distinguished into pressure and friction. By the pressures exerted by the bearings the moving piece is kept in its proper place and path; by friction its motion is resisted. The equal and opposite re-actions of the moving piece on the frame tend to strain the frame; and the making of the frame so as to be capable of bearing them involves questions of strength, belonging to the Third Part of this treatise.

IV. Forces exerted *between connected moving pieces.* These too may be distinguished into pressure and friction.

When exerted along the line of connection, they serve to transmit motion and motive power; when exerted transversely to it, they produce either a straining effect, or a waste of mechanical work, or both. Here the equality of action and re-action is of great importance. The force which is exerted between a driver and a follower along their line of connection is a *driving force*, otherwise called an *effort*, as regards the motion of the follower, and a *resistance* as regards the motion of the driver.

V. Forces exerted between the *different parts of one piece*, whether fixed or moving. These constitute the *stress*, by which the piece resists the tendency of the forces applied to it externally to overstrain it or to break it; and they belong to the subject of the Third Part.

277. **Forces, how Determined and Expressed.**—A force, as respects one of the two bodies between which it acts, is determined, or made known, when the following three things are known respecting it:—*first*, the *place*, or part of the body to which it is applied; *secondly*, the *direction* of its action; *thirdly*, its *magnitude*.

The PLACE of the application of a force to a body may be the whole of its volume, as in the case of gravity; or the surface at which two bodies touch each other, or the bounding surface between two parts of the same body, as in the case of pressure, tension, shearing stress, and friction.

Thus every force has its action distributed over a certain space, either a volume or a surface; and a force concentrated at a single point has no real existence. Nevertheless, in investigations respecting the action of a distributed force upon the position and movements, as a whole, of a rigid body, or of a body which without error may be treated as rigid, like the solid parts of a machine, fixed or moving, that force may be treated as if it were concentrated at a point or points, determined by suitable processes; and such is the use of those numerous propositions in statics which relate to forces concentrated at points; or *single forces*, as they are called.

The DIRECTION of a force is that of the motion which it tends to produce. A straight line drawn through the point of application of a single force, and along its direction, is the LINE OF ACTION of that force.

The MAGNITUDES of two forces are equal when, being applied to the same body in opposite directions along the same line of action, they balance each other.

The magnitude of a force is expressed arithmetically by stating in numbers its ratio to a certain *unit* or *standard* of force, which for practical purposes is usually the *weight* (or attraction towards the earth), at a certain latitude, and at a certain level, of a known mass of a certain material. Thus the British unit of force is the *standard pound avoirdupois;* which is the weight, in the latitude of London, of a certain piece of platinum kept in a public office. (See the Act 18 and 19 Vict., cap. 72; also a paper by Professor W. H. Miller, in the *Philosophical Transactions* for 1856.)

For the sake of convenience, or of compliance with custom, other units of weight are occasionally employed in Britain, bearing certain ratios to the standard pound; such as—

The grain = $\frac{1}{7000}$ of a pound avoirdupois.
The troy pound = 5,760 grains = $0.82\dot{2}8571\dot{4}$ pound avoirdupois.
The hundredweight = 112 pounds avoirdupois.
The ton = 2,240 pounds avoirdupois.

The French standard of weight is the *kilogramme*, which is the weight, in the latitude of Paris, of a certain piece of platinum kept in a public office. It was originally intended to be the weight of a cubic decimètre of pure water, measured at the temperature at which the density of water is greatest—viz., 4°·1 centigrade, or 39°·4 Fahrenheit, and under the pressure which supports a barometric column of 760 millimètres of mercury; but it is in reality a little greater.

A comparison of French and British measures of weight and of size is given in a table at the end of this volume.

A kilogramme is 2·20462125 lbs. avoirdupois.
A pound avoirdupois is 0·4535926525 of a kilogramme.

For scientific purposes, forces are sometimes expressed in *Absolute Units*. The "Absolute Unit of Force" is a term used to denote the force which, acting on an unit of mass for an unit of time, produces an unit of velocity.

The unit of time employed is always a second.

The unit of velocity is in Britain one foot per second; in France one mètre per second.

The unit of mass is the mass of so much matter as weighs one unit of weight near the level of the sea, and in some definite latitude.

In Britain the latitude chosen is that of London; in France, that of Paris.

In Britain the unit of weight chosen is sometimes a grain, sometimes a pound avoirdupois; and it is equal to 32·187 of the corresponding absolute units of force. In France the unit of weight chosen is either a gramme or a kilogramme, and it is equal to 9·8087 of the corresponding absolute units of force. Each of those co-efficients is denoted by the letter g.

A single force may be represented in a drawing by a straight line; the position of the line showing the line of action of the force, and an arrow-head its direction; a point in the line marking the point of application of the force; and the length of the line representing the magnitude of the force.

277 A. **Measures of Force and Mass.**—If by the unit of force is understood the weight of a certain standard, such as the avoirdupois pound, then the mass of that standard is $1 \div g$; and

the unit of mass is g times the mass of the standard; and this is the most convenient system for calculations connected with mechanical engineering, and is therefore followed in the present work.

But if we take for the unit of mass, the mass of the standard itself, then the unit of force is the *absolute unit;* and the weight of the standard in such units is expressed by g; for g is the velocity which a body's own weight, acting unbalanced, impresses on it in a second. This is the system employed in many scientific writings, and in particular, in Thomson and Tait's *Natural Philosophy*. It has great advantages in a scientific point of view; but its use in calculations for practical purposes would be inconvenient, because of the prevailing custom of expressing forces in terms of the standard of weight.

278. **Resultant and Component Forces—Their Magnitude.**—The RESULTANT of any combination of forces applied to one body is a single force capable of balancing that single force which balances the combined forces; that is to say, the resultant of the combined forces is equal and directly opposed to the force which balances the combined forces, and is *equivalent* to the combined forces so far as the balance of the body is concerned. The combined forces are called *components* of their resultant.

The resultant of a set of mutually balanced forces is nothing.

The *magnitudes* and *directions* of a resultant force and of its components are related to each other exactly in the same manner with the velocities and directions of resultant and component motions; and all the rules of Article 41, pages 18 to 21, are applicable to forces as well as to motions, and need not be repeated here.

As to the *position* of the resultant, if the components act through one point, the resultant acts through that point also; but if the components do not act through one point, the position of the resultant is to be found by methods which will be stated further on.

The following are additional rules as to resultant and component forces not explicitly given in Article 41:—

I. If the component forces act *along one line*, all in the same direction, their resultant is equal to their sum; if some act in one direction and some in the contrary direction, the resultant is their *algebraical sum;* that is to say, add together separately the forces which act in the two contrary directions respectively; the difference of the two sums will be the amount of the resultant, and its direction will be the same with that of the forces whose sum is the greater. This principle applies also to the *magnitudes* and *directions* of parallel forces not acting in one line.

II. *Triangle of Forces.*—Given, the directions of three forces which balance each other, acting in one plane and through one point; construct a triangle whose sides make the same angles with each other that the directions of the forces do; the proportions of the forces to each other will be the same with those of the corresponding sides of that triangle. Unless three forces act in one plane and through one point, they cannot balance each other.

To solve the same question by calculation; let A, B, C, stand for the magnitudes of the three forces; A O B, B O C, C O A, for the angles between their directions; then

$$\sin BOC : \sin COA : \sin AOB :: A : B : C.$$

Each of those three forces is equal and opposite to the resultant of the other two.

III. *Polygon of Forces.*—To find the resultant of any number (F_1, F_2, F_3, &c., fig. 224), of forces in different directions, acting through one point, O. Commence at the point of application, and construct a chain of lines representing the forces in magnitude, and parallel to them in direction, (O A = and || F_1, A B = and || F_2, B C = and || F_3, &c.) Let D be the end of that chain; join O D; this will represent the required resultant; and a force (F_5) equal and opposite to O D will balance the given forces. This rule applies to the *projections* of the forces on any given plane.

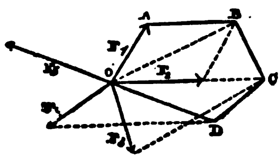

Fig. 224.

To solve the same question by calculation instead of by construction:—

IV. (When the forces act in *one plane.*) Assume any two directions at right angles to each other as axes; resolve each force into two components (X, Y) along those axes; take the resultants of those components along the two axes separately (ΣX, ΣY); these will be the *rectangular components of the resultant* R *of all the forces;* that is to say,

$$R = \sqrt{\left\{(\Sigma X)^2 + (\Sigma Y)^2\right\}};$$

and if α be the angle which R makes with X,

$$\cos \alpha = \frac{\Sigma X}{R}; \sin \alpha = \frac{\Sigma Y}{R}.$$

V. (When the forces act in *different planes.*) Assume any three directions at right angles to each other as axes; resolve each force into three components (X, Y, Z) along those axes; take the resultants of the components along the three axes separately (ΣX, ΣY, ΣZ); these will be the *rectangular components of the resultant* *of all the forces;* and its magnitude and direction will be given by the following equations:—

$$R = \sqrt{\left\{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2\right\}}.$$

$$\cos \alpha = \frac{\Sigma X}{R}; \cos \beta = \frac{\Sigma Y}{R}; \cos \gamma = \frac{\Sigma Z}{R}.$$

279. **Couples.**—In fig. 225, let F, F represent a couple of equal, parallel, and opposite forces, applied to a rigid body, and not acting in the same line; L, the perpendicular distance between their lines of action; then F is the *force* of the couple, L the *arm*, *span*, or *leverage;* and the product force × leverage = F L is the *statical moment* of the couple, which is right or left-handed according as the couple tends to impress right-handed or left-handed rotation on the body (Article 48, page 25). All the forces which produce and resist the motion of rotating pieces in a machine act in couples. Couples of equal moment acting in the same direction and in the same plane, or in parallel planes, are *equivalent* to each other.

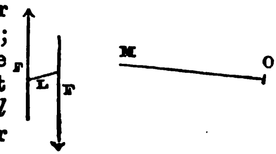

Fig. 225.

Comparison of Measures of Statical Moment.

						Kilogrammètres.
Inch-lb. =					....	0·011521
12 =	1 Ft.-lb. =				....	0·138254
112 =	$9\frac{1}{3}$ =	1 Inch-cwt. =			....	1·29037
1,344 =	112 =	12 =	1 Foot-cwt. =		....	15·4844
2,240 =	$186\frac{2}{3}$ =	20 =	$1\frac{2}{3}$ =	1 Inch-ton =	......	25·8074
26,880 =	2,240 =	240 =	20 =	12 =	1 Foot-ton =	309·689

I. To find the resultant moment of any number of couples acting on a rigid body in the same plane, or in parallel planes. Take the sums of the right-handed and left-handed moments separately; the difference between those sums will be the resultant moment, which will be right-handed or left-handed according to the direction of the moments whose sum is the greater.

II. To represent the moment of a couple by a single line. Upon any line perpendicular to the plane of the couple, set off a length proportional to the moment (O M, fig. 225), in such a direction that to a spectator looking from O towards M the couple shall seem right-handed. The line O M is called the *axis* of the couple.

Couples as represented by their axes are compounded and resolved like velocities, and like single forces, by the Rules of

Article 41, pages 18 to 21, and by those of Article 278, page 319.

III. To find the resultant of a single force, F, applied to a rigid body at O, and a couple, M, acting on the same body in the same or in a parallel plane. Conceive the force F to be shifted in that plane, parallel to itself, to the left if the couple is right-handed, to the right if the couple is left-handed, through a distance, O A, found by dividing M by F. The shifted single force, F acting through A, will be the resultant required.

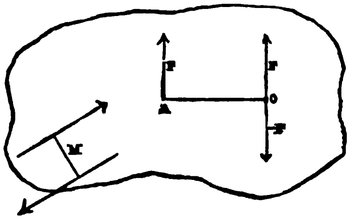

Fig. 226.

(The combination of a single force with a couple acting in a plane perpendicular to the line of action of the force cannot be further simplified.)

IV. To resolve a single force into a single force acting in a different but parallel line, and a couple. In fig. 227, let F be the given force acting in the line E D, and B a given point not in E D. Through B conceive a pair of equal and contrary forces to act in a line parallel to E D; viz., + F' equal to F and in the same direction; and − F' equal to F and in the contrary direction; also, let fall B A perpendicular to E D. Then the original force F acting through A is resolved into the equal and parallel force F' acting through B, and the couple of forces F and − F', with the arm A B and moment F × A B; which couple is right or left-handed according as B lies to the right or left of F, relatively to a spectator looking in the direction towards which F acts.

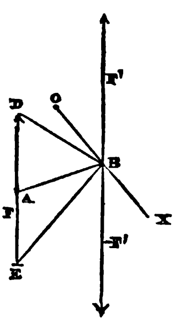

Fig. 227.

F × A B is called *the moment of the force* F *relatively to the point* B; or relatively to the *axis* O X traversing B in a direction perpendicular to the plane of F and A B; or relatively to a *plane* traversing B perpendicularly to A B.

280. **Parallel Forces.**—I. To find the resultant of two parallel forces. The resultant is in the same plane with, and parallel to, the components. It is their sum or difference, according as they act in the same or contrary directions; and in the latter case its direction is that of the greater component. To find its line of action by construction, proceed as follows :—Fig. 228 representing the case in which the components act in the same direction, fig. 229 that in which they act in contrary directions. Let A D and B E be the components. Join A E and B D, cutting each other

in F. In B D (produced in fig. 229) take B G = D F. Through G draw a line parallel to the components; this will be the line of

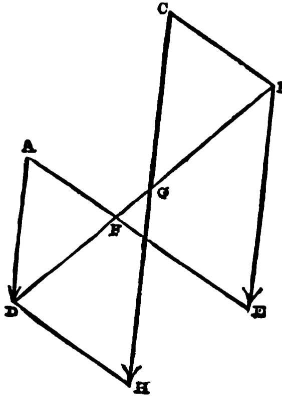

Fig. 228.

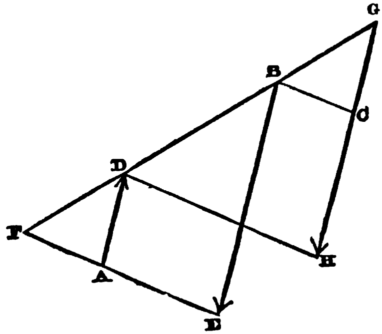

Fig. 229.

action of the resultant. To find its magnitude by construction: parallel to A E, draw B C and D H, cutting the line of action of the resultant in C and H; C H will represent the resultant required; and a force equal and opposite to C H will balance A D and B E.

To find the line of action of the resultant by calculation; make either

$$B\,G = \frac{A\,D \cdot D\,B}{C\,H};\ \text{or}\ D\,G = \frac{B\,E \cdot D\,B}{C\,H}.$$

When the two given parallel forces are opposite and equal, they form a couple, and have no single resultant.

II. To find the relative proportions of three parallel forces which balance each other, acting in one plane: their lines of action being given. Across the three lines of action, in any convenient position, draw a straight line A C B, fig. 230, and measure the distances between the points where it cuts the lines of action. Then each force will be proportional to the distance between the lines of action of the other two. The direction of the middle force, C, is contrary to that of the other two forces, A and B.

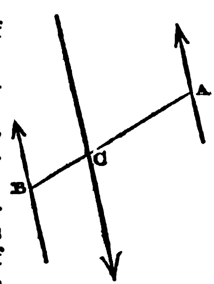

Fig. 230.

In symbols, let A, B, and C be the forces; then,

$$A + B + C = 0;\ AB : BC : CA :: C : A : B.$$

Each of the three forces is equal and opposite to the resultant of the other two; and each pair of forces are equal and opposite to the components of the third. Hence this rule serves to resolve a given force into two parallel components acting in given lines in the same plane.

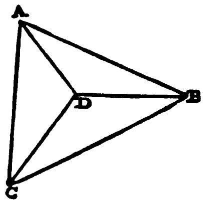

Fig. 231.

III. To find the relative proportions of four parallel forces which balance each other, not acting in one plane: their lines of action being given. Conceive a plane to cross the lines of action in any convenient position; and in fig. 231 or fig. 232, let A, B, C, D represent the points where the four lines of action cut the plane. Draw the six straight lines joining those four points by pairs. Then the force which acts through each point will be proportional to the area of the triangle formed by the other three points.

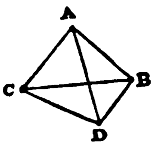

Fig. 232.

In fig. 231 the directions of the forces at A, B, and C are the same, and are contrary to that of the force at D. In fig. 232 the forces at A and D act in one direction, and those at B and C in the contrary direction.

In symbols,

$$A + B + C + D = 0;$$

$$BCD : CDA : DAB : ABC$$

$$:: \quad A \quad : \quad B \quad : \quad C \quad : \quad D.$$

Each of the four forces is equal and opposite to the resultant of the other three; and each set of three forces are equal and opposite to the components of the fourth. Hence the rule serves to resolve a force into three parallel components not acting in one plane.

IV. To find the *Resultant of any number of Parallel Forces.* For the magnitude and direction of the resultant take the algebraical sum of the components as if they acted along one line (Article 278, page 319). This may be denoted by $R = \Sigma F$. For the position of the resultant proceed as follows:—In any plane perpendicular to the lines of action of the parallel forces take an *axis of moments* in any convenient position. Multiply each component force (F) by its perpendicular distance (x) from that axis, so as to obtain its moment ($F\,x$) relatively to the axis. Mark those moments as positive or negative according to the direction in which they tend to turn the body to which they are applied about the axis; and take their algebraical sum, which will be the *resultant* *moment* ($M = \Sigma \cdot F x$). Divide the resultant moment by the resultant force; the quotient will be the perpendicular distance of the line of action of the resultant from the axis of moments; viz:—

$$x_0 = \frac{M}{R} = \frac{\Sigma \cdot F x}{\Sigma F}.$$

The algebraical sign of this distance will indicate its direction.

Take another axis of moments in the same plane, and perpendicular to the first axis; and by a similar operation find the perpendicular distance of the resultant from the second axis. The position of the resultant will then be completely determined.

If $R = 0$, the resultant is a couple. If $M = 0$, the line of action of the resultant traverses the axis of moments.

281. **Specific Gravity—Heaviness—Density—Bulkiness.**—I. *Specific Gravity* is the ratio of the weight of a given bulk of a given substance to the weight of the same bulk of pure water at a standard temperature. In Britain the standard temperature is 62° Fahr. = 16°·67 Cent. In France it is the temperature of the maximum density of water = 3°·94 Cent. = 39°·1 Fahr.

In rising from 39°·1 Fahr. to 62° Fahr., pure water expands in the ratio of 1·001118 to 1; but that difference is of no consequence in calculations of specific gravity for engineering purposes.

II. The *Heaviness* of any substance is the weight of an unit of volume of it in units of weight. In British measures heaviness is most conveniently expressed in *lbs. avoirdupois to the cubic foot;* in French measures, in *kilogrammes to the cubic decimètre* (or to the litre). The values of the heaviness of water at 39°·1 Fahr., and at 62° Fahr., are respectively 62·425 and 62·355 lbs. to the cubic foot.

III. The *Density* of a substance is either the number of units of *mass* in an unit of volume, in which case it is equal to the heaviness,—or the ratio of the mass of a given volume of the substance to the mass of an equal volume of water, in which case it is equal to the specific gravity. In its application to *gases*, the term "Density" is often used to denote the ratio of the heaviness of a given gas to that of air, at the same temperature and pressure.

IV. The *Bulkiness* of a substance is the number of units of volume which an unit of weight fills; and is the *reciprocal of the heaviness.* In British measures bulkiness is most conveniently expressed in *cubic feet to the lb. avoirdupois;* in French measures, in *cubic decimètres* (or in litres) *to the kilogramme.*

Rise of temperature produces (with certain exceptions) increase of bulkiness. The following are examples of rates of expansion in bulk, in rising from the freezing to the boiling point of water: that is from 0° Cent. or 32° Fahr., to 100° Cent. or 212° Fahr. The linear expansion of a solid body is one-third of its expansion in bulk.

Perfect gases, 0·365; air at ordinary pressures, 0·366; water, 0·04775; spirit of wine, 0·1112; mercury, 0·018153; oil, linseed and olive, 0·08; wrought iron and steel, 0·0036; cast iron, 0·0033; copper, 0·0055; bronze, 0·0054; brass, 0·0065: brick, common, 0·0106; fire-brick, 0·0015; glass, 0·0027.

Table of Heaviness and Specific Gravity.

Gases, at 32° Fahr., and under one atmosphere:—	Weight of a cubic foot in lb. avoirdupois.
Air,	0·080728
Carbonic acid,	0·12344
Hydrogen,	0·005592
Oxygen,	0·089256
Nitrogen,	0·078596
Steam (ideal),	0·05022

Liquids, at 32° Fahr. (except Water, which is taken at 39°·1 Fahr.):—	Weight of a cubic foot in lbs. avoirdupois.	Specific gravity, pure water = 1.
Water, pure, at 39°·1,	62·425	1·000
„ sea, ordinary,	64·05	1·026
Alcohol, pure,	49·38	0·791
„ proof spirit,	57·18	0·916
Æther,	44·70	0·716
Mercury,	848·75	13·596
Naphtha,	52·94	0·848
Oil, linseed,	58·68	0·940
„ olive,	57·12	0·915
„ whale,	57·62	0·923
„ of turpentine,	54·31	0·870
Petroleum,	54·81	0·878
Solid Mineral Substances, non-metallic:—		
Brick,	125 to 135	2 to 2·167
Brickwork,	112	1·8
Coal, anthracite,	100	1·602
„ bituminous,	77·4 to 89·9	1·24 to 1·44
Coke,	62·43 to 103·6	1·00 to 1·66
Glass, crown, average,	156	2·5
„ flint, „	187	3·0
„ green, „	169	2·7
„ plate, „	169	2·7
Granite,	164 to 172	2·63 to 2·76
Limestone (including marble),	169 to 175	2·7 to 2·8
„ magnesian,	178	2·86
Masonry,	116 to 144	1·85 to 2·3

	Weight of a cubic foot in lbs. avoirdupois.	Specific gravity, pure water = 1.
SOLID MINERAL SUBSTANCES, non-metallic—*continued.*		
Mortar,	109	1·75
Sand (damp),	118	1·9
„ (dry),	88·6	1·42
Sandstone, average,	144	2·3
„ various kinds,	130 to 157	2·08 to 2·52
METALS, solid:—		
Brass, cast,	487 to 524·4	7·8 to 8·4
„ wire,	533	8·54
Bronze,	524	8·4
Copper, cast,	537	8·6
„ sheet,	549	8·8
„ hammered,	556	8·9
Gold,	1186 to 1224	19 to 19·6
Iron, cast, various,	434 to 456	6·95 to 7·3
„ average,	444	7·11
Iron, wrought, various,	474 to 487	7·6 to 7·8
„ average,	480	7·69
Lead,	712	11·4
Platinum,	1311 to 1373	21 to 22
Silver,	655	10·5
Steel,	487 to 493	7·8 to 7·9
Tin,	456 to 468	7·3 to 7·5
Zinc,	424 to 449	6·8 to 7·2
TIMBER:—*		
Ash,	47	0·753
Bamboo,	25	0·4
Beech,	43	0·69
Box,	60	0·96
Elm,	34	0·544
Fir: Red Pine,	30 to 44	0·48 to 0·7
„ Spruce,	30 to 44	0·48 to 0·7
„ American Yellow Pine,	29	0·46
„ Larch,	31 to 35	0·5 to 0·56
Mahogany, Honduras,	35	0·56
„ Spanish,	53	0·85
Oak, European,	43 to 62	0·69 to 0·99
„ American, Red,	54	0·87
Teak,	41 to 55	0·66 to 0·88
Willow,	25	0·4
Yew,	50	0·8

* The Timber in every case is supposed to be dry.

WEIGHT OF CUBES, RODS, PLATES, BARS, AND SPHERES.

	A. Cubic Inch.	B. Round Rod, 1 ft. long 1 in. diam.	C. Square Bar, 1 ft. × 1 in. × 1 in.	D. Plate, 1 ft. × 1 ft. × 1 in.	E. Cubic foot.	F. Sphere, 1 inch diam.
	lbs.	lbs.	lbs.	lbs.	lbs.	
Brass, cast, average,...	0·298	2·81	3·58	43·0	516	0·156
„ wire,............	0·308	2·91	3·70	44·4	533	0·162
Bronze,..................	0·303	2·86	3·64	43·7	524	0·159
Copper, sheet,..........	0·318	2·99	3·81	45·75	549	0·166
„ hammered,....	0·322	3·03	3·86	46·3	556	0·168
Iron, cast, average,....	0·257	2·42	3·08	37·0	444	0·134
Iron, wrought, average,	0·278	2·62	3·33	40·0	480	0·146
Lead,.....................	0·412	3·88	4·94	59·3	712	0·216
Steel, average,..........	0·283	2·67	3·40	40·8	490	0·148
Tin, average,............	0·267	2·52	3·21	38·5	462	0·140
Zinc, average,..........	0·252	2·38	3·03	36·3	436	0·132

282. **Centre of Gravity—Moment of Weight.**—RULE I.—The centre of gravity of a body of uniform heaviness is its centre of magnitude. (See supplement to this Chapter, page 334.)

RULE II.—To find the moment of a body's weight relatively to a given *plane of moments;* multiply the weight by the perpendicular distance of the body's centre of gravity from the given plane.

NOTE.—In comparing together or combining the moments of weights which lie some at one side and some at the other side of a plane of moments, those moments are to be distinguished into positive and negative, according to the sides of the plane at which the weights lie.

RULE III.—To find the common centre of gravity of a set of detached bodies; find their several moments relatively to a convenient fixed plane; find the *resultant* of those moments by adding together, separately, the positive and negative moments, and taking the difference between the two sums, which will be positive or negative according as the positive or negative sum is the greater. Divide that resultant moment by the total weight; the quotient will be the perpendicular distance of the common centre of gravity from the fixed plane; and its positive or negative sign will show at which side of the plane that centre lies. If necessary, repeat the same process for a second and a third fixed plane, so as to determine the position of the required centre completely. The two or three planes (as the case may be) are usually taken perpendicular to each other.

RULE IV.—To find the centre of gravity of a body consisting of parts of unequal heaviness; find separately the centres of those parts, and treat them as detached weights by Rule III.

283. The **Centre of Pressure** in a plane surface is the point traversed by the resultant of a pressure that is exerted at that surface.

RULE.—Conceive that upon the pressed surface as a base, there stands a prismatic solid of a height at each point of that surface proportional to the intensity of the pressure; the point in the pressed surface at the foot of a perpendicular from the centre of magnitude of the solid (see supplement to this Chapter) will be the centre of pressure.

When the intensity is uniform, the centre of pressure is at the *centre of magnitude* of the pressed surface.

284. The **Centre of Buoyancy** of a solid wholly or partly immersed in a liquid is the centre of gravity of the mass of liquid displaced. The resultant pressure of the liquid on the solid is equal to the weight of liquid displaced, and is exerted vertically upwards through the centre of buoyancy.

285. The **Resultant of a Distributed Force**.—I. To find the resultant of a body's weight; find the centre of gravity of the body; the resultant will be a single force equal to the weight, acting vertically downwards through the centre of gravity.

II. To find the resultant of a pressure; find the centre of pressure (as in Article 283); the resultant will be a single force equal in amount to the pressure, and acting in the same direction and through the centre of pressure. The *amount* of the pressure is equal to the area of the pressed surface, multiplied by the *mean intensity* of the pressure, and is also equal to the weight of the imaginary prismatic solid mentioned in Article 283.

286. The **Intensity of Pressure** is expressed in units of weight on the unit of area: as pounds on the square inch, or kilogrammes on the square mètre; or by the height of a column of some fluid; or in *atmospheres*, the unit in this case being the average pressure of the atmosphere at the level of the sea. (See Article 302.)

287. **Principles Relating to Varied Motion.**—An unbalanced force applied to a body produces change of momentum equal in amount to and coincident in direction with the impulse exerted by the force. *Impulse* is the product of the force into the time during which it acts in seconds. *Momentum* is the product of the mass of a body into its velocity in units of distance per second. As to the units of force and of mass, see Article 277A, page 318. A body receiving an impulse *re-acts* against the body giving the impulse, with an equal and opposite impulse. The following are rules based on the equality of impulse and momentum:—

I. To find what *impulse* is required to produce a given change in the velocity of a given mass; multiply the mass by the change in its velocity, in units of distance per second.

(If the change consists in acceleration, the impulse must be forward; if in retardation, backward.)

II. To calculate what unbalanced effort or unbalanced resistance, as the case may be, is required to produce a given increase or diminution of a body's speed in a given time or in a given distance.

Case I.—If the *time* is given; multiply the weight of the mass by its change of velocity; divide by g, and by the time in seconds.

Case II.—If the *distance* is given; multiply the weight of the mass by the change in the half-square of its velocity, and divide by g, and by the distance. (For values of g, see page 318.)

III. To find the *re-action* of an accelerated or retarded body; find the force required to produce the change of velocity; the re-action will be equal and opposite.

The momentum and re-action of a body of any figure undergoing *translation* are the same as if its whole mass were concentrated at its centre of gravity.

The principles of this and the following Article will be further explained and exemplified in the next Chapter.

288. **Deviated Motion and Centrifugal Force.**—To make a body move in a curve, some other body must guide it by exerting on it a *deviating force* directed towards the centre of curvature. The revolving body re-acts on the guiding body with an equal and opposite *centrifugal force.*

To find the deviating and centrifugal force of a given mass revolving with a given velocity in a circle of a given radius:—multiply the weight of the mass by the square of its linear velocity, and *divide* by the radius;—or *otherwise:* multiply the mass by the square of its angular velocity of revolution, and *multiply* by the radius:—the result will be the value of the deviating and centrifugal forces in absolute units, which may be converted into units of weight by dividing by g.

The *resultant centrifugal force* of a rigid body of any shape is the same in amount and direction (though not the same in distribution) as if the whole mass were collected at its centre of gravity.

288 A. **Falling Bodies.**—The following rules apply to a body falling without sensible resistance from the air:—

I. To find the velocity acquired at the end of a given time; multiply the time in seconds by g (see page 318).

II. To find the height of fall in a given time; multiply the square of the time in seconds by $\frac{1}{2}\,g = 16{\cdot}1$ feet $= 4{\cdot}904$ mètres.

III. To find the height of fall corresponding (or "due") to a given velocity; divide the half-square of the velocity by g.

IV. To find the velocity due to a given height; multiply the height by $2\,g$, and extract the square root.

$$\sqrt{2\,g} = 8{\cdot}025 \text{ feet} = 4{\cdot}429 \text{ mètres.}$$

TABLE OF HEIGHTS DUE TO VELOCITIES.

v = Velocity in feet per second.
h = Height in feet $= v^2 \div 64{\cdot}4$.

This table is exact for latitude $54\frac{3}{4}°$, and near enough to exactness for practical purposes in all parts of the earth's surface.

v	h	v	h	v	h
1	·01553	27	11·320	54	45·280
2	·06211	28	12·174	56	48·695
3	·13975	29	13·059	58	52·235
4	·24845	30	13·975	60	55·901
5	·38820	31	14·922	62	59·688
6	·55901	32	15·901	64	63·602
7	·76087	32·2	16·100	64·4	64·400
8	·99379	33	16·910	66	67·640
9	1·2578	34	17·950	68	71·800
10	1·5528	35	19·022	70	76·087
11	1·8789	36	20·124	72	80·496
12	2·2360	37	21·257	74	85·029
13	2·6242	38	22·422	76	89·688
14	3·0435	39	23·618	78	94·472
15	3·4938	40	24·845	80	99·379
16	3·9752	41	26·102	82	104·41
17	4·4876	42	27·391	84	109·56
18	5·0311	43	28·711	86	114·84
19	5·6056	44	30·062	88	120·25
20	6·2112	45	31·444	90	125·78
21	6·8478	46	32·857	92	131·43
22	7·5155	47	34·301	94	137·20
23	8·2143	48	35·776	96	143·10
24	8·9441	49	37·283	98	149·13
25	9·7050	50	38·820	100	155·28
26	10·497	52	41·987		

SUPPLEMENT TO CHAPTER I.

Rules for the Mensuration of Figures and finding of Centres of Magnitude.

289. **To Measure any Plane Area.**—Draw an axis or base-line, A X, in a convenient position. The most convenient position is usually parallel to the greatest length of the area to be measured. Divide the length of the figure into a convenient number of equal intervals, and measure breadths in a direction perpendicular to the axis at the two ends of that length, and at the points of division,

which breadths will, of course, be one more in number than the intervals. (For example, in fig. 233, the length of the figure is divided into ten equal intervals, and eleven breadths are measured at b_0, b_1, &c.) Then the following rules are exact, if the sides of the figure are bounded by straight lines, and by parabolic curves not exceeding the third degree, and are approximate for boundaries of any other figures.

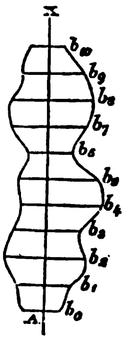

Fig. 233.

Rule A. ("*Simpson's First Rule*," to be used when the number of intervals is even.)—Add together the two endmost breadths, *twice* every second intermediate breadth, and *four times* each of the remaining intermediate breadths; multiply the sum by the common interval between the breadths, and divide by 3; the result will be the area required.

For two intervals the multipliers for the breadths are 1, 4, 1; for four intervals, 1, 4, 2, 4, 1; for six intervals, 1, 4, 2, 4, 2, 4, 1; and so on. These are called "Simpson's Multipliers."

Rule B. ("*Simpson's Second Rule*," to be used when the number of intervals is a multiple of 3.)—Add together the two endmost breadths, *twice* every third intermediate breadth, and *thrice* each of the remaining intermediate breadths; multiply the sum by the common interval between the breadths, and by 3; divide the product by 8; the result will be the area required.

"Simpson's multipliers" in this case are, for three intervals, 1, 3, 3, 1; for six intervals, 1, 3, 3, 2, 3, 3, 1; for nine intervals, 1, 3, 3, 2, 3, 3, 2, 3, 3, 1; and so on.

Rule C. ("*Merrifield's Trapezoidal Rule*," for calculating separately the areas of the parts into which a figure is subdivided by its equidistant ordinates or breadths.)—Write down the breadths in their order. Then take the *differences* of the successive breadths, distinguishing them into positive and negative according as the breadths are increasing or diminishing, and write them opposite the intervals between the breadths. Then take the differences of those differences, or *second differences*, and write them opposite the intervals between the first differences, distinguishing them into positive and negative according to the following principles:—

First Differences.	Second Difference.
Positive increasing, or Negative diminishing,	.........Positive.
Negative increasing, or Positive diminishing,	.........Negative.

In the column of second differences there will now be two blanks opposite the two endmost breadths; those blanks are to be filled up with numbers each forming an arithmetical progression with the two adjoining second differences if these are unequal, or equal to them if they are equal.

Divide each second difference by 12; this gives a *correction*, which is to be *subtracted* from the breadth opposite it if the second difference is *positive*, and *added* to that breadth if the second difference is negative.

Then to find the area of the division of the figure contained between a given pair of ordinates or breadths; *multiply the half sum of the corrected breadths by the interval between them.*

The area of the whole figure may be found either by adding together the areas of all its divisions, or by adding together the halves of the endmost corrected breadths, and the whole of the intermediate breadths, and multiplying the sum by the common interval.

In symbols, let y be an actual breadth, and y' the corresponding corrected breadth; then $y' = y - \frac{1}{12} \Delta^2 y$.

RULE D. ("*Common Trapezoidal Rule*," to be used when a rough approximation is sufficient.)—Add together the halves of the endmost breadths and the whole of the intermediate breadths, and multiply the sum by the common interval.

290. **To Measure the Volume of any Solid.**—METHOD I. *By Layers.* —Choose a straight axis in any convenient position. (The most convenient is usually parallel to the greatest length of the solid.) Divide the whole length of the solid, as marked on the axis, into a convenient number of equal intervals, and measure the sectional area of the solid upon a series of planes crossing the axis at right angles at the two ends and at the points of division. Then treat those areas as if they were the breadths of a plane figure, applying to them Rule A, B, or C of Article 289, page 332; and the result of the calculation will be the volume required. If Rule C is used, the volume will be obtained in separate layers.

METHOD II. *By Prisms or Columns* ("*Woolley's Rule*").—Assume a plane in a convenient position as a base, divide it into a network of equal rectangular divisions, and conceive the solid to be built of a set of rectangular prismatic columns, having those rectangular divisions for their sectional areas. Measure the thickness of the solid at the *centre* and at the *middle of each of the sides* of each of those rectangular columns; add together the doubles of all the thicknesses before-mentioned, which are in the interior of the solid, and the simple thicknesses which are at its boundaries; divide the sum by *six*, and multiply by the area of one rectangular division of the base.

291. **To Measure the Length of any Curve.**—Divide it into short arcs, and measure each of them by Rule I of Article 51, page 28.

292. **Centre of Magnitude—General Principles.**—By the *magnitude* of a figure is to be understood its length, area, or volume, according as it is a line, a surface, or a solid.

The *Centre of Magnitude* of a figure is a point such that, if the figure be divided in any way into equal parts, the distance of the centre of magnitude of the whole figure from any given plane is the mean of the distances of the centres of magnitude of the several equal parts from that plane.

The *Geometrical Moment* of any figure relatively to a given plane is the product of its magnitude into the perpendicular distance of its centre from that plane.

I. *Symmetrical Figure.*—If a plane divides a figure into two symmetrical halves, the centre of magnitude of the figure is in that plane; if the figure is symmetrically divided in the like manner by two planes, the centre of magnitude is in the line where those planes cut each other; if the figure is symmetrically divided by three planes, the centre of magnitude is their point of intersection; and if a figure has a *centre of figure* (for example, a circle, a sphere, an ellipse, an ellipsoid, a parallelogram, &c.), that point is its centre of magnitude.

II. *Compound Figure.*—To find the perpendicular distance from a given plane of the centre of a compound figure made up of parts whose centres are known. Multiply the magnitude of each part by the perpendicular distance of its centre from the given plane; distinguish the products (or *geometrical moments*) into positive or negative, according as the centres of the parts lie to one side or to the other of the plane; add together, separately, the positive moments and the negative moments: take the difference of the two sums, and call it positive or negative according as the positive or negative sum is the greater; this is the *resultant moment* of the compound figure relatively to the given plane; and its being positive or negative shows at which side of the plane the required centre lies. Divide the resultant moment by the magnitude of the compound figure; the quotient will be the distance required.

The centre of a figure in three dimensions is determined by finding its distances from three planes that are not parallel to each other. The best position for those planes is perpendicular to each other; for example, one horizontal, and the other two cutting each other at right angles in a vertical line. To determine the centre of a plane figure, its distances from two planes perpendicular to the plane of the figure are sufficient.

293. **Centre of a Plane Area.**—To find, approximately, the centre of any plane area.

Rule A.—Let the plane area be that represented in fig. 233 (of Article 289, page 332). Draw an axis, A X, in a convenient position, divide it into equal intervals, measure breadths at the ends and at the points of division, and calculate the area, as in Article 289.

Then multiply each breadth by its distance from one end of the axis (as A); consider the products as if they were the breadths of a new figure, and proceed by the rules of Article 289 to calculate the area of that new figure. The result of the operation will be the *geometrical moment* of the original figure relatively to a plane perpendicular to A X at the point A.

Divide the *moment* by the *area* of the original figure; the quotient will be the distance of the centre required from the plane perpendicular to A X at A.

Draw a second axis intersecting A X (the most convenient position being in general perpendicular to A X), and by a similar process find the distance of the centre from a plane perpendicular to the second axis at one of its ends; the centre will then be completely determined.

RULE B.—If convenient, the distance of the required centre from a plane cutting an axis at one of the intermediate points of division, instead of at one of its ends, may be computed as follows:—Take separately the moments of the two parts into which that plane divides the figure; the required centre will lie in the part which has the greater moment. Subtract the less moment from the greater; the remainder will be the *resultant moment* of the whole figure, which being divided by the whole area, the quotient will be the distance of the required centre from the plane of division.

REMARK.—When the resultant moment is = 0, the centre is in the plane of division.

RULE C.—To find the perpendicular distance of the centre from the axis A X. Multiply each breadth by the distance of the middle point of that breadth from the axis, and by the proper "Simpson's Multiplier," Article 289, page 331; distinguish the products into right-handed and left-handed, according as the middle points of the breadths lie to the right or left of the axis; take separately the sum of the right-handed products and the sum of the left-handed products; the required centre will lie to that side of the axis for which the sum is the greater; subtract the less sum from the greater, and multiply the remainder by $\frac{1}{3}$ of the common interval if Simpson's first rule is used, or by $\frac{3}{8}$ of the common interval if Simpson's second rule is used; the product will be the *resultant moment* relatively to the axis A X, which being divided by the area, the quotient will be the required distance of the centre from that axis.*

* The rules of this Article are expressed in symbols, as follows:—Let x and y be the perpendicular distances of any point in the plane area from two planes perpendicular to the area and to each other, and x_0 and y_0 the per-

294. **Centre of a Volume.**—To find the perpendicular distance of the centre of magnitude of any solid figure from a plane perpendicular to a given axis at a given point, proceed as in Rule A of the preceding Article to find the moment relatively to the plane, substituting *sectional areas* for *breadths;* then divide the moment by the volume (as found by Article 290); the quotient will be the required distance.

To determine the centre completely, find its distances from three planes, no two of which are parallel. In general it is best that those planes should be perpendicular to each other.

295. **Centre of Magnitude of a Curved Line.**—RULE A.—*To find approximately the Centre of Magnitude of a very Flat Curved Line.*—In fig. 234, let A D B be the arc. Draw the straight chord A B, which bisect in C; draw C D (the *deflection* of the arc) perpendicular to A B; from D lay off D E $= \frac{1}{3}$ C D; E will be very nearly the centre required.

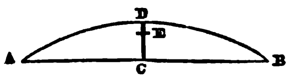

Fig. 234.

This process is exact for a cycloidal arc whose chord, A B, is parallel to the base of the cycloid. For other curves it is approximate. For example, in the case of a circular arc, it gives D E too small; the error, for an arc subtending 60°, being about $\frac{1}{300}$ of the deflection, and its proportion to the deflection varying nearly as the square of the angular extent of the arc.

RULE B.—*When the Curved Line is not very flat,* divide it into very flat arcs; find their several centres of magnitude by Rule A, and measure their lengths; then treat the whole curve as a compound figure, agreeably to Rule II. of Article 292, page 334.

296. **Special Figures.**—I. TRIANGLE (fig. 235).—From any two of the angles draw straight lines to the middle points of the opposite sides; these lines will cut each other in the centre required;—or otherwise,—from any one of the angles draw a straight line to the middle of the opposite side, and cut off one-third part from that line, commencing at the side.

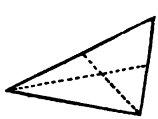
Fig. 235.

II. QUADRILATERAL (fig. 236).—Draw the two diagonals A C and B D, cutting each other in E. If the quadrilateral is a parallelogram, E will divide each diagonal into two equal parts, and will itself be the centre. If not, one or both of the diagonals will be divided into unequal parts by the point E. Let B D be a diagonal

pendicular distances of the centre of magnitude of the area from the same planes; then

$$x_0 = \frac{\iint x\,dx\,dy}{\iint dx\,dy};\ y_0 = \frac{\iint y\,dx\,dy}{\iint dx\,dy}.$$

that is unequally divided. From D lay off D F in that diagonal = B E. Then the centre of the triangle F A C, found as in the preceding rule, will be the centre required

III. PLANE POLYGON.—Divide it into triangles; find their centres, and measure their areas; then treat the polygon as a compound figure made up of the triangles, by Rule II. of Article 292, page 334.

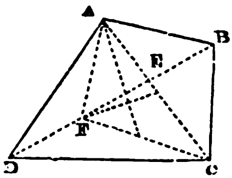

Fig. 236.

IV. PRISM OR CYLINDER WITH PLANE PARALLEL ENDS.—Find the centres of the ends; a straight line joining them will be the axis of the prism or cylinder, and the middle point of that line will be the centre required.

V. TETRAHEDRON, OR TRIANGULAR PYRAMID (fig. 237).—Bisect any two opposite edges, as A D and B C, in E and F; join E F, and bisect it in G; this point will be the centre required.

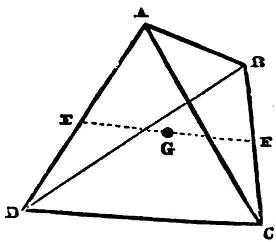

Fig. 237.

VI. ANY PYRAMID OR CONE WITH A PLANE BASE.—Find the centre of the base, from which draw a straight line to the summit; this will be the axis of the pyramid or cone. From the axis cut off one-fourth of its length, beginning at the base; this will give the centre required.

VII. ANY POLYHEDRON OR PLANE-FACED SOLID.—Divide it into pyramids; find their centres and measure their volumes; then treat the whole solid as a compound figure by Rule II. of Article 290.

VIII. CIRCULAR ARC.—In fig. 238, let A B be the arc, and C the centre of the circle of which it is part. Bisect the arc in D, and join C D and A B. Multiply the radius C D by the chord A B, and divide by the length of the arc A D B; lay off the quotient C E upon C D; E will be the centre of magnitude of the arc.

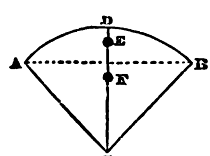

Fig. 238.

IX. CIRCULAR SECTOR, C A D B, fig. 238.—Find C E as in the preceding rule, and make C F = $\frac{2}{3}$ C E; F will be the centre required.

X. SECTOR OF A FLAT RING.—Let r be the external and r' the internal radius of the ring. Draw a circular arc of the same angular extent with the sector, and of the radius $\frac{2}{3} \cdot \frac{r^3 - r'^3}{r^2 - r'^2}$, and find its centre of magnitude by Rule VIII.

CHAPTER II.

OF THE PERFORMANCE OF WORK BY MACHINES.

SECTION I.—*Of Resistance and Work.*

297. The **Action of a Machine** is to produce Motion against Resistance. For example, if the machine is one for lifting solid bodies, such as a crane, or fluid bodies, such as a pump, its action is to produce upward motion of the lifted body against the resistance arising from gravity; that is, against its own weight: if the machine is one for propulsion, such as a locomotive engine, its action is to produce horizontal or inclined motion of a load against the resistance arising from friction, or from friction and gravity combined: if it is one for shaping materials, such as a planing machine, its action is to produce relative motion of the tool and of the piece of material shaped by it, against the resistance which that material offers to having part of its surface removed; and so of other machines.

298. **Work.** (*A. M.*, 513.)—The action of a machine is measured, or expressed as a definite quantity, by multiplying the motion which it produces into the resistance, or force directly opposed to that motion, which it overcomes; the product resulting from that multiplication being called WORK.

In Britain, the distances moved through by pieces of mechanism are usually expressed in feet; the resistances overcome, in pounds avoirdupois; and quantities of work, found by multiplying distances in feet by resistances in pounds, are said to consist of so many *foot-pounds*. Thus the work done in lifting a weight of one pound, through a height of one foot, is *one foot-pound;* the work done in lifting a weight of twenty pounds, through a height of one hundred feet, is $20 \times 100 = 2,000$ foot-pounds.

In France, distances are expressed in mètres, resistances overcome in kilogrammes, and quantities of work in what are called *kilogrammètres,* one kilogrammètre being the work performed in lifting a weight of one kilogramme through a height of one mètre.

The following are the proportions amongst those units of distance, resistance, and work, with their logarithms:—

		Logarithms.
One mètre	= 3·2808693 feet,	0·515989
One foot	= 0·30479721 mètres,	$\bar{1}$·484011
One kilogramme	= 2·20462 lbs. avoirdupois,	0·343334
One lb. avoirdupois	= 0·453593 kilogramme,	$\bar{1}$·656666
One kilogrammètre	= 7·23308 foot-pounds,	0·859323
One foot-pound	= 0·138254 kilogrammètres,	$\bar{1}$·140677

299. The **Rate of Work** of a machine means, the quantity of work which it performs in some given interval of time, such as a second, a minute, or an hour (*A. M.*, 661). It may be expressed in units of work (such as foot-pounds) per second, per minute, or per hour, as the case may be; but there is a peculiar unit of power appropriated to its expression, called a HORSE-POWER, which is, in Britain,

550 foot-pounds per second,
or 33,000 foot-pounds per minute,
or 1,980,000 foot-pounds per hour.

This is also called an *actual* or *real* horse-power, to distinguish it from a *nominal* horse-power, the meaning of which will afterwards be explained. It is greater than the performance of any ordinary horse, its name having a conventional value attached to it.

In France, the term FORCE DE CHEVAL, or CHEVAL-VAPEUR, is applied to the following rate of work :—

		Foot-lbs.
75 kilogrammètres per second	=	542½
or 4,500 kilogrammètres per minute	=	32,549
or 270,000 kilogrammètres per hour	=	1,952,932

being about one-seventieth part less than the British horse-power.

300. **Velocity.**—If the *velocity of the motion* which a machine causes to be performed against a given resistance be given, then the product of that velocity into the resistance obviously gives the rate of work, or effective power. If the velocity is given in feet per second, and the resistance in pounds, then their product is the rate of work in foot-pounds per second, and so of minutes, or hours, or other units of time.

It is usually most convenient, for purposes of calculation, to express the velocities of the parts of machines either in feet per second or in feet per minute. For certain dynamical calculations to be afterwards referred to, the second is the more convenient unit of time: in stating the performance of machines for practical purposes, the minute is the unit most commonly employed.

Comparison of Different Measures of Velocity.

	Miles per hour.		Feet per second.		Feet per minute.		Feet per hour.
	1	=	1·4$\dot{6}$	=	88	=	5280·
	0·68$\dot{1}\dot{8}$	=	1	=	60	=	3600
	0·011$\dot{3}\dot{6}$	=	0·01$\dot{6}$	=	1	=	60
	0·00018$\dot{9}\dot{3}$	=	0·0002$\dot{7}$	=	0·01$\dot{6}$	=	1
1 nautical mile per hour, or "knot,"......	= 1·1508	=	1·688	=	101·27	=	6076

The units of time being the same in all civilized countries, the proportions amongst their units of velocity are the same with those amongst their linear measures.

301. **Work in Terms of Angular Motion.** (*A. M.*, 593.)—When a resisting force opposes the motion of a part of a machine which moves round a fixed axis, such as a wheel, an axle, or a crank, the product of the amount of that resistance into its *leverage* (that is, the perpendicular distance of the line along which it acts from the fixed axis) is called the *moment*, or *statical moment*, of the resistance. If the resistance is expressed in pounds, and its leverage in feet, then its moment is expressed in terms of a measure which may be called a *foot-pound*, but which, nevertheless, is a quantity of an entirely different kind from a foot-pound of work. (See p. 321.)

Suppose now that the body to whose motion the resistance is opposed turns through any number of revolutions, or parts of a revolution; and let T denote the angle through which it turns, expressed in revolutions, and parts of a revolution; also, let

$$2\pi = 6{\cdot}2832$$

denote, as is customary, the ratio of the circumference of a circle to its radius. Then the distance through which the given resistance is overcome is expressed by

$$\text{the leverage} \times 2\pi \times \text{T};$$

that is, by the product of the circumference of a circle whose radius is the leverage, into the number of turns and fractions of a turn made by the rotating body.

The distance thus found being multiplied by the resistance overcome, gives the work performed; that is to say,

The work performed
$= \textit{the resistance} \times \textit{the leverage} \times 2\pi \times \text{T}.$

But the product of the resistance into the leverage is what is called the *moment* of the resistance, and the product $2\pi T$ is called the *angular motion* of the rotating body; consequently,

The work performed
= the moment of the resistance × the angular motion.

The mode of computing the work indicated by this last equation is often more convenient than the direct mode already explained in Article 298.

The angular motion $2\pi T$ of a body during some definite unit of time, as a second or a minute, is called its *angular velocity;* that is to say, *angular velocity is the product of the turns and fractions of a turn made in an unit of time into the ratio* ($2\pi = 6{\cdot}2832$) *of the circumference of a circle to its radius.* Hence it appears that

The rate of work
= the moment of the resistance × the angular velocity.

302. **Work in Terms of Pressure and Volume.** (*A. M.*, 517.)—If the resistance overcome be a pressure uniformly distributed over an area, as when a piston drives a fluid before it, then the amount of that resistance is equal to the intensity of the pressure, expressed in units of force on each unit of area (for example, in pounds on the square inch, or pounds on the square foot) multiplied by the area of the surface at which the pressure acts, if that area is perpendicular to the direction of the motion; or, if not, then by the projection of that area on a plane perpendicular to the direction of motion. In practice, when the *area of a piston* is spoken of, it is always understood to mean the projection above mentioned.

Now, when a plane area is multiplied into the distance through which it moves in a direction perpendicular to itself, if its motion is straight, or into the distance through which its centre of gravity moves, if its motion is curved, the product is the *volume of the space traversed* by the piston.

Hence the work performed by a piston in driving a fluid before it, or by a fluid in driving a piston before it, may be expressed in either of the following ways:—

Resistance × distance traversed
= intensity of pressure × area × distance traversed;
= intensity of pressure × volume traversed.

In order to compute the work in foot-pounds, if the pressure is stated in pounds on the square foot, the area should be stated in square feet, and the volume in cubic feet; if the pressure is stated in pounds on the square inch, the area should be stated in square inches, and the volume in units, each of which is a prism of one foot in length and one square inch in area; that is, of $\frac{1}{144}$ of a cubic foot in volume.

The following table gives a comparison of various units in which the intensities of pressures are commonly expressed. (*A. M.*, 86.)

	Pounds on the square foot.	Pounds on the square inch.
One pound on the square inch,.....	144	1
One pound on the square foot,.....	1	$\frac{1}{144}$
One inch of mercury (that is, weight of a column of mercury at 32° Fahr., one inch high),............	70·73	0·4912
One foot of water (at 39°·1 Fahr.),	62·425	0·4335
One inch of water,....................	5·2021	0·036125
One atmosphere, of 29·922 inches of mercury, or 760 millimètres,	2116·3	14·7
One foot of air, at 32° Fahr., and under the pressure of one atmosphere,...............................	0·080728	0·0005606
One kilogramme on the square mètre, ...	0·204813	0·00142231
One kilogramme on the square millimètre, ...	204813	1422·31
One millimètre of mercury,.........	2·7847	0·01934

303. **Algebraical Expressions for Work.** (*A. M.*, 515, 517, 593.)—To express the results of the preceding articles in algebraical symbols, let

s denote the distance in feet through which a resistance is overcome in a given time;

R, the amount of the resistance overcome in pounds.

Also, supposing the resistance to be overcome by a piece which turns about an axis, let

T be the number of turns and fractions of a turn made in the given time, and $i = 2\pi T = 6{\cdot}2832\ T$ the angular motion in the given time; and let

l be the leverage of the resistance; that is, the perpendicular distance of the line along which it acts from the axis of motion; so that $s = i\,l$, and $R\,l$ is the statical moment of the resistance. Supposing the resistance to be a pressure, exerted between a piston and a fluid, let A be the area or projected area of the piston, and p the intensity of the pressure in pounds per unit of area.

Then the following expressions all give quantities of work in the given time in foot-pounds:—

$$\mathrm{R}s;\ i\mathrm{R}l;\ p\mathrm{A}s;\ ip\mathrm{A}l.$$

The last of these expressions is applicable to a piston turning on an axis, for which l denotes the distance from the axis to the centre of gravity of the area A.

304. **Work against an Oblique Force.** (*A. M.*, 511.)—The resistance directly due to a force which acts against a moving body in a direction oblique to that in which the body moves, is found by resolving that force into two components, one at right angles to the direction of motion, which may be called a *lateral force*, and which must be balanced by an equal and opposite lateral force, unless it takes effect by altering the direction of the body's motion, and the other component directly opposed to the body's motion, which is the *resistance* required. That resolution is effected by means of the well known principle of the parallelogram of forces as follows:—

In fig. 239, let A represent the point at which a resistance is overcome, A B the direction in which that point is moving, and let $\overline{\mathrm{AF}}$ be a line whose direction and length represent the direction and magnitude of a force obliquely opposed to the motion of A.

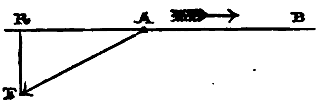

Fig. 239.

From F upon B A produced, let fall the perpendicular $\overline{\mathrm{FR}}$; the length of that perpendicular will represent the magnitude of the lateral component of the oblique force, and the length $\overline{\mathrm{AR}}$ will represent the direct component or resistance.

The work done against an oblique resisting force may also be calculated by resolving the motion into a direct component in the line of action of the force, and a transverse component, and multiplying the whole force by the direct component of the motion.

305. **Summation of Quantities of Work.**—In every machine, resistances are overcome during the same interval of time, by different moving pieces, and at different points in the same moving piece; and the whole work performed during the given interval is found by adding together the several products of the resistances into the respective distances through which they are simultaneously overcome. It is convenient, in algebraical symbols, to denote the result of that summation by the symbol—

$$\Sigma \cdot \mathrm{R}s; \qquad (1.)$$

in which Σ denotes the operation of taking the sum of a set of quantities of the kind denoted by the symbols to which it is prefixed.

When the resistances are overcome by pieces turning upon axes, the above sum may be expressed in the form—

$$\Sigma \cdot i \, R \, l \, ; \quad \text{.............................(2.)}$$

and so of other modes of expressing quantities of work.

The following are particular cases of the summation of quantities of work performed at different points:—

I. In a *shifting piece*, or one which has the kind of movement called *translation* only, the velocities of every point at a given instant are equal and parallel; hence, in a given interval of time, the motions of all the points are equal; and the work performed is to be found by multiplying the *sum of the resistances* into the motion as a common factor; an operation expressed algebraically thus—

$$s \, \Sigma \, R \, ; \quad \text{.............................(3.)}$$

II. For a *turning piece*, the angular motions of all the points during a given interval of time are equal; and the work performed is to be found by multiplying the *sum of the moments of the resistances relatively to the axis* into the angular motion as a common factor—an operation expressed algebraically thus—

$$i \, \Sigma \cdot R \, l \, ; \quad \text{.............................(4.)}$$

The sum denoted by $\Sigma \cdot R \, l$ is the *total moment of resistance* of the piece in question.

III. In every *train of mechanism*, the *proportions* amongst the motions performed during a given interval of time by the several moving pieces, can be determined from the mode of connection of those pieces, independently of the absolute magnitudes of those motions, by the aid of the science called by Mr. Willis, *Pure Mechanism*. This enables a calculation to be performed which is called *reducing the resistances to the driving point;* that is to say, determining the resistances, which, if they acted directly at the point where the motive power is applied to the machine, would require the same quantity of work to overcome them with the actual resistances.

Suppose, for example, that by the principles of pure mechanism it is found, that a certain point in a machine, where a resistance R is to be overcome, moves with a velocity bearing the ratio $n : 1$ to the velocity of the driving point. Then the work performed in overcoming that resistance will be the same as if a resistance $n \, R$ were overcome directly at the driving point. If a similar calculation be made for each point in the machine where resistance is overcome, and the results added together, as the following symbol denotes:—

$$\Sigma \cdot n \, R, \ldots\ldots\ldots\ldots(5.)$$

that sum is the *equivalent resistance at the driving point;* and if in a given interval of time the driving point moves through the distance s, then the work performed in that time is—

$$s \, \Sigma \cdot n \, R. \ldots\ldots\ldots\ldots(6.)$$

The process above described is often applied to the steam engine, by reducing all the resistances overcome to equivalent resistances acting directly against the motion of the piston.

A similar method may be applied to the moments of resistances overcome by rotating pieces, so as to reduce them to *equivalent moments at the driving axle.* Thus, let a resistance R, with the leverage l, be overcome by a piece whose angular velocity of rotation bears the ratio $n : 1$ to that of the driving axle. Then the equivalent moment of resistance at the driving axle is $n \, R \, l$; and if a similar calculation be made for each rotating piece in the machine which overcomes resistance, and the results added together, the sum—

$$\Sigma \cdot n \, R \, l \ldots\ldots\ldots\ldots(7.)$$

is the total *equivalent moment of resistance* at the driving axle; and if in a given interval of time the driving axle turns through the arc i to radius unity, the work performed in that time is—

$$i \, \Sigma \cdot n \, R \, l. \ldots\ldots\ldots\ldots(8.)$$

IV. *Centre of Gravity.*—The work performed in lifting a body is *the product of the weight of the body into the height through which its centre of gravity is lifted.* (See Article 282, page 328.)

If a machine lifts the centres of gravity of several bodies at once to heights either the same or different, the whole quantity of work performed in so doing is the sum of the several products of the weights and heights; but that quantity can also be computed by *multiplying the sum of all the weights into the height through which their common centre of gravity is lifted.*

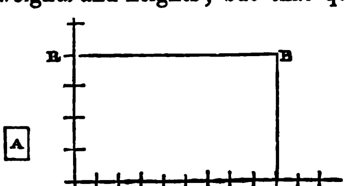

Fig. 240.

306. **Representation of Work by an Area.**—As a quantity of work is the product of two quantities, a force and a motion, it may be represented by the area of a plane figure, which is the product of two dimensions.

Let the base of the rectangle A, fig. 240, represent *one foot* of motion, and its height *one pound* of resistance; then will its area represent one foot-pound of work.

In the larger rectangle, let the base $\overline{\mathrm{OS}}$ represent a certain motion *s*, on the same scale with the base of the unit-area A; and let the height $\overline{\mathrm{OR}}$ represent a certain resistance R, on the same scale with the height of the unit-area A; then will the number of times that the rectangle $\overline{\mathrm{OS}} \cdot \overline{\mathrm{OR}}$ contains the unit-rectangle A, express the number of foot-pounds in the quantity of work R *s*, which is performed in overcoming the resistance R through the distance *s*.

307. **Work against Varying Resistance.** (*A. M.*, 515.)—In fig. 241, let distances as before, be represented by lengths measured along the base line O X of the figure; and let the magnitudes of the resistance overcome at each instant be represented by the lengths of ordinates drawn perpendicular to O X, and parallel to O Y:—For example, when the working body has moved through the distance represented by $\overline{\mathrm{OS}}$, let the resistance be represented by the ordinate $\overline{\mathrm{SR}}$.

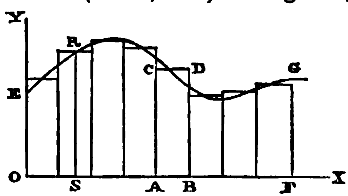

Fig. 241.

If the resistance were constant, the summits of those ordinates would lie in a straight line parallel to O X, like R B in fig. 240; but if the resistance varies continuously as the motion goes on, the summits of the ordinates will lie in a line, straight or curved, such as that marked E R G, fig. 241, which is not parallel to O X.

The values of the resistance at each instant being represented by the ordinates of a given line E R G, let it now be required to determine the work performed against that resistance during a motion represented by $\overline{\mathrm{OF}} = s$.

Suppose the area O E G F to be divided into bands by a series of parallel ordinates, such as A C and B D, and between the upper ends of those ordinates let a series of short lines, such as C D, be drawn parallel to O X, so as to form a stepped or serrated outline, consisting of lines parallel to O X and O Y alternately, and *approximating* to the given continuous line E G.

Now conceive the resistance, instead of varying continuously, to remain constant during each of the series of divisions into which the motion is divided by the parallel ordinates, and to change abruptly at the instants between those divisions, being represented for each division by the height of the rectangle which stands on that division: for example, during the division of the motion represented by A B, let the resistance be represented by A C, and so for other divisions.

Then the work performed during the division of the motion represented by $\overline{\mathrm{A\,B}}$, on the supposition of alternate constancy and abrupt variation of the resistance, is represented by the rectangle $\overline{\mathrm{AB}}\cdot\overline{\mathrm{AC}}$; and the whole work performed, on the same supposition, during the whole motion $\overline{\mathrm{O\,F}}$, is represented by the sum of all the rectangles lying between the parallel ordinates; and inasmuch as the supposed mode of variation of the resistance represented by the stepped outline of those rectangles is an approximation to the real mode of variation represented by the continuous line E G, and is a closer approximation the closer and the more numerous the parallel ordinates are, so the sum of the rectangles is an approximation to the exact representation of the work performed against the continuously varying resistance, and is a closer approximation the closer and more numerous the ordinates are, and by making the ordinates numerous and close enough, can be made to differ from the exact representation by an amount less than any given difference.

But the sum of those rectangles is also an approximation to the area O E G F, bounded above by the continuous line E G, and is a closer approximation the closer and the more numerous the ordinates are, and by making the ordinates numerous and close enough, can be made to differ from the area O E G F by an amount less than any given difference.

Therefore the area O E G F, *bounded by the straight line* O F, *which represents the motion, by the line* E G, *whose ordinates represent the values of the resistance, and by the two ordinates* O E *and* F G, *represents exactly the work performed.* (See Article 289, page 331.)

The MEAN RESISTANCE during the motion is found by dividing the area O E G F by the motion $\overline{\mathrm{O\,F}}$.

308. **Useful Work and Lost Work.**—The useful work of a machine is that which is performed in effecting the purpose for which the machine is designed. The lost work is that which is performed in producing effects foreign to that purpose. The resistances overcome in performing those two kinds of work are called respectively *useful resistance* and *prejudicial resistance.*

The useful work and the lost work of a machine together make up its *total* or *gross work.*

In a pumping engine, for example, the useful work in a given time is the product of the weight of water lifted in that time into the height to which it is lifted: the lost work is that performed in overcoming the friction of the water in the pumps and pipes, the friction of the plungers, pistons, valves, and mechanism, and the resistance of the air pump and other parts of the engine.

For example, the useful work of a marine steam engine in a given time is the product of the resistance opposed by the water to the motion of the ship, into the distance through which she moves: the lost work is that performed in overcoming the resistance of the water to the motion of the propeller through it, the friction of the mechanism, and the other resistances of the engine, and in raising the temperature of the condensation water, of the gases which escape by the chimney, and of adjoining bodies.

There are some cases, such as those of muscular power and of windmills, in which the useful work of a prime mover can be determined, but not the lost work.

309. **Friction.** (Partly extracted and abridged from *A. M.*, 189, 190, 191, 204, and 669 to 685).—The most frequent cause of loss of work in machines is friction—being that force which acts between two bodies at their surface of contact so as to resist their sliding on each other, and which depends on the force with which the bodies are pressed together. The following law respecting the friction of solid bodies has been ascertained by experiment:—

The friction which a given pair of solid bodies, with their surfaces in a given condition, are capable of exerting, is simply proportional to the force with which they are pressed together.

There is a limit to the exactness of the above law, when the pressure becomes so intense as to crush or grind the parts of the bodies at and near their surface of contact. At and beyond that limit the friction increases more rapidly than the pressure; but that limit ought never to be attained at the bearings of any machine. For some substances, especially those whose surfaces are sensibly indented by a moderate pressure, such as timber, the friction between a pair of surfaces which have remained for some time at rest relatively to each other, is somewhat greater than that between the same pair of surfaces when sliding on each other. That excess, however, of the *friction of rest* over the *friction of motion*, is instantly destroyed by a slight vibration; so that the *friction of motion* is alone to be taken into account as causing continuous loss of work.

As to *materials for bearings*, see pages 462, 463, 464.

The friction between a pair of bearing surfaces is calculated by multiplying the force with which they are directly pressed together, by a factor called the *co-efficient of friction*, which has a special value depending on the nature of the materials and the state of the surfaces as to smoothness and lubrication. Thus, let R denote the friction between a pair of surfaces; Q, the force, in a direction perpendicular to the surfaces, with which they are pressed together; and f the co-efficient of friction; then

$$R = f Q \text{.............................(1.)}$$

The co-efficient of friction of a given pair of surfaces is the tangent of an angle called the *angle of repose*, being the greatest angle which an oblique pressure between the surfaces can make with a perpendicular to them, without making them slide on each other.

The following is a table of the angle of repose φ, the co-efficient of friction $f = \tan \varphi$, and its reciprocal $1 : f$, for the materials of mechanism—condensed from the tables of General Morin, and other sources, and arranged in a few comprehensive classes. The values of those constants which are given in the table have reference to the *friction of motion*.* (See page 399.)

No.	SURFACES.	φ	f	$1 : f$
1	Wood on wood, dry,........................	14° to 26½°	·25 to ·5	4 to 2
2	" " soaped,....................	11½° to 2°	·2 to ·04	5 to 25
3	Metals on oak, dry, ...	26½° to 31°	·5 to ·6	2 to 1·67
4	" " wet, ...	13½° to 14½°	·24 to ·26	4·17 to 3·85
5	" " soapy,......................	11½°	·2	5
6	Metals on elm, dry,....	11½° to 14°	·2 to ·25	5 to 4
7	Hemp on oak, dry,..........................	28°	·53	1·89
8	" " wet,.......................	18½°	·33	3
9	Leather on oak,..............................	15° to 19½°	·27 to ·38	3·7 to 2·86
10	Leather on metals, dry,....................	29½°	·56	1·79
11	" " wet,.....................	20°	·36	2·78
12	" " greasy,..............	13°	·23	4·35
13	" " oily,..................	8½°	·15	6·67
14	Metals on metals, dry,.....................	8½° to 11½°	·15 to ·2	6·67 to 5
15	" " wet, ...	16½°	·3	3·33
16	Smooth surfaces, occasionally greased,	4° to 4½°	·07 to ·08	14·3 to 12·5
17	" " continually greased,	3°	·05	20
18	" " best results, ...	1¾° to 2°	·03 to ·036	33·3 to 27·6
19	Bronze on lignum vitæ, constantly wet,	3° ?	·05 ?	20 ?

* In a paper, of which an abstract appeared in the *Comptes Rendus* of the French Academy of Sciences for the 26th of April, 1858, M. H. Bochet describes a series of experiments which have led him to the conclusion, that the friction between a pair of surfaces of iron is not, as had formerly been believed, absolutely independent of the velocity of sliding, but that it diminishes slowly as that velocity increases, according to a law expressed by the following formula. Let

R denote the friction;

Q, the pressure;

v, the velocity of sliding, in mètres per second = velocity in feet per second × 0·3048;

f, a, γ, constant co-efficients; then

$$\frac{R}{Q} = \frac{f + \gamma a v}{1 + a v}.$$

The following are the values of the co-efficients deduced by M. Bochet from

310. **Unguents.**—Three results in the preceding table, Nos. 16, 17. and 18, have reference to smooth firm surfaces of any kind, greased or lubricated to such an extent that the friction depends chiefly on the continual supply of unguent, and not sensibly on the nature of the solid surfaces; and this ought almost always to be the case in machinery. Unguents should be thick for heavy pressures, that they may resist being forced out, and thin for light pressures, that their viscidity may not add to the resistance.

Unguents may be divided into four classes, as follows :—

I. *Water*, which acts as an unguent on surfaces of wood and leather. It is not, however, an unguent for a *pair* of metallic surfaces; for when applied to them, it increases their friction.

II. *Oily unguents*, consisting of animal and vegetable fixed oils, as tallow, lard, lard oil, seal oil, whale oil, olive oil. The vegetable drying oils, such as linseed oil, are unfit for unguents, as they absorb oxygen, and become hard. The animal oils are on the whole better than the vegetable oils.

III. *Soapy unguents*, composed of oil, alkali, and water. For a temporary purpose, such as lubricating the ways for the launch of a ship, one of the best unguents of this class is soft soap, made from whale oil and potash, and used either alone or mixed with tallow. But for a permanent purpose, such as lubricating railway carriage axles, it is necessary that the unguent should contain less water and more oil or fatty matter than soft soap does, otherwise it would dry and become stiff by the evaporation of the water. The best grease for such purposes does not contain more than from 25 to 30 per cent. of water; that which contains 40 or 50 per cent. is bad.

IV. *Bituminous unguents*, in which liquid mineral hydrocarbons are used to dissolve and dilute oily and fatty substances.

The *intensity of the pressure* between a pair of greased surfaces ought not to be so great as to force out the unguent. The following formula agrees very fairly with the results of practice :—

Let v be the velocity of sliding, in feet per second; p, the greatest proper intensity of pressure, in lbs. on the square inch; then

$$p = \frac{44800}{60\,v + 20}.$$

p ought not in any case to exceed 1200.

his experiments, for iron surfaces of wheels and skids rubbing longitudinally on iron rails :—

f, for dry surfaces, 0·3, 0·25, 0·2; for damp surfaces, 0·14.
a, for wheels sliding on rails, 0·03; for skids sliding on rails, 0·07.
γ, not yet determined, but treated meanwhile as inappreciably small.

As to the measurement of friction, see Article 348, page 395.

310 A. **Friction of a Band.**—A flexible band may be used either to exert an effort or a resistance upon a drum or pulley. In either case, the tangential force, whether effort or resistance, exerted between the band and the pulley, is their mutual friction, caused by and proportional to the normal pressure between them.

In fig. 242, let C be the axis of a pulley A B, round an arc of which there is wrapped a band, T_1 A B T_2; let the outer arrow represent the direction in which the band slides, or tends to slide, relatively to the pulley, and the inner arrow the direction in which the pulley slides, or tends to slide, relatively to the band.

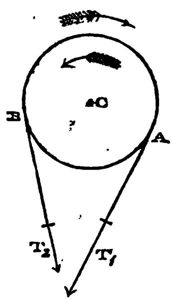

Fig. 242.

Let T_1 be the tension of the free part of the band at that side *towards* which it tends to draw the pulley, or *from* which the pulley tends to draw it; T_2, the tension of the free part at the other side; T, the tension of the band at any intermediate point of its arc of contact with the pulley; θ, the ratio of the length of that arc to the radius of the pulley; $d\,\theta$, the ratio of an indefinitely small element of that arc to the radius; $R = T_1 - T_2$, the total friction between the band and the pulley; $d\,R$, the elementary portion of that friction due to the elementary arc $d\,\theta$; f, the co-efficient of friction between the materials of the band and pulley.

Then it is known that the normal pressure at the elementary arc $d\,\theta$ is $T\,d\,\theta$; T being the mean tension of the band at that elementary arc; consequently, the friction on that arc is

$$d\,R = f\,T\,d\,\theta.$$

Now, that friction is also the difference between the tensions of the band at the two ends of the elementary arc; or

$$d\,T = d\,R = f\,T\,d\,\theta;$$

which equation being integrated throughout the entire arc of contact, gives the following formulæ:—

$$\left.\begin{aligned} \text{hyp log}\,\frac{T_1}{T_2} = f\,\theta;\ T_1 \div T_2 = e^{f\theta}; \\ R = T_1 - T_2 = T_1\,(1 - e^{-f\theta}) = T_2\,(e^{f\theta} - 1). \end{aligned}\right\}\ldots\ldots(1.)$$

When a belt connecting a pair of pulleys has the tensions of its two sides originally equal, the pulleys being at rest; and when the pulleys are set in motion, so that one of them drives the other by means of the belt; it is found that the advancing side of the belt is exactly as much tightened as the returning side is slackened, so that the *mean* tension remains unchanged. The ratio which it bears to the force, R, to be transmitted, is given by this formula:—

$$\frac{T_1 + T_2}{2\,R} = \frac{e^{f\theta} + 1}{2\,(e^{f\theta} - 1)}. \quad \text{.....................(2.)}$$

If the arc of contact between the band and pulley, expressed in turns and fractions of a turn, be denoted by n,

$$\theta = 2\,\pi\,n;\; e^{f\theta} = 10^{2\cdot7288\,f\,n}; \quad \text{..................(3.)}$$

that is to say, $e^{f\theta}$ is the *antilogarithm*, or natural number, corresponding to the common logarithm $2\cdot7288\,f\,n$.

The value of the co-efficient of friction, f, depends on the state and material of the rubbing surfaces. For leather belts on iron pulleys, the table of Article 309, page 349, shows that it ranges from ·56 to ·15. In calculating, by equation 2 of this Article, the proper mean tension for a belt, the smallest value, $f = \cdot15$, is to be taken, if there is a probability of the belt becoming wet with oil. The experiments of Messrs. Henry R. Towne and Robert Briggs, however (published in the Journal of the Franklin Institute for 1868), show that such a state of lubrication is not of ordinary occurrence; and that in designing machinery, we may in most cases safely take $f = 0\cdot42$. Professor Reuleaux (*Constructionslehre für Maschinenbau*) takes $f = 0\cdot25$. The following table shows the values of the co-efficient $2\cdot7288\,f$, by which n is multiplied in equation 3, corresponding to different values of f; also the corresponding values of various ratios amongst the forces, when the arc of contact is half a circumference:—

$f =$	0·15	0·25	0·42	0·56
$2\cdot7288\,f =$	0·41	0·68	1·15	1·53

Let $\theta = \pi$, and $n = \frac{1}{2}$; then

$T_1 \div T_2 =$	1·603	2·188	3·758	5·821
$T_1 \div R =$	2·66	1·84	1·36	1·21
$T_1 + T_2 \div 2\,R =$	2·16	1·34	0·86	0·71

In ordinary practice, it is usual to assume $T_2 = R$; $T_1 = 2\,R$; $T_1 + T_2 \div 2\,R = 1\cdot5$. This corresponds to $f = 0\cdot22$ nearly.

For a wire rope on cast-iron, f may be taken as $= 0\cdot15$ nearly; and if the groove of the pulley is bottomed with gutta-percha, 0·25.

When an endless band runs at a very high velocity, its centrifugal force has an indirect effect on the friction, which will be considered further on. (See page 441.)

311. **The Work Performed against Friction** in a given time, between a pair of rubbing surfaces, is the product of that friction into the distance through which one surface slides over the other.

When the motion of one surface relatively to the other consists in rotation about an axis, the work performed may also be calculated by multiplying the relative *angular motion* of the surfaces to radius unity into the *moment of friction;* that is, the product of the friction into its leverage, which is the mean distance of the rubbing surfaces from the axis.

For a cylindrical journal, the leverage of the friction is simply the radius of the journal.

For a *flat pivot,* the leverage is two-thirds of the radius of the pivot.

For a *collar,* let r and r' be the outer and inner radii; then the leverage of the friction is

$$\frac{2}{3} \cdot \frac{r^3 - r'^3}{r^2 - r'^2} \quad \text{.............................(1.)}$$

For "*Schiele's anti-friction pivot,*" whose longitudinal section is the curve called the "tractrix," the moment of friction is $f \times$ the load $\times$ the external radius. This is greater than the moment for an equally smooth flat pivot of the same radius; but the anti-friction pivot has the advantage, inasmuch as the wear of the surfaces is uniform at every point, so that they always fit each other accurately, and the pressure is always uniformly distributed, and never becomes, as is the case in other pivots, so intense at certain points as to force out the unguent and grind the surfaces.

In the *cup and ball* pivot, the end of the shaft, and the step on which it presses, present two recesses facing each other, into which are fitted two shallow cups of steel or hard bronze. Between the concave spherical surfaces of those cups is placed a steel ball, being either a complete sphere, or a lens having convex surfaces of a somewhat less radius than the concave surfaces of the cups. The moment of friction of this pivot is at first almost inappreciable, from the extreme smallness of the radius of the circles of contact of the ball and cups; but as they wear, that radius and the moment of friction increase.

By the rolling of two surfaces over each other without sliding, a resistance is caused, which is called sometimes "rolling friction," but more correctly *rolling resistance.* It is of the nature of a *couple* resisting rotation; its *moment* is found by multiplying the normal pressure between the rolling surfaces by an *arm* whose length depends on the nature of the rolling surfaces; and the work lost in an unit of time in overcoming it is the product of its moment by the *angular velocity* of the rolling surfaces relatively to each other. The following are approximate values of the arm in *decimals of a foot*:—

Oak upon oak,........................0·006 (Coulomb).
Lignum-vitæ on oak,..................0·004 —
Cast-iron on cast-iron,..............0·002 (Tredgold).

The work lost in friction produces HEAT in the proportion of one British thermal unit, being so much heat as raises the temperature of a pound of water one degree of Fahrenheit, for every 772 foot-pounds of lost work.

The heat produced by friction, when moderate in amount, is useful in softening and liquefying unguents; but when excessive, it is prejudicial by decomposing the unguents, and sometimes even by softening the metal of the bearings, and raising their temperature so high, as to set fire to neighbouring combustible matters.

Excessive heating is prevented by a constant and copious supply of a good unguent. When the velocity of rubbing is about four or five feet per second, the elevation of temperature is found to be, with good fatty and soapy unguents, 40° to 50° Fahrenheit, with good mineral unguents, about 30°. The effect of friction upon the efficiency of machines will be considered further on, in Section IV.

312. **Work of Acceleration.** (*A. M.*, 12, 521–33, 536, 547, 549, 554, 589, 591, 593, 595–7.)—In order that the velocity of a body's motion may be changed, it must be acted upon by some other body with a force in the direction of the change of velocity, which force is proportional directly to the change of velocity, and to the mass of the body acted upon, and inversely to the time occupied in producing the change. If the change is an acceleration or increase of velocity, let the first body be called the *driven body*, and the second the *driving body*. Then the force must act upon the driven body in the direction of its motion. Every force being a pair of equal and opposite actions between a pair of bodies, the same force which accelerates the driven body is a *resistance* as respects the driving body. (See Article 287, page 329.)

For example, during the commencement of the stroke of the piston of a steam engine, the velocity of the piston and of its rod is accelerated; and that acceleration is produced by a certain part of the pressure between the steam and the piston, being the excess of that pressure above the whole resistance which the piston has to overcome. The piston and its rod constitute the driven body; the steam is the driving body; and the same part of the pressure which accelerates the piston, acts as a *resistance* to the motion of the steam, in addition to the resistance which would have to be overcome if the velocity of the piston were uniform.

The resistance due to acceleration is computed in the following manner:—It is known by experiment, that if a body near the earth's surface is accelerated by the attraction of the earth,—that is, by its own weight, or by a force equal to its own weight, its velocity goes on continually increasing very nearly at the rate of 32·2 *feet per second of additional velocity, for each second during which the force acts.* This quantity varies in different latitudes, and at different elevations, but the value just given is near enough to the truth for purposes of mechanical engineering. For brevity's sake, it is usually denoted by the symbol g; so that if at a given instant the velocity of a body is v_1 feet per second, and if its own weight, or an equal force, acts freely on it in the direction of its motion for t seconds, its velocity at the end of that time will have increased to

$$v_2 = v_1 + g\,t \text{..........................(1.)}$$

If the acceleration be at any different rate per second, *the force necessary to produce that acceleration, being the resistance on the driving body due to the acceleration of the driven body, bears the same proportion to the driven body's weight which the actual rate of acceleration bears to the rate of acceleration produced by gravity acting freely.* (In mètres per second, $g = 9·81$ nearly.)

To express this by symbols, let the weight of the driven body be denoted by W. Let its velocity at a given instant be v_1 feet per second; and let that velocity increase at an uniform rate, so that at an instant t seconds later, it is v_2 feet per second.

Let f denote the rate of acceleration; then

$$f = \frac{v_2 - v_1}{t};\text{..........................(2.)}$$

and the force R necessary to produce it will be given by the proportion,

$$g : f :: \mathrm{W} : \mathrm{R};$$

that is to say,

$$\mathrm{R} = \frac{f\,\mathrm{W}}{g} = \frac{\mathrm{W}\,(v_2 - v_1)}{g\,t}\text{....................(3.)}$$

The factor $\frac{\mathrm{W}}{g}$, in the above expression, is called the MASS of the driven body; and being the same for the same body, in what place soever it may be, is held to represent the *quantity of matter* in the body. (See Article 277A, page 318.)

The product $\frac{\mathrm{W}\,v}{g}$ of the mass of a body into its velocity at instant, is called its MOMENTUM; so that the resistance due to a given acceleration is equal to *the increase of momentum divided by the time which that increase occupies.*

If the product of the force by which a body is accelerated, equal and opposite to the resistance due to acceleration, into the time during which it acts, be called IMPULSE, the same principle may be otherwise stated by saying, that the *increase of momentum is equal to the impulse by which it is caused.*

If the rate of acceleration is not constant, but variable, the force R varies along with it. In this case, the value, at a given instant of the rate of acceleration, is represented by $f = \frac{dv}{dt}$, and the corresponding value of the force is

$$R = \frac{fW}{g} = \frac{W}{g} \cdot \frac{dv}{dt} \text{.....................(4.)}$$

The WORK PERFORMED in accelerating a body is the product of the resistance due to the rate of acceleration into the distance moved through by the driven body while the acceleration is going on. The resistance is equal to the mass of the body, multiplied by the increase of velocity, and divided by the time which that increase occupies. The distance moved through is the product ot the mean velocity into the same time. Therefore, the work performed is equal to the mass of the body multiplied by the increase of the velocity, and by the mean velocity; that is, *to the mass o) the body, multiplied by the increase of the half-square of its velocity.*

To express this by symbols, in the case of an uniform rate of acceleration, let s denote the distance moved through by the driven oody during the acceleration; then

$$s = \frac{v_2 + v_1}{2} t; \text{..........................(5.)}$$

which being multiplied by equation 3, gives for the work of acceleration,

$$R s = \frac{W}{g} \cdot \frac{v_2 - v_1}{t} \cdot \frac{v_2 + v_1}{2} \cdot t = \frac{W}{g} \cdot \frac{v_2^2 - v_1^2}{2} \text{.........(6.)}$$

In the case of a variable rate of acceleration, let v denote the mean velocity, and ds the distance moved through, in an interval of time dt so short that the increase of velocity dv is indefinitely small stampared with the mean velocity. Then
com

$$ds = v\,dt; \text{..........................(7.)}$$

which being multiplied by equation 4, gives for the work of acceleration during the interval $d\,t$,

$$\mathrm{R}\,d\,s = \frac{\mathrm{W}}{g} \cdot \frac{d\,v}{d\,t} \cdot v\,d\,t$$

$$= \frac{\mathrm{W}}{g} \cdot v\,d\,v; \quad\text{.......................}(8.)$$

and the *integration* of this expression (see Article 11 A) gives for the work of acceleration during a finite interval,

$$\int \mathrm{R}\,d\,s = \frac{\mathrm{W}}{g}\int v\,d\,v = \frac{\mathrm{W}}{g} \cdot \frac{v_2^2 - v_1^2}{2} \quad\text{.........}(9.)$$

being the same with the result already arrived at in equation 6.

From equation 9 it appears that *the work performed in producing a given acceleration depends on the initial and final velocities,* v_1 *and* v_2, *and not on the intermediate changes of velocity.*

If a body falls freely under the action of gravity from a state of rest through a height h, so that its initial velocity is 0, and its final velocity v, the work of acceleration performed by the earth on the body is simply the product W h of the weight of the body into the height of fall. Comparing this with equation 6, we find—

$$h = \frac{v^2}{2g} \quad\text{..............................}(10.)$$

This quantity is called the *height*, or *fall, due to the velocity* v; and from equations 6 and 9 it appears that *the work performed in producing a given acceleration is the same with that performed in lifting the driven body through the difference of the heights due to its initial and final velocities.*

If work of acceleration is performed by a prime mover upon bodies which neither form part of the prime mover itself, nor of the machines which it is intended to drive, that work is lost; as when a marine engine performs work of acceleration on the water that is struck by the propeller.

Work of acceleration performed on the moving pieces of the prime mover itself, or of the machinery driven by it, is not necessarily lost, as will afterwards appear.

313. **Summation of Work of Acceleration—Moment of Inertia.**—If several pieces of a machine have their velocities increased at the same time, the work performed in accelerating them is the sum of the several quantities of work due to the acceleration of the respective pieces; a result expressed in symbols by

$$\Sigma\left\{\frac{\mathrm{W}}{g} \cdot \frac{v_2^2 - v_1^2}{2}\right\} \quad\text{........................}(1.)$$

The process of finding that sum is facilitated and abridged in certain cases by special methods.

I. *Accelerated Rotation—Moment of Inertia.*—Let a denote the angular velocity of a solid body rotating about a fixed axis;—that is, as explained in Article 46, the velocity of a point in the body whose radius-vector, or distance from the axis, is unity.

Then the velocity of a particle whose distance from the axis is r, is

$$v = a\,r; \qquad (2.)$$

and if in a given interval of time the angular velocity is accelerated from the value a_1 to the value a_2, the increase of the velocity of the particle in question is

$$v_2 - v_1 = r\,(a_2 - a_1) \qquad (3.)$$

Let w denote the weight, and $\frac{w}{g}$ the mass of the particle in question. Then the work performed in accelerating it, being equal to the product of its mass into the increase of the half-square of its velocity, is also equal to *the product of its mass into the square of its radius-vector, and into the increase of the half-square of the angular velocity;* that is to say, in symbols,

$$\frac{w}{g} \cdot \frac{v_2^2 - v_1^2}{2} = \frac{w\,r^2}{g} \cdot \frac{a_2^2 - a_1^2}{2} \qquad (4.)$$

To find the work of acceleration for the whole body, it is to be conceived to be divided into small particles, whose velocities at any given instant, and also their accelerations, are proportional to their distances from the axis; then the work of acceleration is to be found for each particle, and the results added together. But in the sum so obtained, the increase of the half-square of the angular velocity is a common factor, having the same value for each particle of the body; and the rate of acceleration produced by gravity, $g = 32{\cdot}2$, is a common divisor. It is therefore sufficient *to add together the products of the weight of each particle* (w) *into the square of its radius-vector* (r^2), *and to multiply the sum so obtained* ($\Sigma \cdot w\,r^2$) *by the increase of the half-square of the angular velocity* $\left(\frac{1}{2}(a_2^2 - a_1^2)\right)$, *and divide by the rate of acceleration due to gravity* (g). The result, viz.:—

$$\Sigma \left\{ \frac{w}{g} \cdot \frac{v_2^2 - v_1^2}{2} \right\} = \frac{a_2^2 - a_1^2}{2\,g} \cdot \Sigma\, w\,r^2 \qquad (5.)$$

is the work of acceleration sought. In fact, the sum $\Sigma\, w\,r^2$ is *the weight of a body, which, if concentrated at the distance unity from* *the axis of rotation, would require the same work to produce a given increase of angular velocity which the actual body requires.*

The term MOMENT OF INERTIA is applied in some writings to the sum $\Sigma w r^2$, and in others to the corresponding *mass* $\Sigma w r^2 \div g$. For purposes of mechanical engineering, the sum $\Sigma w r^2$ is, on the whole, the most convenient, bearing as it does the same relation to angular acceleration which *weight* does to acceleration of linear velocity.

The *Radius of Gyration*, or *Mean Radius* of a rotating body, is a line whose square is the mean of the squares of the distances of its particles from the axis; and its value is given by the following equation:—

$$\rho^2 = \frac{\Sigma w r^2}{\Sigma w} \quad \text{(6.)}$$

so that if we put $W = \Sigma w$ for the weight of the whole body, the moment of inertia may be represented by

$$I = W \rho^2 \quad \text{(7.)}$$

The following are additional rules relating to moments of inertia and radii of gyration:—

RULE II.—Given, the moment of inertia of a body about an axis traversing its centre of gravity in a given direction; to find its moment of inertia about another axis parallel to the first; multiply the mass (or weight) of the body by the square of the perpendicular distance between the two axes, and to the product add the given moment of inertia.

RULE III.—Given, the separate moments of inertia of a set of bodies about parallel axes traversing their several centres of gravity; required, the combined moment of inertia of those bodies about a common axis parallel to their separate axes; multiply the mass (or weight) of each body by the square of the perpendicular distance of its centre of gravity from the common axis; add together all the products, and all the separate moments of inertia; the sum will be the combined moment of inertia.

RULE IV.—To find the square of the radius of gyration of a body about a given axis; divide the moment of inertia of the body about the given axis by the mass (or weight) of the body.

RULE V.—Given, the square of the radius of gyration of a body about an axis traversing its centre of gravity in a given direction; to find the square of the radius of gyration of the same body about another axis parallel to the first; to the given square add the square of the perpendicular distance between the two axes.

Table of Squares of Radii of Gyration.

Body.	Axis.	Radius.^{2}
I. Sphere of radius r,	Diameter	$\frac{2r^2}{5}$
II. Spheroid of revolution—polar semi-axis a, equatorial radius r,	Polar axis	$\frac{2r^2}{5}$
III. Ellipsoid—semi-axes a, b, c,	Axis, $2a$	$\frac{b^2 + c^2}{5}$
IV. Spherical shell—external radius r, internal r',	Diameter	$\frac{2(r^5 - r'^5)}{5(r^3 - r'^3)}$
V. Spherical shell, insensibly thin—radius r, thickness dr,	Diameter	$\frac{2r^2}{3}$
VI. Circular cylinder—length $2a$, radius r,	Longitudinal axis, $2a$	$\frac{r^2}{2}$
VII. Elliptic cylinder—length $2a$, transverse semi-axes b, c,	Longitudinal axis, $2a$	$\frac{b^2 + c^2}{4}$
VIII. Hollow circular cylinder—length $2a$, external radius r, internal r',	Longitudinal axis, $2a$	$\frac{r^2 + r'^2}{2}$
IX. Hollow circular cylinder, insensibly thin—length $2a$, radius r, thickness dr,	Longitudinal axis, $2a$	r^2
X. Circular cylinder—length $2a$, radius r,	Transverse diameter	$\frac{r^2}{4} + \frac{a^2}{3}$
XI. Elliptic cylinder—length $2a$, transverse semi-axes b, c,	Transverse axis, $2b$	$\frac{c^2}{4} + \frac{a^2}{3}$
XII. Hollow circular cylinder—length $2a$, external radius r, internal r',	Transverse diameter	$\frac{r^2 + r'^2}{4} + \frac{a^2}{3}$
XIII. Hollow circular cylinder, insensibly thin—radius r, thickness dr,	Transverse diameter	$\frac{r^2}{2} + \frac{a^2}{3}$
XIV. Rectangular prism—dimensions $2a$, $2b$, $2c$,	Axis, $2a$	$\frac{b^2 + c^2}{3}$
XV. Rhombic prism—length $2a$, diagonals $2b$, $2c$,	Axis, $2a$	$\frac{b^2 + c^2}{6}$
XVI. Rhombic prism, as above,	Diagonal, $2b$	$\frac{c^2}{6} + \frac{a^2}{3}$

314. **Centre of Percussion—Equivalent Simple Pendulum—Equivalent Fly-wheel.**—In calculations respecting the rotation of a rigid body about a given axis, it is often convenient to conceive that for the actual body there is substituted its *equivalent simple pendulum;* that is, a body having the same total mass, but concentrated at two points, of which one is in the axis: also the same statical moment, and the same moment of inertia.

The following are the rules for doing this:—

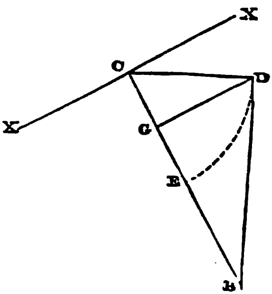

Fig. 243.

I. To find the centre of percussion of a given body turning about a given axis.

In fig. 243, let X X be the given axis, and G the centre of gravity of the body. From G let fall G C perpendicular to X X. Through G draw G D parallel to X X, and equal to the radius of gyration of the body about the axis G D. Join C D. Then will C E = C D = $\sqrt{\text{G D}^2 + \text{C G}^2}$ = the radius of gyration of the body about X X. From D draw D B perpendicular to C D, cutting C G produced in B. Then will B be the centre of percussion of the body for the axis X X.

To find B by calculation; make $\text{G B} = \frac{\text{G D}^2}{\text{G C}}$.

C is the centre of percussion for an axis traversing B parallel to X X.

II. To convert the body into an "equivalent simple pendulum" for the axis X X, or for an axis through B parallel to X X; divide the mass of the body into two parts inversely proportional to G C and G B, and conceive those parts to be concentrated at C and B respectively, and rigidly connected together.

(Let W be the whole mass, and C and B the two parts; then $\text{C} = \frac{\text{W} \cdot \text{G B}}{\text{C B}}$; $\text{B} = \frac{\text{W} \cdot \text{G C}}{\text{C B}}$.)

(The "equivalent simple pendulum" has the same weight with the given body, and also the same moment of weight, and the same moment of inertia, with the given body, relatively to an axis in the given direction X X, traversing either C or B.)

III. *Equivalent Ring, or Equivalent Fly-wheel.*—When the given axis traverses the centre of gravity, G, there is no centre of percussion. The moment of the body's weight is nothing, and its moment of inertia is the same as if its whole mass were concentrated in a ring of a radius equal to the radius of gyration of the body. That ring may be called the "equivalent ring," or "equivalent fly-wheel."

315. **Reduced Inertia.**—If in a certain machine, a moving piece whose weight is W has a velocity always bearing the ratio $n : 1$ to the velocity of the driving point, it is evident that when the driving point undergoes a given acceleration, the work performed in producing the corresponding acceleration in the piece in question is the same with that which would have been required if a weight n^2 W had been concentrated at the driving point.

If a similar calculation be performed for each moving piece in the machine, and the results added together, the sum

$$\Sigma \cdot n^2 \mathrm{W} \quad \ldots\ldots (1.)$$

gives the weight which, being concentrated at the driving point, would require the same work for a given acceleration of the driving point that the actual machine requires; so that if v_1 is the initial, and v_2 the final velocity of the driving point, the work of acceleration of the whole machine is

$$\frac{v_2^2 - v_1^2}{2g} \cdot \Sigma \cdot n^2 \mathrm{W} \quad \ldots\ldots (2.)$$

This operation may be called *the reduction of the inertia to the driving point.* Mr. Moseley, by whom it was first introduced into the theory of machines, calls the expression (1.) the "*co-efficient of steadiness,*" for reasons which will afterwards appear.

In finding the reduced inertia of a machine, the mass of each rotating piece is to be treated as if concentrated at a distance from its axis equal to its radius of gyration ϱ; so that if v represents the velocity of the driving point at any instant, and a the corresponding angular velocity of the rotating piece in question, we are to make

$$n^2 = \frac{a^2 \varrho^2}{v^2} \quad \ldots\ldots (3.)$$

in performing the calculation expressed by the formula (1.)

316. **Summary of Various Kinds of Work.**—In order to present at one view the symbolical expression of the various modes of performing work described in the preceding articles, let it be supposed that in a certain interval of time $d\,t$ the driving point of a machine moves through the distance $d\,s$; that during the same time its centre of gravity is elevated through the height $d\,h$; that resistances, any one of which is represented by R, are overcome at points, the respective ratios of whose velocities to that of the driving point are denoted by n; that the weight of any piece of the mechanism is W, and that n' denotes the ratio of its velocity (or if it rotates, the ratio of the velocity of the end of its radius of gyration) to the velocity of the driving point; and that the driving point, whose mean velocity

is $v = \frac{d s}{d t}$, undergoes the acceleration $d v$. Then the *whole work performed* during the interval in question is

$$d h \cdot \Sigma \mathrm{W} + d s \cdot \Sigma n \mathrm{R} + \frac{v d v}{g} \cdot \Sigma n^2 \mathrm{W} \ldots (1.)$$

The *mean total resistance, reduced to the driving point,* may be computed by dividing the above expression by the motion of the driving point $d s = v d t$, giving the following result:—

$$\frac{d h}{d s} \cdot \Sigma \mathrm{W} + \Sigma n \mathrm{R} + \frac{d v}{g d t} \cdot \Sigma n^2 \mathrm{W} \ldots\ldots (2.)$$

SECTION II.—*Of Deviating and Centrifugal Force.*

317. **Deviating Force of a Single Body.** (*A. M.*, 537.)—It is part of the first law of motion, that if a body moves under no force, or balanced forces, it moves in a straight line. (*A. M.*, 510, 512.)

It is one consequence of the second law of motion, that in order that a body may move in a curved path, it must be continually acted upon by an unbalanced force at right angles to the direction of its motion, the direction of the force being that towards which the path of the body is curved, and its magnitude bearing the same ratio to the weight of the body that the height due to the body's velocity bears to half the radius of curvature of its path.

This principle is expressed symbolically as follows:—

Half radius of curvature.		Height due to velocity.		Body's weight.		Deviating force.
$\frac{r}{2}$	:	$\frac{v^2}{2 g}$	: :	W	:	$\mathrm{Q} = \frac{\mathrm{W} v^2}{g r} \ldots\ldots (1.)$

In the case of projectiles and of the heavenly bodies, deviating force is supplied by that component of the mutual attraction of two masses which acts perpendicular to the direction of their relative motion. In machines, deviating force is supplied by the strength or rigidity of some body, which *guides* the revolving mass, making it move in a curve.

A pair of free bodies attracting each other have both deviated motions, the attraction of each guiding the other; and their deviations of motion relatively to their common centre of gravity are inversely as their masses.

In a machine, each revolving body tends to press or draw the body which guides it away from its position, in a direction from the centre of curvature of the path of the revolving body; and that tendency is resisted by the strength and stiffness of the guiding body, and of the frame with which it is connected.

318. **Centrifugal Force** (*A. M.*, 538) is the force with which a revolving body reacts on the body that guides it, and is equal and opposite to the deviating force with which the guiding body acts on the revolving body.

In fact, as has been already stated, every force is an action between two bodies; and *deviating force* and *centrifugal force* are but two different names for the same force, applied to it according as the condition of the revolving body or that of the guiding body is under consideration at the time.

319. **A Revolving Pendulum** is one of the simplest practical applications of the principles of deviating force, and is described here because its use in regulating the speed of prime movers will afterwards have to be referred to. It consists of a ball A, suspended from a point C by a rod C A of small weight as compared with the ball, and revolving in a circle about a vertical axis C B. The tension of the rod is the resultant of the weight of the ball A, acting vertically, and of its centrifugal force, acting horizontally; and therefore the rod will assume such an inclination that

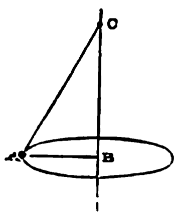

Fig. 244.

$$\frac{\text{height } \overline{\mathrm{B\,C}}}{\text{radius } \overline{\mathrm{A\,B}}} = \frac{\text{weight}}{\text{centrifugal force}} = \frac{g\,r}{v^2} \ldots (1.)$$

where $r = \mathrm{A\,B}$. Let T be the *number of turns per second* of the pendulum, and a its angular velocity; then

$$v = a\,r = 2\,\pi\,\mathrm{T}\,r;$$

and therefore, making $\overline{\mathrm{B\,C}} = h$,

$$h = \frac{g\,r^2}{v^2} = \frac{g}{a^2} = \frac{g}{4\,\pi^2\,\mathrm{T}^2}$$

$$= \text{(in the latitude of London)} \ \frac{0{\cdot}8154 \text{ foot}}{\mathrm{T}^2} = \frac{9{\cdot}7848 \text{ inches}}{\mathrm{T}^2} \ldots (2.)$$

320. **Deviating Force in Terms of Angular Velocity.** (*A. M.*, 540.)—When a body revolves in a circular path round a fixed axis, as is almost always the case with the revolving parts of machines, the radius of curvature of its path, being the perpendicular distance of the body from the axis, is constant; and the velocity v of the body is the product of that radius into the angular velocity; or symbolically,

$$v = a\,r = 2\,\pi\,\mathrm{T}\,r.$$

If these values of the velocity be substituted for v in equation 1 of Article 317, it becomes—

$$Q = \frac{W a^2 r}{g} = \frac{W \cdot 4 \pi^2 T^2 r}{g} \quad \text{.................(1.)}$$

321. **Resultant Centrifugal Force.** (*A. M.*, 603.)—The whole centrifugal force of a body of any figure, or of a system of connected bodies, rotating about an axis, is the same in *amount* and *direction* as if the whole mass were concentrated at the centre of gravity of the system. That is to say, in the formula of Article 320, W is to be held to represent the weight of the whole body or system, and r the perpendicular distance of its centre of gravity from the axis; and the line of action of the resultant centrifugal force Q is always *parallel* to r, although it does not in every case *coincide* with r.

When the axis of rotation *traverses* the centre of gravity of the body or system, the amount of the centrifugal force is *nothing;* that is to say, the rotating body does not tend to pull its axis as a whole out of its place.

The centrifugal forces exerted by the various rotating pieces of a machine against the bearings of their axles are to be taken into account in determining the lateral pressures which cause friction, and the strength of the axles and framework.

As those centrifugal forces cause increased friction and stress, and sometimes, also, by reason of their continual change of direction, produce detrimental or dangerous vibration, it is desirable to reduce them to the smallest possible amount; and for that purpose, unless there is some special reason to the contrary, the axis of rotation of every piece which rotates rapidly ought to traverse its centre of gravity, that the resultant centrifugal force may be nothing.

322. **Centrifugal Couple—Permanent Axis.**—It is not, however, sufficient to annul the effect of centrifugal force, that there should be no tendency to *shift* the axis as a whole; there should also be no tendency to *turn* it into a new angular position.

To show, by the simplest possible example, that the latter tendency may exist without the former, let the axis of rotation of the system shown in fig. 245 be the centre line of an axle resting in bearings at E and F. At B and D let two arms project perpendicularly to that axle, in opposite directions in the same plane, carrying at their extremities two heavy bodies A and C. Let the weights of the arms be

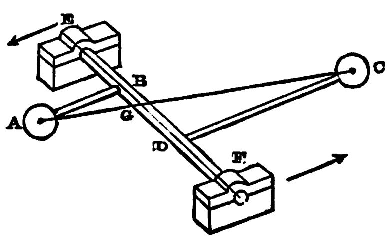

Fig. 245.

insensible as compared with the weights of those bodies; and let the weights of the bodies be inversely as their distances from the axis; that is, let

$$A \cdot \overline{AB} = C \cdot \overline{CD}.$$

Let A C be a straight line joining the centres of gravity of A and C, and cutting the axis in G; then G is the common centre of gravity of A and C, and being in the axis, the resultant centrifugal force is nothing.

In other words, let a be the angular velocity of the rotation; then

The centrifugal force exerted on the axis by A

$$= \frac{a^2 \, A \cdot \overline{AB}}{g};$$

The centrifugal force exerted on the axis by C

$$= \frac{a^2 \, C \cdot \overline{CD}}{g};$$

and those forces are equal in magnitude and opposite in direction; so that there is no tendency to remove the point G in any direction.

There is, however, a tendency to *turn the axis about* the point G, being the product of the common magnitude of the *couple* of centrifugal forces above stated, into their leverage; that is, the perpendicular distance $\overline{BD}$ between their lines of action. That product is called *the moment of the centrifugal couple;* and is represented by

$$Q \cdot \overline{BD}; \quad \ldots\ldots\ldots\ldots (1.)$$

Q being the common magnitude of the equal and opposite centrifugal forces.

That couple causes a couple of equal and opposite pressures of the journals of the axle against their bearings at E and F, in the directions represented by the arrows, and of the magnitude given by the formula—

$$Q \cdot \frac{\overline{BD}}{\overline{EF}}; \quad \ldots\ldots\ldots\ldots (2.)$$

these pressures continually change their directions as the bodies A and C revolve; and they are resisted by the strength and rigidity of the bearings and frame. It is desirable, when practicable, to reduce them to nothing; and for that purpose, the points B, G, and D should coincide; in which case the centre line of the axle E F is said to be *a permanent axis.*

When there are more than two bodies in the rotating system, the centrifugal couple is found as follows:—

Let X X′, fig. 246, represent the axis of rotation; G, the centre of gravity of the rotating body or system, situated in that axis; so that the resultant centrifugal force is nothing.

Let W be any one of the parts of which the body or system is composed, so that, the weight of that part being denoted by W, the weight of the whole body or system may be denoted by $\Sigma \cdot W$.

Let r denote the perpendicular distance of the centre of W from the axis; then

$$\frac{W a^2 r}{g},$$

is the centrifugal *force* of W, pulling the axis in the direction x W.

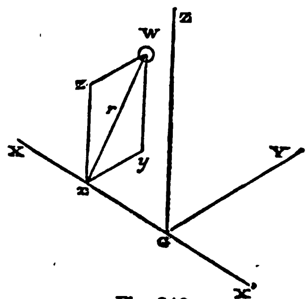

Fig. 246.

Assume a pair of axes of co-ordinates, G Z, G Y, perpendicular to X X′ and to each other, and fixed relatively to the rotating body or system—that is, rotating along with it.

From W let fall $\overline{Wy}$ perpendicular to the plane of G X and G Y, and parallel to G Z; also $\overline{Wz}$, perpendicular to the plane of G X and G Z, and parallel to G Y; and make

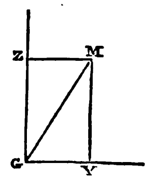

Fig. 247.

$$\overline{xy} = \overline{Wz} = y; \ \overline{xz} = \overline{Wy} = z; \ \overline{Gx} = x.$$

Then the centrifugal force which W exerts on the axis, and which is proportional to r, may be resolved into two components, in the direction of, and proportional to, y and z respectively, viz.:—

$$\frac{W a^2 y}{g} \text{ parallel to G Y, and}$$

$$\frac{W a^2 z}{g} \text{ parallel to G Z;}$$

and those two component forces, being both applied at the end of the lever $\overline{Gx} = x$, exert *moments*, or tendencies to turn the axis X X′ about the point Z, expressed as follows:—

$\frac{W a^2 y x}{g}$, tending to turn G X about G Z towards G Y;

$\frac{W a^2 z x}{g}$, tending to turn G X about G Y towards G Z.

In the same manner are to be found the several moments of the centrifugal forces of all the other parts of which the body or system consists; and care is to be taken to distinguish moments which tend to turn the axis *towards* G Y or G Z from those which tend to turn it *from* those positions, by treating one of these classes of quantities as positive, and the other as negative.

Then by adding together the positive moments and subtracting the negative moments for all the parts of the body or system, are to be found the two sums,

$$\frac{a^2}{g} \cdot \Sigma \cdot W y x ; \frac{a^2}{g} \cdot \Sigma \cdot W z x ; \dots\dots\dots\dots (3.)$$

which represent the total tendencies of all the centrifugal forces to turn the axis in the planes of G Y and G Z respectively.

In fig. 247 lay down $\overline{GY}$ to represent the former moment, and $\overline{GZ}$, perpendicular to G Y, to represent the latter. Then the diagonal $\overline{GM}$ of the rectangle G Z M Y will represent the resultant moment of what is called the Centrifugal Couple, and the direction of that line will indicate the direction in which that couple tends to turn the axis G X about the point G. Its value, and its angular position, are given by the equations,

$$\left. \begin{array}{r} \overline{GM} = \sqrt{(\overline{GY}^2 + \overline{GZ}^2)}; \\ \tan \angle YGM = \overline{GZ} \div \overline{GY} \end{array} \right\} \dots\dots\dots\dots (4.)$$

The condition which it is desirable to fulfil in all rapidly rotating pieces of machines, that the axis of rotation shall be a *permanent axis*, is fulfilled when each of the sums in the formula 3 is nothing; that is, when

$$\Sigma \cdot W y x = 0 \cdot \Sigma \cdot W z x = 0, \dots\dots\dots\dots (5.)$$

The question, whether the axis of a rotating piece is a permanent axis or not, is tested experimentally by making the piece spin round rapidly with its shaft resting in bearings which are suspended by chains or cords, so as to be at liberty to swing. If the axis is not a permanent axis, it oscillates; if it is, it remains steady.

When the axis of rotation traverses the centre of gravity of the piece, there is said to be a STANDING BALANCE; when it is also a permanent axis, there is said to be a RUNNING BALANCE.

Section III.—*Of Effort, Energy, Power, and Efficiency.*

323. **Effort** is a name applied to a force which acts on a body in the direction of its motion (*A. M.*, 511).

If a force is applied to a body in a direction making an acute angle with the direction of the body's motion, the component of that oblique force along the direction of the body's motion is an effort. That is to say, in fig. 248, let A B represent the direction in which A is moving; let $\overline{A F}$ represent a force applied to A, obliquely to that direction; from F draw $\overline{F P}$ perpendicular to A B; then $\overline{A P}$ is the *effort* due to the force $\overline{A F}$. The transverse component $\overline{P F}$ is a *lateral force*, like the transverse component of the oblique resisting force in Article 304.

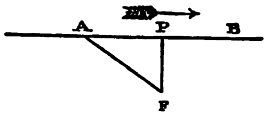

Fig. 248.

To express this algebraically, let the entire force $\overline{A F} = F$, the effort $\overline{A P} = P$, the lateral force $\overline{P F} = Q$, and the angle of obliquity $P A F = \theta$. Then

$$\left. \begin{aligned} P &= F \cdot \cos \theta; \\ Q &= F \cdot \sin \theta \end{aligned} \right\} \text{.........................(1.)}$$

324. **Condition of Uniform Speed.** (*A. M.*, 510, 512, 537.)—According to the first law of motion, in order that a body may move uniformly, the forces applied to it, if any, must balance each other; and the same principle holds for a machine consisting of any number of bodies.

When the *direction* of a body's motion varies, but not the *velocity*, the lateral force required to produce the change of direction depends on the principles set forth in Section 2; but the condition of balance still holds for the forces which act *along* the direction of the body's motion, that is, for the *efforts* and *resistances;* so that, whether for a single body or for a machine, the condition of *uniform velocity* is, that the *efforts shall balance the resistances.*

In a machine, this condition must be fulfilled for each of the single moving pieces of which it consists.

It can be shown from the principles of statics (that is, the science of balanced forces), that in any body, system, or machine, that condition is fulfilled when *the sum of the products of the efforts into the velocities of their respective points of action is equal to the sum of the products of the resistances into the velocities of the points where they are overcome.*

Thus, let v be the velocity of a *driving point*, or point where an effort P is applied; v' the velocity of a *working point*, or point where a resistance R is overcome; the condition of uniform velocity for any body, system, or machine is

$$\Sigma \cdot P v = \Sigma \cdot R v' \text{..........................(1.)}$$

If there be only one driving point, or if the velocities of all the driving points be alike, then P being the total effort, the single product $P v$ may be put in in place of the sum $\Sigma \cdot P v$; reducing the above equation to

$$P v = \Sigma \cdot R v' \dots\dots\dots (2.)$$

Referring now to Article 305, let the machine be one in which the *comparative* or *proportionate* velocities of all the points at a given instant are known independently of their absolute velocities, from the construction of the machine; so that, for example, the velocity of the point where the resistance R is overcome bears to that of the driving point the ratio

$$\frac{v'}{v} = n;$$

then the condition of uniform speed may be thus expressed:—

$$P = \Sigma \cdot n R; \dots\dots\dots (3.)$$

that is, *the total effort is equal to the sum of the resistances reduced to the driving point.*

325. **Energy—Potential Energy.** (*A. M.*, 514, 517, 593, 660.)—*Energy* means *capacity for performing work*, and is expressed, like work, by the product of a force into a space.

The energy of an effort, sometimes called "*potential energy*" (to distinguish it from another form of energy to be afterwards referred to), is the *product of the effort into the distance through which it is capable of acting.* Thus, if a weight of 100 pounds be placed at an elevation of 20 feet above the ground, or above the lowest plane to which the circumstances of the case admit of its descending, that weight is said to possess potential energy to the amount of $100 \times 20 = 2{,}000$ *foot-pounds;* which means, that in descending from its actual elevation to the lowest point of its course, the weight is *capable of performing work* to that amount.

To take another example, let there be a reservoir containing 10,000,000 gallons of water, in such a position that the centre of gravity of the mass of water in the reservoir is 100 feet above the lowest point to which it can be made to descend while overcoming resistance. Then as a gallon of water weighs 10 lbs., the weight of the store of water is 100,000,000 lbs., which being multiplied by the height through which that weight is capable of acting, 100 feet, gives 10,000,000,000 foot-pounds for the potential energy of the weight of the store of water.

326. **Equality of Energy Exerted and Work Performed.**—When an effort actually does drive its point of application through a certain distance, energy to the amount of the product of the effort into that distance is said to be *exerted;* and the potential energy,

or energy which remains *capable of being exerted*, is to that amount diminished.

When the energy is exerted in driving a machine at an uniform speed, it is *equal to the work performed*.

To express this algebraically, let t denote the time during which the energy is exerted, v the velocity of a driving point at which an effort P is applied, s the distance through which it is driven, v' the velocity of any working point at which a resistance R is overcome, s' the distance through which it is driven; then

$$s = v\ t;\ s' = v'\ t;$$

and multiplying equation 1 of Article 324 by the time t, we obtain the following equation:—

$$\Sigma \cdot \mathrm{P}\, v\, t = \Sigma \cdot \mathrm{R}\, v'\, t = \Sigma \cdot \mathrm{P}\, s = \Sigma \cdot \mathrm{R}\, s'; \ldots\ldots\ldots (1.)$$

which expresses the equality of energy exerted, and work performed, for constant efforts and resistances.

When the efforts and resistances vary, it is sufficient to refer to Article 307, to show that the same principle is expressed as follows:—

$$\Sigma \int \mathrm{P}\, d\, s = \Sigma \int \mathrm{R}\, d\, s'; \ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

where the symbol $\int$ expresses the operation of finding the work performed against a varying resistance, or the energy exerted by a varying effort, as the case may be; and the symbol Σ expresses the operation of adding together the quantities of energy exerted, or work performed, as the case may be, at different points of the machine.

327. **Various Factors of Energy.**—A quantity of energy, like a quantity of work, may be computed by multiplying either a force into a distance, or a statical moment into an angular motion, or the intensity of a pressure into a volume. These processes have already been explained in detail in Articles 301 and 302, pages 340 to 341.

328. The **Energy Exerted in Producing Acceleration** (*A. M.*, 549) is equal to the work of acceleration, whose amount has been investigated in Articles 312 and 313, pages 354 to 357.

329. The **Accelerating Effort** (*A. M.*, 554) by which a given increase of velocity in a given mass is produced, and which is exerted by the *driving body* against the *driven body*, is equal and opposite to the resistance due to acceleration which the driven body exerts against the driving body, and whose amount has been given in Articles 312 and 313. Referring, therefore, to equations 4 and 8 of Article 312, we find the two following expressions, the first of which gives the accelerating effort required to produce a given

acceleration $d\,v$ in a body whose weight is W, when the *time* $d\,t$ in which that acceleration is to be produced is given, and the second, the same accelerating effort, when the *distance* $d\,s = v\,d\,t$ in which the acceleration is to be produced is given :—

$$\mathrm{P} = \frac{\mathrm{W}}{g} \cdot \frac{d\,v}{d\,t} \text{.........................(1.)}$$

$$= \frac{\mathrm{W}}{g} \cdot \frac{v\,d\,v}{d\,s} = \frac{\mathrm{W}}{g} \cdot \frac{d\,(v^2)}{2\,d\,s} \text{.................(2.)}$$

Referring next to Article 313, page 357, we find from equations 5, 6, and 7, that the work of acceleration corresponding to an increase $d\,a$ in the angular velocity of a rotating body whose moment of inertia is I, is

$$\frac{\mathrm{I} \cdot d\,(a^2)}{2\,g} = \frac{\mathrm{I}\,a\,d\,a}{g}.$$

Let $d\,t$ be the *time*, and $d\,i = a\,d\,t$ the *angular motion* in which that acceleration is to be produced; let P be the accelerating effort, and l its *leverage*, or the perpendicular distance of its line of action from the axis; then, according as the time $d\,t$, or the angle $d\,i$, is given, we have the two following expressions for the *accelerating couple* :—

$$\mathrm{P}\,l = \frac{\mathrm{I}}{g} \cdot \frac{d\,a}{d\,t} \text{.........................(3.)}$$

$$= \frac{\mathrm{I}}{g} \cdot \frac{a\,d\,a}{d\,i} = \frac{\mathrm{I}}{g} \cdot \frac{d\,(a^2)}{2\,d\,i} \text{....................(4.)}$$

Lastly, referring to Article 315, page 362, equation 2, we find, that if a train of mechanism consists of various parts, and if W be the weight of any one of those parts, whose velocity v' bears to that of the driving point v the ratio $\frac{v'}{v} = n$, then the accelerating effort which must be applied to the driving point, in order that, during the interval $d\,t$, in which the driving point moves through the distance $d\,s = v\,d\,t$, that point may undergo the acceleration $d\,v$, and each weight W the corresponding acceleration $n\,d\,v$, is given by one or other of the two formulæ—

$$\mathrm{P} = \frac{\Sigma\,n^2\,\mathrm{W}}{g} \cdot \frac{d\,v}{d\,t} \text{.......................(5.)}$$

$$= \frac{\Sigma\,n^2\,\mathrm{W}}{g} \cdot \frac{v\,d\,v}{d\,s} = \frac{\Sigma\,n^2\,\mathrm{W}}{g} \cdot \frac{d\,(v^2)}{2\,d\,s} \text{............(6.)}$$

330. **Work During Retardation—Energy Stored and Restored.** (*A. M.*, 528, 549, 550.)—In order to cause a given retardation, or diminution of the velocity of a given body, in a given time, or while it traverses a given distance, resistance must be opposed to its motion equal to the effort which would be required to produce in the same time, or in the same distance, an acceleration equal to the retardation.

A moving body, therefore, while being retarded, *overcomes resistance* and *performs work;* and that work is equal to the energy exerted in producing an acceleration of the same body equal to the retardation.

It is for this reason that it has been stated, in Article 312, that the work performed in accelerating the speed of the moving pieces of a machine is not necessarily lost; for those moving pieces, by returning to their original speed, are capable of performing an equal amount of work in overcoming resistance; so that the performance of such work is not prevented, but only deferred. Hence energy exerted in acceleration is said to be *stored;* and when by a subsequent and equal retardation an equal amount of work is performed, that energy is said to be *restored.*

The algebraical expressions for the relations between a retarding resistance, and the retardation which it produces in a given body by acting during a given time or through a given space, are obtained from the equations of Article 329 simply by putting R, the symbol for a resistance, instead of P, the symbol for an effort, and $-d\,v$, the symbol for a retardation, instead of $d\,v$, the symbol for an acceleration.

331. The **Actual Energy** (*A. M.*, 547, 589) of a moving body is the work which it is capable of performing against a retarding resistance before being brought to rest, and is equal to the energy which must be exerted on the body to bring it from a state of rest to its actual velocity. The value of that quantity is the *product of the weight of the body into the height from which it must fall to acquire its actual velocity;* that is to say,

$$\frac{\mathrm{W}\,v^2}{2\,g} \dots\dots\dots\dots (1.)$$

The total actual energy of a system of bodies, each moving with its own velocity, is denoted by

$$\frac{\Sigma \cdot \mathrm{W}\,v^2}{2\,g}; \dots\dots\dots\dots (2.)$$

and when those bodies are the pieces of a machine, whose velocities bear definite ratios (any one of which is denoted by n) to the velocity of the driving point v, their total actual energy is

$$\frac{v^2}{2g} \cdot \Sigma n^2 W, \text{......................... (3.)}$$

being *the product of the reduced inertia* (or co-efficient of steadiness, as Mr. Moseley calls it) *into the height due to the velocity of the driving point.*

The actual energy of a rotating body whose angular velocity is a, and moment of inertia $\Sigma W r^2 = I$, is

$$\frac{a^2 I}{2g}; \text{............................... (4.)}$$

that is, *the product of the moment of inertia into the height due to the velocity,* a, *of a point, whose distance from the axis of rotation is unity.*

When a given amount of energy is alternately stored and restored by alternate increase and diminution in the speed of a machine, the actual energy of the machine is alternately increased and diminished by that amount.

Actual energy, like motion, is *relative* only. That is to say, in computing the actual energy of a body, which is the capacity it possesses of performing work upon certain other bodies by *reason of its motion,* it is the motion *relatively to those other bodies* that is to be taken into account.

For example, if it be wished to determine how many turns a wheel of a locomotive engine, rotating with a given velocity, would make, before being stopped *by the friction of its bearings only,* supposing it lifted out of contact with the rails,—the actual energy of that wheel is to be taken *relatively to the frame of the engine* to which those bearings are fixed, and is simply the actual energy due to the rotation. But if the wheel be supposed to be detached from the engine, and it is inquired *how high it will ascend up a perfectly smooth inclined plane before being stopped by the attraction of the earth,* then its actual energy is to be taken *relatively to the earth;* that is to say, to the energy of rotation already mentioned, is to be added the energy due to the *translation* or forward motion of the wheel along with its axis.

332. **A Reciprocating Force** (*A. M.*, 556) is a force which acts alternately as an effort and as an equal and opposite resistance, according to the direction of motion of the body. Such a force is the weight of a moving piece whose centre of gravity alternately rises and falls; and such is the elasticity of a perfectly elastic body.

The work which a body performs in moving against a reciprocating force is employed in increasing its own potential energy, and is not lost by the body; so that by the motion of a body alternately against and with a reciprocating force, energy is *stored* and *restored*, as well as by alternate acceleration and retardation.

Let Σ W denote the weight of the whole of the moving pieces of any machine, and h a height through which the common centre of gravity of them all is alternately raised and lowered. Then the quantity of energy—

$$h \Sigma W,$$

is stored while the centre of gravity is rising, and restored while it is falling.

These principles are illustrated by the action of the plungers of a single acting pumping steam engine. The weight of those plungers acts as a resistance while they are being lifted by the pressure of the steam on the piston; and the same weight acts as effort when the plungers descend and drive before them the water with which the pump barrels have been filled. Thus, the energy exerted by the steam on the piston is stored during the up-stroke of the plungers; and during their down-stroke the same amount of energy is restored, and employed in performing the work of raising water and overcoming its friction.

333. **Periodical Motion.** (*A. M.*, 553.)—If a body moves in such a manner that it periodically returns to its original velocity, then at the end of each period, the entire variation of its actual energy is nothing; and if, during any part of the period of motion, energy has been stored by acceleration of the body, the same quantity of energy exactly must have been during another part of the period restored by retardation of the body.

If the body also returns in the course of the same period to the same position relatively to all bodies which exert reciprocating forces on it—for example, if it returns periodically to the same elevation relatively to the earth's surface—any quantity of energy which has been stored during one part of the period by moving against reciprocating forces must have been exactly restored during another part of the period.

Hence *at the end of each period, the equality of energy and work, and the balance of* mean effort *and* mean resistance, *holds with respect to the driving effort and the resistances, exactly as if the speed were uniform and the reciprocating forces null;* and all the equations of Articles 324 and 326 are applicable to periodic motion, provided that in the equations of Article 324, and equation 1 of Article 326, P, R, and v are held to denote the *mean values* of the efforts, resistances, and velocities,—that s and s' are held to denote spaces moved through in one or more *entire periods*,—and that in equation 2 of Article 326, the integrations denoted by $\int$ be held to extend to one or more *entire periods.*

These principles are illustrated by the steam engine. The velocities of its moving parts are continually varying, and those of some of them, such as the piston, are periodically reversed in direction. But at the end of each period, called a *revolution*, or *double-stroke*, every part returns to its original position and velocity; so that the *equality of energy and work*, and the *equality of the mean effort to the mean resistance reduced to the driving point*,—that is, the equality of the mean effective pressure of the steam on the piston to the mean total resistance reduced to the piston—hold for one or any whole number of *complete revolutions*, exactly as for uniform speed.

It thus appears that (as stated at the commencement of this Part) there are two fundamentally different ways of considering a periodically moving machine, each of which must be employed in succession, in order to obtain a complete knowledge of its working.

"I. In the first place is considered the action of the machine during one or more whole periods, with a view to the determination of the relation between the mean resistances and mean efforts, and of the EFFICIENCY; that is, the ratio which the *useful* part of its work bears to the whole expenditure of energy. The motion of every ordinary machine is either uniform or periodical.

"II. In the second place is to be considered the action of the machine during intervals of time less than its period, in order to determine the law of the periodic changes in the motions of the pieces of which the machine consists, and of the periodic or reciprocating forces by which such changes are produced."

334. **Starting and Stopping.** (*A. M.*, 691.)—The *starting* of a machine consists in setting it in motion from a state of rest, and bringing it up to its proper mean velocity. This operation requires the exertion, besides the energy required to overcome the mean resistance, of an additional quantity of energy equal to the actual energy of the machine when moving with its mean velocity, as found according to the principles of Article 331, page 373.

If, in order to *stop* a machine, the effort of the prime mover is simply suspended, the machine will continue to go until work has been performed in overcoming resistances equal to the actual energy due to the speed of the machine at the time of suspending the effort of the prime mover.

In order to diminish the time required by this operation, the resistance may be increased by means of the friction of a *brake*. Brakes will be further described in the sequel.

335. The **Efficiency** of a machine is a fraction expressing the ratio

of the useful work to the whole work, which is equal to the energy expended. The COUNTER-EFFICIENCY is the reciprocal of the efficiency, and is the ratio in which the energy expended is greater than the useful work. The object of improvements in machines is to bring their efficiency and counter-efficiency as near to unity as possible.

As to useful and lost work, see Article 308. The algebraical expression of the efficiency of a machine having uniform or periodical motion, is obtained by introducing the distinction between useful and lost work into the equations of the conservation of energy. Thus, let P denote the mean effort at the driving point; s, the space described by it in a given interval of time, being a whole number of periods of revolutions; R_1, the mean useful resistance; s_1, the space through which it is overcome in the same interval; R_2, any one of the wasteful resistances; s_2, the space through which it is overcome; then

$$P\,s = R_1\,s_1 + \Sigma \cdot R_2\,s_2; \qquad (1.)$$

and the efficiency of the machine is expressed by

$$\frac{R_1\,s_1}{P\,s} = \frac{R_1\,s_1}{R_1\,s_1 + \Sigma \cdot R_2\,s_2} \qquad (2.)$$

In many cases the lost work of a machine, $R_2\,s_2$, consists of a constant part, and of a part bearing to the useful work a proportion depending in some definite manner on the sizes, figures, arrangement, and connection of the pieces of the train, on which also depends the constant part of the lost work. In such cases the whole energy expended and the efficiency of the machine are expressed by the equations

$$\left.\begin{array}{l} P\,s = (1 + A)\,R_1\,s_1 + B; \\ \dfrac{R_1\,s_1}{P\,s} = \dfrac{1}{1 + A + \dfrac{B}{R_1\,s_1}} \end{array}\right\} \qquad (3.)$$

and the first of these is the mathematical expression of what Mr. Moseley calls the "modulus" of a machine.

The useful work of an intermediate piece in a train of mechanism consists in driving the piece which follows it, and is less than the energy exerted upon it by the amount of the work lost in overcoming its own friction. Hence the efficiency of such an intermediate piece is the ratio of the work performed by it in driving the following piece, to the energy exerted on it by the preceding piece; and it is evident that *the efficiency of a machine is the product of the efficiencies of the series of moving pieces which transmit energy from the driving point to the working point.* The same principle applies to a train of *successive machines*, each driving that which follows it; and to counter-efficiency as well as to efficiency.

336. **Power and Effect—Horse-Power.**—The *power* of a machine is the energy exerted, and the *effect*, the useful work performed, in some interval of time of definite length, such as a second, a minute, an hour, or a day.

The unit of power called conventionally a *horse-power*, is 550 foot-pounds per second, or 33,000 foot-pounds per minute, or 1,980,000 foot-pounds per hour. The effect is equal to the power multiplied by the efficiency; and the power is equal to the effect multiplied by the counter-efficiency. The loss of power is the difference between the effect and the power. As to the French "Force de Cheval," see Article 299, page 339. It is equal to 0·9863 of a British horse-power; and a British horse-power is 1·0139 force de cheval.

337. **General Equation.**—The following general equation presents at one view the principles of the action of machines, whether moving uniformly, periodically, or otherwise:—

$$\int P\, d\, s = \Sigma \int R\, d\, s' \pm h \Sigma W + \Sigma \cdot \frac{W(v_2^2 - v_1^2)}{2\, g};$$

where W is the weight of any moving piece of the machine;

h, when positive, the elevation, and when negative, the depression, which the common centre of gravity of all the moving pieces undergoes in the interval of time under consideration; v_1 the velocity at the beginning, and v_2 the velocity at the end, of the interval in question, with which a given particle of the machine of the weight W is moving;

g, the acceleration which gravity causes in a second, or 32·2 feet per second, or 9·81 mètres per second.

$\int R\, d\, s'$, the work performed in overcoming any resistance during the interval in question;

$\int P\, d\, s$, the energy exerted during the interval in question.

The second and third terms of the right-hand side, when positive, are *energy stored;* when negative, *energy restored.*

The principle represented by the equation is expressed in words as follows:—

The energy exerted, added to the energy restored, is equal to the energy stored added to the work performed.

338. **The Principle of Virtual Velocities**, when applied to the uniform motion of a machine, is expressed by equation 3 of Article 324, already given in page 369; or in words as follows:—*The effort is equal to the sum of the resistances reduced to the driving point;*

that is, each multiplied by the ratio of the velocity of its working point to the velocity of the driving point. The same principle, when applied to reciprocating forces and to re-actions due to varying speed, as well as to passive resistances, is expressed by means of a modified form of the general equation of Article 337, obtained in the following manner:—Let n denote either the ratio borne at a given instant by the velocity of a given working point, where the resistance R is overcome, to the velocity of the driving point, or the mean value of that ratio during a given interval of time; let n'' denote the corresponding ratio for the vertical ascent or descent (according as it is positive or negative) of a moving piece whose weight is W; let n' denote the corresponding ratio for the mean velocity of a mass whose weight is W, undergoing acceleration or retardation, and $\frac{d v'}{d t}$ either the rate of acceleration of that mass, if the calculation relates to an instant, or the mean value of that rate, if to a finite interval of time. Then the effort at the instant, or the mean effort during the given interval, as the case may be, is given by the following equation:—

$$P = \Sigma \cdot n \, R + \Sigma \cdot n'' \, W + \Sigma \cdot \frac{n' \, W \, d v'}{g \, d t}.$$

If the ratio n', which the velocity of the mass W bears to that of the driving point, is constant, we may put $\frac{d v'}{d t} = \frac{n' \, d v}{d t}$, where $\frac{d v}{d t}$ denotes the rate of acceleration of the driving point; and then the third term of the foregoing expression becomes $\frac{d v}{g \, d t} \Sigma \cdot n'^2 \, W$, as in formula 2 of Article 316, page 363.

339. **Forces in the Mechanical Powers, neglecting Friction—Purchase.**—The mechanical powers, considered as means of modifying motion only, have been considered in Articles 221 to 224, pages 231 to 234. When friction is neglected, any one of the mechanical powers may be regarded as *an uniformly-moving simple machine, in which one effort balances one resistance;* and in which, consequently, according to the principle of virtual velocities, or of the equality of energy exerted and work done, *the effort and resistance are to each other inversely as the velocities along their lines of action of the points where they are applied.*

In the older writings on mechanics, the effort is called the *power*, and the resistance the *weight;* but it is desirable to avoid the use of the word "power" in this sense, because of its being very commonly used in a different sense—viz., the rate at which energy is exerted by a prime mover; and the substitution of "resistance" for "weight" is made in order to express the fact that the principle just stated applies to the overcoming of all sorts of resistance, and not to the lifting of weights only.

The weight of the moving piece itself in a mechanical power may either be wholly supported at the bearing, if the piece is balanced; or if not, it is to be regarded as divided into two parallel components, one supported directly at the bearing, and the other being included in the effort or in the resistance, as the case may be.

The relation between the effort and the resistance in any mechanical power may be deduced from the principles of statics; viz.:—In the case of the LEVER (including the *wheel and axle*), from the balance of couples of equal and opposite moments; in the case of the INCLINED PLANE (including the *wedge* and the *screw*), from the parallelogram of forces; and in the case of the pulley, from the composition of parallel forces. The principle of virtual velocities, however, is more convenient in calculation.

The *total load* in a mechanical power is the resultant of the effort, the resistance, the lateral components of the forces acting at the driving and working points, and the weight directly carried at the bearings; and it is equal and directly opposed to the re-action of the bearings or supports of the machine.

By the *purchase* of a mechanical power is to be understood the ratio borne by the resistance to the effort, which is equal to the ratio borne by the velocity of the driving point to that of the working point. This term has already been explained in connection with the pulley, in Article 201, pages 215, 216.

The following are the results of the principle of virtual velocities, as applied to determine the purchase in the several mechanical powers:—

I. LEVER.—The effort and resistance are to each other in the inverse ratio of the perpendicular distances of their lines of action from the axis of rotation or fulcrum; so that the *purchase* is the ratio which the perpendicular distance of the effort from the axis bears to the perpendicular distance of the resistance from the axis.

Under the head of the lever may be comprehended all turning or rocking primary pieces in mechanism which are connected with their drivers and followers by linkwork.

II. WHEEL AND AXLE.—The purchase is the same as in the case of the lever; and the perpendicular distances of the lines of action of the effort and of the resistance from the axis are the radii of the pitch-circles of the wheel and of the axle respectively.

Under the head of the wheel and axle may be comprehended all turning or rocking primary pieces in mechanism which are connected with their drivers and followers by means of rolling contact, of teeth, or of bands. By the "wheel" is to be understood the pitch-cylinder of that part of the piece which is driven; and by the "axle," the pitch-cylinder of that part of the piece which drives.

III. INCLINED PLANE, and IV. WEDGE.—Here the purchase, or ratio of the resistance to the effort, is the ratio borne by the whole velocity of the sliding body (represented by B C in fig. 165, page 233, and C *c* in fig. 166, page 234) to that component of the velocity (represented by B D in fig. 165, page 233, and C *e* in fig. 166, page 234) which is directly opposed to the resistance: it being understood that the effort is exerted in the direction of motion of the sliding body.

The term *inclined plane* may be used when the resistance to the motion of a body that slides along a guiding surface consists of its own weight, or of a force applied to a point in it by means of a link; and the term *wedge*, when that resistance consists of a pressure applied to a plane surface of the moving body, oblique to its direction of motion.

V. SCREW.—Let the resistance (R) to the motion of a screw be a force acting along its axis, and directly opposed to its advance; and let the effort (P) which drives the screw be applied to a point rigidly attached to the screw, and at the distance r from the axis, and be exerted in the direction of motion of that point. Then, while the screw makes one revolution, the working point advances against the resistance through a distance equal to the pitch (p); and at the same time the driving point moves in its helical path through the distance $\sqrt{(4 \pi^2 r^2 + p^2)}$; therefore the purchase of the screw, neglecting friction, is expressed as follows:—

$$\frac{\mathrm{R}}{\mathrm{P}} = \frac{\sqrt{4 \pi^2 r^2 + p^2}}{p}$$

$$= \frac{\text{length of one coil of path of driving point}}{\text{pitch}}.$$

VI. PULLEY. (See Articles 200 and 201, pages 214 to 216.)—In the pulley without friction, the purchase is the ratio borne by the resistance which opposes the advance of the running block to the effort exerted on the hauling part of the rope; and it is expressed by the number of plies of rope by which the running block is connected with the fixed block.

VII. The HYDRAULIC PRESS, when friction is neglected, may be included amongst the mechanical powers, agreeably to the definition of them given at the beginning of this Article. By the resistance is to be understood the force which opposes the outward motion of the press-plunger, A, fig. 159, page 224; and by the effort, the force which drives inward the pump-plunger, A′. The intensity of the pressure exerted between each of the two plungers and the fluid is the same; therefore the amount of the pressure exerted between each plunger and the fluid is proportional to the area of that plunger; so that the purchase of the hydraulic press is expressed as follows:—

$$\frac{R}{P} = \frac{A}{A'} = \frac{\text{transverse area of press-plunger}}{\text{transverse area of pump-plunger}};$$

and this is the reciprocal of the ratio of the velocities of those plungers, as already shown in Article 209, page 223.

The purchase of a train of mechanical powers is the product of the purchases of the several elementary parts of that train.

The object of producing a purchase expressed by a number greater than unity is, to enable a resistance to be overcome by means of an effort smaller than itself, but acting through a greater distance; and the use of such a purchase is found chiefly in machines driven by muscular power, because of the effort being limited in amount.

SECTION IV.—*Of Dynamometers.*

340. **Dynamometers** are instruments for measuring and recording the energy exerted and work performed by machines. They may be classed as follows:—

I. Instruments which merely *indicate the force* exerted between a driving body and a driven body, leaving the *distance* through which that force is exerted to be observed independently. The following are examples of this class:—

a. The weight of a solid body may be so suspended as to balance the resistance, as in Scott Russell's experiments on the resistance of boats. (*Edin. Trans.*, xiv.)

b. The weight of a column of liquid may be employed to balance and measure the effort required to drag a carriage or other body, as in Milne's mercurial dynamometer.

c. The available energy of a prime mover may be wholly expended in overcoming friction, which is measured by a weight, as in Prony's dynamometer (described further on).

d. A spring balance may be interposed between a prime mover and a body whose resistance it overcomes.

II. Instruments which *record* at once the *force*, *motion*, and *work* of a machine, by drawing a line, straight or curved, as the case may be, whose abscissæ represent the distances moved through, its ordinates the resistances overcome, and its area the work performed (as in fig. 241, page 346).

A dynamometer of this class consists essentially of two principal parts: a spring whose deflection indicates the force exerted between a driving body and a driven body; and a band of paper, or a card, moving at right angles to the direction of deflection of the spring with a velocity bearing a known constant proportion to the velocity with which the resistance is overcome. The spring carries a pen or pencil, which marks on the paper or card the required line. The following are examples of this class of instruments:—

a. Morin's Traction Dynamometer.

b. Morin's and Hirn's Rotatory Dynamometers.

c. The Steam Engine Indicator.

III. Instruments called *Integrating Dynamometers*, which record the work performed, but not the resistance and motion separately.

341. **Prony's Friction Dynamometer** measures the useful work performed by a prime mover, by causing the whole of that work to be expended in overcoming the friction of a brake. In fig. 249, A represents a cylindrical drum, driven by the prime mover. The block D, attached to the lever B C, and the smaller blocks with which the chain E is shod, form a brake which embraces the drum, and which is tightened by means of the screws F, F, until its friction is sufficient to cause the drum to rotate at an uniform speed. The end C of the lever carries a scale G, in which weights are placed to an amount just sufficient to balance the friction, and keep the lever horizontal. The lever ought to be so loaded at B that when there are no weights in the scale, it shall be balanced upon the axis. The lever is prevented from deviating to any inconvenient extent from a horizontal position by means of safety-stops or guards, H, K.

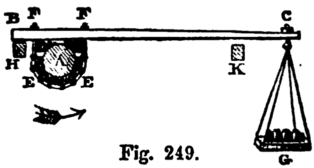

Fig. 249.

The weight of the load in the scale which balances the friction being multiplied into the horizontal distance of the point of suspension C from the axis, gives the *moment of friction*, which being multiplied into the angular velocity of the drum, gives the *rate of useful work* or *effective power* of the prime mover.

As the whole of that power is expended in overcoming the friction between the drum and the brake, the heat produced is in general considerable; and a stream of water must be directed on the rubbing surfaces to abstract that heat.

The friction dynamometer is simple and easily made; but it is ill adapted to measure a variable effort; and it requires that when the power of a prime mover is measured, its ordinary work should be interrupted, which is inconvenient and sometimes impracticable.

342. **Morin's Traction Dynamometer.**—The descriptions of this and some other dynamometers invented by General Morin are abridged from his works, entitled *Sur quelques Appareils dynamométriques* and *Notions fondamentales de Mécanique.*

Fig. 250 is a plan and fig. 250 *a* an elevation of a dynamometer for recording by a diagram the work of dragging a load horizontally. *a a*, *b b* are a pair of steel springs, through which the tractive force is transmitted, and which serve by their deflection to measure

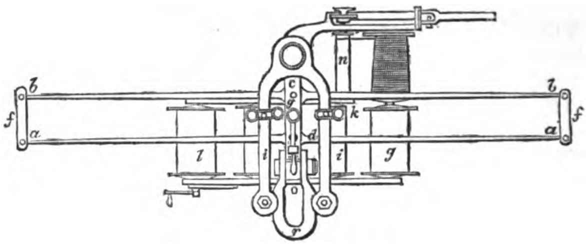

Fig. 250.

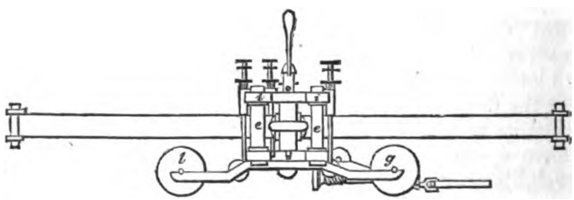

Fig. 250 A.

that force. They are connected together at the ends by the steel links *f*, *f*. The effort of the prime mover is applied, through the link *r*, to the gland *d*, which is fixed on the middle of the foremost spring; the equal and opposite resistance of the vehicle is applied to the gland *c*, which is fixed on the middle of the aftermost spring. When no tractive force is exerted, the inward faces of the springs are straight and parallel; when a force is exerted, the springs are bent, and are drawn apart, through a distance proportional to the force. The springs are protected against being bent so far as to injure them by means of the safety bridles *i*, *i*, with their bolts *e*, *e*. Those bridles are carried by the after-gland, and their bolts serve to stop the foremost spring when it is drawn forward as far as is consistent with the preservation of elasticity and strength.

The frame of the apparatus for giving motion to the paper band is carried by the after-gland. The principal parts of that apparatus are the following:—

l, store drum on which the paper band is rolled, before the commencement of the experiment, and off which it is drawn as the experiment proceeds;

g, taking-up drum, to which one end of the paper band is glued, and which draws along and rolls up the paper band with a velocity proportional to that of the vehicle. Fixed on the axis of this drum is a fusee having a spiral groove round it, whose radius gradually increases at the same rate as that at which the effective radius of the drum *g* is increased during its motion by the rolling of successive coils of paper upon it. The object of this is to prevent that increase of the effective radius of the drum from accelerating the speed of the paper band;

n is a drum which receives through a train of wheelwork and endless screws a velocity proportional to that of the wheels of the vehicle, and which, by means of a cord, drives the fusee. The mechanism is usually so designed that the paper moves at one-fiftieth of the speed of the vehicle.

Between the drums *l* and *g* there are three small rollers to support the paper band and keep it steady.

One of the safety bridles carries a pencil, *k*, which, being at rest relatively to the frame of the recording apparatus, traces a straight line on the band of paper as the latter travels below the pencil. That line is called the *zero line*, and corresponds to O X in fig. 241, page 346.

An arm fixed to the forward gland carries another pencil, whose position is adjusted before the experiment, so that when there is no tractive force its point rests on the zero line. During the experiment, this pencil traces on the paper band a line such as E R G, fig. 241, whose ordinate or distance from any given point in the zero line is the deflection of the pair of springs, and proportional to the tractive force, at the corresponding point in the journey of the vehicle.

The areas of the diagrams drawn by this apparatus, representing quantities of work, may be found either by the method described in Article 289, page 331, or by an instrument for measuring the areas of plane figures, called the *Planimeter*, or *Platometer*, of which various forms have been invented by Ernst, Sang, Clerk Maxwell, Amstler, and others.

A third pencil, actuated by a clock, is sometimes caused to mark a series of dots on the paper band at equal intervals of time, and so to record the changes of velocity.

When one vehicle (such as a locomotive engine) drags one or more others, the apparatus may, if convenient, be turned hind side before, and carried by the foremost vehicle. In such a case the motion of the band of paper ought to be derived, not from a driving-wheel, which is liable to slip, but from a bearing-wheel.

When the apparatus is used to record the tractive force and work performed in towing a vessel, the apparatus for moving the paper band may be driven by means of a wheel or fan, acted upon by the water; in which case the ratio of the velocity of the band to that of the vessel should be determined by experiment.

Owing to the varieties which exist in the elasticity of steel, the relation between the deflections of the springs and the tractive forces can only be roughly calculated beforehand, and should be determined exactly by direct experiment—that is, by hanging known weights to the springs, and noting the deflections.

The best form of longitudinal section for each spring is that which gives the greatest flexibility for a given strength, and consists of two parabolas, having their vertices at the two ends of the spring, and meeting base to base in the middle; that is to say, the thickness of the spring at any given point of its length should be proportional to the square root of the distance of that point from the nearest end of the spring. To express this by a formula, let

c be the half-length of the spring;

h, the thickness in the middle;

x, the distance of any point in the spring from the end nearest to it;

h', the thickness at that point; then

$$h' = h \cdot \sqrt{\frac{x}{c}} \text{ .... (1.)}$$

The breadth of each spring should be uniform, and, according to General Morin, should not exceed from $1\frac{1}{2}$ to 2 inches. Let it be denoted by b.

The following is the formula for calculating beforehand the *probable* joint deflection of a given pair of springs under a given tractive force:—

Let the dimensions c, h, b be stated in inches, and the force P in pounds.

Let y denote the deflection in inches.

Let E denote the *modulus of elasticity* of steel, in pounds on the square inch. Its value, for different specimens of steel, varies from 29,000,000 to 42,000,000, the smaller values being the most common. Then

$$y = \frac{8 \, \mathrm{P} \, c^3}{\mathrm{E} \, b \, h^3} \text{ .... (2.)}$$

The deflection should not be permitted to exceed about one-tenth part of the length of the springs.

343. **Morin's Rotatory Dynamometer** is represented in figs. 251, 251 *a*, and is designed to record the work performed by a prime mover in transmitting rotatory motion to any machine. A is a fast pulley, and C a loose pulley, on the same shaft. A belt transmits motion from the prime mover to one or other of those pulleys according as it is desired to transmit motion to the shaft or not.

A third pulley, B, on the same shaft, carries the belt which transmits motion to the machine to be driven. This pulley is also loose on the shaft to a certain extent, so that it is capable of mov-

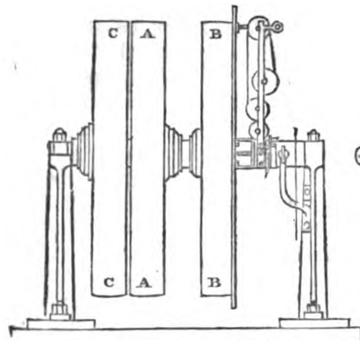

Fig. 251.

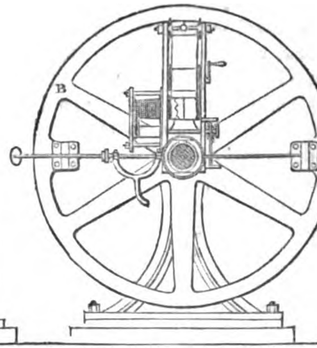

Fig. 251 A.

ing, relatively to the shaft, backwards and forwards through a small arc, sufficient to admit of the deflection of a steel spring by which motion is transmitted from the shaft to the pulley.

One end of that spring is fixed to the shaft, so that the spring projects from the shaft like an arm, and rotates along with it. The other end of the spring is connected with the pulley B near its circumference, and is the means of driving that pulley; so that the spring undergoes deflection proportional to the effort exerted by the shaft on the pulley.

A frame projecting radially like an arm from the shaft, and rotating along with it, carries an apparatus, similar to that used in the traction dynamometer, for making a band of paper move radially with respect to the shaft with a velocity proportional to the speed with which the shaft rotates. A pencil carried by this frame traces a zero line on the paper band; and another pencil carried by the end of the spring, traces a line whose ordinates represent the forces exerted, just as in the traction dynamometer.

The mechanism for moving the paper band is driven by a toothed ring surrounding the shaft, and kept at rest while the shaft rotates by means of a catch. When that catch is drawn back, the toothed ring is set free, rotates along with the shaft, and ceases to drive the mechanism; and thus the motion of the paper band can be stopped if necessary. (See page 446.)

344. In the **Torsion Dynamometer** (otherwise called "Pandynamometer") of M. G. A. Hirn, the torsion of the rotating shaft which transmits power is made the means of measuring and record-

ing, by a self-acting apparatus, the moment of the couple by which the shaft is driven, and the work done by that couple. Two trains of wheels, driven from the shaft at two different points, communicate rotations of equal speed in opposite directions about one axis to two bevel-wheels which gear with an intermediate bevel-wheel at opposite sides of its rim, forming a combination like that shown in fig. 176, page 245. The axis of the third wheel, corresponding to the train-arm A in fig. 176, indicates by its position one-half of the angle through which the shaft is twisted between the spur-wheels, and communicates its motion to the pencil of the recording apparatus; which pencil, as in other recording dynamometers, draws a line on a strip of paper that is moved at a speed proportional to the speed of the shaft that transmits the power. (See *Annales des Mines*, 1867, vol. xi.)

The only perfectly accurate way of determining the relation between the displacement of the pencil and the moment transmitted by the shaft, is to ascertain by direct experiment the twisting effect of a known couple when applied to the shaft. But a probable approximate value of that relation may be calculated as follows:—Let M be the twisting moment; x, the length of that part of the shaft whose angular torsion is to be determined; h, its diameter; C, the co-efficient of transverse elasticity of the material; θ, the angle of torsion, in circular measure; then,*

$$\frac{\mathrm{M}}{\theta} = \frac{\pi}{32} \cdot \frac{\mathrm{C}\, h^4}{x} = 0{\cdot}098\, \frac{\mathrm{C}\, h^4}{x} \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

Let n be the ratio which each of the contrary angular velocities of the bevel-wheels corresponding to B and C in fig. 176 bears to the angular velocity of the shaft, and y the length of an index corresponding to the train-arm, A, in that figure; then the angular displacement of that index is $\frac{n\,\theta}{2}$; and the linear displacement of its end (which may be denoted by z) is

$$z = \frac{n\,\theta\,y}{2};$$

therefore the following formula expresses the relation between the moment M, and the displacement z;

$$\frac{\mathrm{M}}{z} = \frac{2\,\mathrm{M}}{n\,y\,\theta} = \frac{\pi}{16} \cdot \frac{\mathrm{C}\, h^4}{n\,x\,y} = 0{\cdot}196\, \frac{\mathrm{C}\, h^4}{n\,x\,y} \ldots\ldots\ldots\ldots (2.)$$

Should the shaft be hollow, let h' be its internal diameter; then in each of the preceding formulæ for h^4 substitute $h^4 - h'^4$.

The following are values of the co-efficient C:—

* *Manual of Applied Mechanics*, Article 322, page 357.

Dimensions in..........................	Inches,....	Millimètres.
Forces in..............................	Lbs.,.....	Kilogrammes.
Cast Iron, about....................	2,850,000	2,000
Wrought Iron, from..............	8,500,000	6,000
" to..............	10,000,000	7,000
Steel, from...............	10,000,000	7,000
" to...............	12,000,000	8,400

Calculation may be used preliminary to the designing of the apparatus, in order to find approximately the extent of the displacement of the recording pencil; but the exact relation of that displacement to the twisting moment exerted through the shaft ought always to be determined by experiment.

345. **Elasticity of Spiral Springs.**—As spiral or helical springs are much used in dynamometric apparatus, it is convenient here to state the laws of their resistance to extension and compression.

In order that such a spring may be an accurate instrument for measuring forces—that is, in order that the proportion borne by the load acting on the spring to the extension or compression which it produces may be constant—the figure of the spring should be a true helix, as described in Article 58, page 36.

It is more favourable to accuracy to measure a force by the extension than by the compression of a spiral spring; because during extension it preserves almost exactly a truly helical form, and the coils remain in a cylindrical surface; whereas during compression the middle coils are apt to swerve sideways, so as to make the spring lose the proper figure. There are cases, however, in which the use of the compression of the spring is unavoidable; and then it is kept approximately in its proper figure by being enclosed in a cylindrical casing, which ought to be so large as not to impede the longitudinal motion of the spring.

The pair of equal and opposite forces by which a spiral spring is stretched should act exactly along the axis of the helix; for which purpose the ends of the spring should be made fast to a pair of strong and stiff arms, each of which should be perpendicular to the helix, and should lie along a radius of the cylinder on which the helix is described, so that the inner ends of the arms may be in the axis of the helix; and at those inner ends the forces to be measured should be applied. The best form of section for the wire of which the spring is made is circular; because the extension of the spring depends on the torsion of the wire; and the laws of torsion are known with greater precision for a circular form of section than for any other.

The following formulæ show the relations between the load and the extension or compression of the spring:—

Let r be the radius of the cylinder containing the helical centre line of the spiral spring, as measured from the axis to the centre of the wire; n, the number of coils of which the spring consists; d, the diameter of the wire; C, the co-efficient of rigidity or transverse elasticity of the material; f, the greatest safe shearing stress upon it; W, any load not exceeding the greatest safe load; v, the corresponding extension or compression; W_1, the greatest safe *steady* load; v_1, the greatest safe extension or compression; then

$$\frac{W}{v} = \frac{C\,d^4}{64\,n\,r^3};\ W_1 = \frac{0{\cdot}196\,f\,d^3}{r};\ v_1 = \frac{12{\cdot}566\,n\,f\,r^2}{C\,d}.$$

The greatest safe *sudden* load is $\frac{W_1}{2}$.

If the wire of which the spring is made is square, and of the dimensions $d \times d$, the load for a given deflection is greater than for a round wire of the diameter d, in the ratio of 281 to 196, or of 1·43 to 1, or of 10 to 7, nearly.

The values of the co-efficient, C, of transverse elasticity of steel and charcoal iron wire, in lbs. on the square inch, range between 10,500,000 and 12,000,000; and in kilogrammes on the square millimètre, from, 7,400 to 8,400, nearly.

By the greatest safe stress is to be understood the greatest stress which is certain not to impair the elasticity of the spring by frequent repetition; say 30,000 lbs. on the square inch.

The value of the ratio $\frac{W}{v}$ borne by the load to the extension ought to be ascertained by direct experiment for every spring that is used in dynamometers or indicators.

346. **Steam Engine Indicator.**—This instrument was invented by Watt, and has been improved by other inventors, especially M'Naught and Richards. Its object is to record, by means of a diagram, the intensity of the pressure exerted by steam against one of the faces of a piston at each point of the piston's motion, and so to afford the means of computing, according to the principles of Articles 302 and 307, first, the energy exerted by the steam in driving the piston during the forward stroke; secondly, the work lost by the piston in expelling the steam from the cylinder during the return stroke; and thirdly, the difference of those quantities, which is the *available* or *effective* energy exerted by the steam on the piston, and which, being multiplied by the number of strokes per minute and divided by 33,000 foot-pounds, gives the INDICATED HORSE-POWER.

The indicator in a common form is represented by fig. 252. A B is a cylindrical case. Its lower end, A, contains a small cylinder, fitted with a piston, which cylinder, by means of the screwed nozzle at its lower end, can be fixed in any convenient position on a tube communicating with that end of the engine-cylinder where the work of the steam is determined. The communication between the engine-cylinder and the indicator-cylinder can be opened and shut at will by means of the cock K. When it is open, the intensity of the pressure of the steam on the engine-piston and on the indicator-piston is the same, or nearly the same.

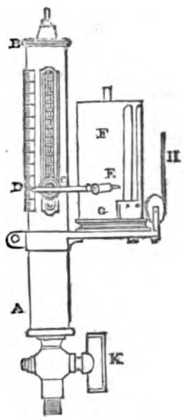

Fig. 252.

The upper end, B, of the cylindrical case contains a spiral spring, one end of which is attached to the piston, or to its rod, and the other to the top of the casing. The indicator-piston is pressed from below by the steam, and from above by the atmosphere. When the pressure of the steam is equal to that of the atmosphere, the spring retains its unstrained length, and the piston its original position. When the pressure of the steam exceeds that of the atmosphere, the piston is driven outwards, and the spring compressed; when the pressure of the steam is less than that of the atmosphere, the piston is driven inwards, and the spring extended. The compression or extension of the spring indicates the difference, upward or downward, between the pressure of the steam and that of the atmosphere.

A short arm, C, projecting from the indicator piston-rod carries at one side a pointer, D, which shows the pressure on a scale whose zero denotes the *pressure of the atmosphere*, and which is graduated into pounds on the square inch both upwards and downwards from that zero. At the other side the short arm has a longer arm jointed to it, carrying a pencil, E.

F is a brass drum, which rotates backward and forward about a vertical axis, and which, when about to be used, is covered with a piece of paper called a "card." It is alternately pulled round in one direction by the cord H, which wraps on the pulley G, and pulled back to its original position by a spring contained within itself. The cord H is to be connected with the mechanism of the steam engine in any convenient manner which shall ensure that the velocity of rotation of the drum shall at every instant bear a constant ratio to that of the steam engine piston: the back and forward motion of the surface of the drum representing that of the steam engine piston on a reduced scale. This having been done, and before opening the cock K, the pencil is to be placed in contact with the drum during a few strokes, when it will mark on the card a line which, when the card is afterwards spread out flat, becomes a straight line. This line, whose position indicates the pressure of the atmosphere, is called the *atmospheric line*. In fig. 253 it is represented by A A.

Then the cock K is opened, and the pencil, moving up and down with the variations of the pressure of the steam, traces on the card during each complete or double stroke a curve such as B C D E B. The ordinates drawn to that curve from any point in the atmospheric line, such as $\overline{HK}$ and $\overline{HG}$, indicate the differences between the pressure of the steam and the atmospheric pressure at the corresponding point of the motion of the piston. The ordinates of the part B C D E represent the pressures of the steam during the forward stroke, when it is driving the piston; those of the part E B represent the pressures of the steam when the piston is expelling it from the cylinder.

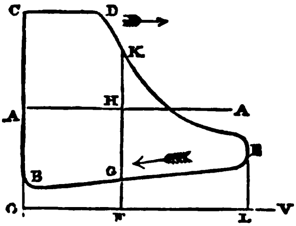

Fig. 258.

To found exact investigations on the indicator-diagrams of steam engines, the atmospheric pressure at the time of the experiment ought to be ascertained by means of a barometer; but this is generally omitted; in which case the atmospheric pressure may be assumed at its mean value, being 14·7 lbs. on the square inch, or 2116·3 lbs. on the square foot, at and near the level of the sea.

Let $\overline{AO} = \overline{HF}$ be ordinates representing the pressure of the atmosphere. Then O F V parallel to A A is the *absolute* or *true* zero line of the diagram, corresponding to *no pressure;* and ordinates drawn to the curve from that line represent the absolute intensities of the pressure of the steam. Let O B and L E be ordinates touching the ends of the diagram; then

$\overline{OL}$ represents the *volume* traversed by the piston at each single stroke (= s A, where s is the length of the stroke and A the area of the piston);

The area O B C D E L O represents the energy exerted by the steam on the piston during the forward stroke;

The area O B E L O represents the work lost in expelling the steam during the return stroke;

The area B C D E B, being the difference of the above areas, represents the *effective work* of the steam on the piston, during the complete stroke.

Those areas can be found by the Rules of Article 289, page 331; and the common trapezoidal rule, D, page 333, is in general sufficiently accurate. The number of intervals is usually ten, and of ordinates eleven.

The *mean forward pressure*, the *mean back pressure*, and the *mean effective pressure*, are found by dividing those three areas respectively by the volume s A, which is represented by $\overline{OL}$.

Those mean pressures, however, can be found by a direct process, without first measuring the areas, viz.:—having multiplied each ordinate, or breadth, of the area under consideration by the proper multiplier, divide the sum of the products by the sum of the multipliers, which process, when the common trapezoidal rule is used, takes the following form: add together the halves of the endmost ordinates, and the whole of the other ordinates, and divide by the number of intervals. That is, let b_0 be the first, b_n the last, and b_1, b_2, &c., the intermediate breadths; then let n be the number of intervals, and b_m the mean breadth; then

$$b_m = \frac{1}{n}\left(\frac{b_0 + b_n}{2} + b_1 + b_2 + \text{\&c.}\right); \quad\quad (1.)$$

and this represents the mean forward pressure, mean back pressure, or mean effective pressure, as the case may be. Let p_e be the mean effective pressure; then the effective energy exerted by the steam on the piston during each double stroke is the product of the mean effective pressure, the area of the piston, and the length of stroke, or

$$p_e \text{A } s; \quad\quad (2.)$$

and if N be the number of double strokes in a minute, the *indicated power in foot-pounds per minute*, in a single-acting engine, is

$$p_e \text{ A N } s; \quad\quad (3.)$$

from which the *indicated horse-power* is found by dividing by 33,000.

In a *double-acting engine* the steam acts alternately on either side of the piston; and to measure the power accurately, two indicators should be used at the same time, communicating respectively with the two ends of the cylinder. Thus a pair of diagrams will be obtained, one representing the action of the steam on each face of the piston. The mean effective pressure is to be found as above for each diagram separately, and then, if the areas of the two faces of the piston are sensibly equal, *the mean of those two results* is to be taken as the *general mean effective pressure;* which being multiplied by the area of the piston, the length of stroke, and *twice* the number of double strokes or revolutions in a minute, gives the indicated power per minute; that is to say, if p'' denotes the general mean effective pressure, the indicated power per minute is

$$p'' \text{A} \cdot 2 \text{ N } s; \quad\quad (4.)$$

If the two faces of the piston are sensibly of unequal areas (as in "trunk engines"), the indicated power is to be computed separately for each face, and the results added together.

If there are two or more cylinders, the quantities of power indicated by their respective diagrams are to be added together.

The re-actions of the moving parts of the indicator, combined with the elasticity of the spring, cause oscillations of its piston. In order that the errors thus produced in the indicated pressures at particular instants may be as small as possible, and may neutralize each other's effects on the whole indicated power, the moving masses ought to be as small as practicable, and the spring as stiff as is consistent with showing the pressures on a visible scale. In Richards's indicator this is effected by the help of a train of very light linkwork, which causes the pencil to show the movements of the spring on a magnified scale.

The *friction* of the moving parts of the indicator tends on the whole to make the indicated power and indicated mean effective pressure less than the truth, but to what extent is uncertain.

Every indicator should have the accuracy of the graduation of its scale of pressures frequently tested by comparison with a standard pressure gauge.

The indicator may obviously be used for measuring the energy exerted by any fluid, whether liquid or gaseous, in driving a piston; or the work performed by a pump, in lifting, propelling, or compressing any fluid.

347. **Integrating Dynamometers** record simply the work performed in dragging a vehicle or driving a machine, without recording separately the force and the motion. In that of Morin this is effected by means of a combination which has already been described in Article 270, page 311, and illustrated in fig. 221. In that figure (which see) A represents a plane circular disc, made to rotate with an angular velocity proportional to the speed of the motion of the vehicle or machine, and B a small wheel driven by the friction of the disc against its edge, and having its axis parallel to a radius of the disc. The wheel B, and some mechanism which it drives, are carried by a frame which is carried by a dynamometer spring, and so adjusted that the distance of the edge of B from the centre of A is equal to the deflection of the spring, and proportional to the effort.

The velocity of the edge of B at any instant being the product of its distance from the centre of A into the angular velocity of A, is proportional to the product of the effort into the velocity of the vehicle or machine—that is, to the *rate at which work is performed;* therefore the motion of the wheel B, in any interval of time, is *proportional to the work performed in that time;* and that work can be recorded by means of dial-plates, with indexes moved by a train of wheelwork driven by the wheel B.

In Moison's integrating dynamometer a ratchet-wheel is driven by the strokes of a click. (See Articles 194 to 196, pages 206 to 211.) The number of these strokes in a given time is proportional to the speed of the machine whose work is to be measured; and by means of a dynamometer-spring the length of each stroke of the click is adjusted so as to be proportional to the effort exerted at the time. The result is that the total extent of motion of the ratchet-wheel in a given time is proportional to the work performed. It is obvious that the frictional catch might be applied to this apparatus (Article 197, page 211).

348. **Measurement of Friction.**—Under the head of Dynamometers may be classed apparatus for the experimental measurement of friction.

If by means of any kind of dynamometer whose use does not involve the interruption of the performance of the ordinary work of a train of mechanism, we measure the power transmitted at two parts of that train, the difference will be the power expended in overcoming the friction of the intermediate parts. Hirn's Pandynamometer (Article 344, page 387) seems well adapted for experiments of this class. The power of a steam engine, as exerted in the cylinder, may be measured by means of the indicator, and the power transmitted to machinery which that engine drives, by a suitable dynamometer; and the difference will be power expended chiefly in overcoming the friction of the intermediate mechanism.

Special apparatus for measuring the friction of axles is used, not only for purposes of scientific investigation as to the co-efficients of friction of different pairs of surfaces in different states, but for practically testing the lubricating properties of oil and grease. Two forms of apparatus may be described.

I. *Statical Apparatus.*—A short cylindrical axle, of a convenient diameter (say 2, 3, or 4 inches), is supported at its ends by bearings on the top of a pair of strong fixed standards. The ends of the axle overhang their bearings, and carry a pair of equal and similar pulleys, by means of which it is driven at a speed equal, or nearly equal, to the greatest intended working speed of the axles with which the unguents to be tested are to be used in practice. The object of driving the axle at both ends is to ensure great steadiness of motion. The driving-gear ought to be capable of reversing the direction of rotation. At the middle of its length the axle is turned so as to form a very accurate and smooth journal, of a length equal to from 1½ to 2½ times its diameter. Upon that journal there hangs a plumber-block or axle-box, fitted with a suitable bush or bearing. That plumber-block is rigidly connected with a heavy mass of suitable material, such as cast iron, so as to form as it were a *pendulum* hanging from the journal in the middle of the axle, and of a weight suited to produce a pressure on the journal equal to the greatest pressure to which the unguent is to be exposed in practice (see Article 310, page 353). The pendulum is furnished with an index and graduated arc, to show its deviation from a vertical position.

The hanging plumber-block having been supplied with the unguent to be tested, the axle is to be driven at full speed, first in one direction, and then in the contrary direction, and the two contrary deviations of the pendulum observed. Let θ denote the *half-sum* of those deviations, expressed in circular measure to radius unity; c, the distance from the axis of rotation to the centre of gravity of the pendulum; r, the radius of the journal; let W be the weight of the pendulum; then the mean statical moment of the pendulum is

$$\mathrm{W}\, c \sin \theta = \mathrm{W}\, c\, \theta \text{ nearly};$$

and that moment balances the moment of friction (Article 311, page 356), whose value is f W r nearly, and will be afterwards shown to be exactly

$$\mathrm{W}\, r \sin \varphi,$$

φ being the angle of repose. Equating, therefore, those two equal moments, we find

$$r \sin \varphi = c \sin \theta; \text{ and}$$

$$\sin \varphi = f \text{ nearly} = \frac{c}{r} \sin \theta = \frac{c\, \theta}{r} \text{ nearly} \ldots\ldots\ldots\ldots (1.)$$

The distance, c, of the centre of gravity of the pendulum from the axis may be found experimentally, by applying a known weight at a known horizontal distance from the axis, so as to make the pendulum deviate, and observing the deviation. Let P be the weight so applied, x its leverage, Θ the deviation which it produces; then, if there were no friction, we should have

$$c = \frac{\mathrm{P}\, x}{\mathrm{W} \sin \Theta}.$$

In order to eliminate the effects of friction from the determination of c, the load P with the leverage x should be applied at contrary sides, so as to increase the deviation of the pendulum, while the axle is rotating in the two contrary directions.

Let $\sin \theta$ be the mean of the sines of the deviations produced by friction alone, and $\sin \Theta$ the mean of the sines of the deviations produced by the friction and the load P together; then we shall have

$$c = \frac{\mathrm{P}\, x}{\mathrm{W}\, (\sin \Theta - \sin \theta)}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

II. *Dynamic* or *Kinetic Apparatus.*—To measure the friction of an axle by means of its retarding effect upon a rotating mass, the axle is supported on suitable bearings at its ends, as in the Statical Apparatus just described; and at the middle of its length it has fitted on it, and accurately balanced, a round disc acting as a fly-wheel, of weight sufficient to produce the required pressure on the bearings. (See Article 310, page 353.) The numbers of turns made by the axle are counted and indicated by means of a light and easily-driven train of small wheels, with dial-plates and indexes.

The axle is provided with driving-gear of a kind which can be instantly disengaged when required; for example, a fast pulley on one overhanging end, with a loose pulley alongside of it, the loose pulley being carried, not by the fly-axle itself, but by a separate axle in the same straight line with the fly-axle.

After the axle with its fly-disc has been set in motion at a speed greater than the working speed of the axles to which the unguent to be tested is to be applied in practice, the driving-gear is to be disengaged; when the speed of rotation will undergo a gradual retardation through the friction of the journals. The numbers of turns made in a series of equal intervals of time (for example, intervals of thirty seconds, or of sixty seconds, or of a hundred seconds) are to be observed on the counting dials, and noted down.

Let W denote the weight of the whole rotating mass, consisting of the axle with its fly-disc; ρ, the radius of gyration of that mass. (See Article 313, page 357). Let t be the uniform length in seconds of the intervals of time during which the numbers of revolutions are recorded; and in one of those intervals let the disc make n revolutions, and in the next interval n' revolutions. Then the mean angular velocity is, during the first interval, $\frac{2 \pi n}{t}$, and during the second interval, $\frac{2 \pi n'}{t}$; and treating the rate of retardation as sensibly uniform, the retardation which takes place during the t seconds which elapse from the middle of the first interval to the middle of the second interval is

$$\frac{2 \pi (n - n')}{t};$$

and to produce that retardation in the course of t seconds in a body whose moment of inertia is $W \rho^2$, there is required a retarding moment of the following value:—

$$M = \frac{2 \pi (n - n') W \rho^2}{g t^2} \text{ .... (1.)}$$

Part of the retarding moment is due to the resistance of the air; but if the fly is a smooth round disc without arms, this may be neglected for the purpose of the experiments, and the whole moment treated as due to axle-friction. Let r be the radius of the journals, and f the co-efficient of friction: then, as before, the moment of friction is very nearly $f\,W\,r$; and by equating this to the retarding moment, and dividing both sides of the equation by $W\,r$, we obtain the following formula for the co-efficient of friction:—

$$f = \frac{2\,\pi\,(n - n')\,\rho^2}{g\,t^2\,r}. \qquad \text{(2.)}$$

When the numbers of revolutions have been observed during a series of more than two equal intervals of time, the formula 2 for the co-efficient of friction is to be applied to each consecutive pair of intervals, and a mean of the results taken.

The radius of gyration ρ and the radius of the journals r should of course be expressed in the same units of measure. In British measures, feet are the most convenient for the present purpose. The constant factor has the following values:—

$$\frac{2\,\pi}{g} = \frac{1}{5{\cdot}125 \text{ feet}} = \frac{1}{1{\cdot}56 \text{ mètre}}. \qquad \text{(3.)}$$

Similar experiments may be made with a disc rotating about a vertical axis, and supported by a pivot; regard being had to the value of the moment of friction of a pivot, as stated in Article 311, page 356.

To find the square ρ^2 of the radius of gyration by experiment, fix a pair of slender pins in the two faces of the disc at two points opposite each other, and near its circumference; hang up the disc with its axle by these pins, and make it swing like a pendulum in a plane perpendicular to its axis; count the number of single swings in some convenient interval of time; calculate their number per second, and let N denote that number. Then calculate the length L of the equivalent simple pendulum, by the following formula:—

$$L = \frac{g}{\pi^2\,N^2}. \qquad \text{(4.)}$$

The constant factor of this expression, being the length of the seconds pendulum, has approximately the following values:—

$$\frac{g}{\pi^2} = 3{\cdot}26 \text{ feet} = 0{\cdot}992 \text{ mètre}. \qquad \text{(5.)}$$

Let C be the distance from the point of suspension to the axis of figure of the disc and axle; then the square of the radius of gyration is calculated as follows:—

$$\rho^2 = C\,(L - C). \qquad \text{(6.)}$$

When the object of the experiments is not to obtain absolute values of the co-efficient of friction, but merely to compare one specimen of unguent with another, it is sufficient to compare together the rates of retardation with the two unguents in equal intervals of time.

III. *Comparison of Heating Effects.*—For the same purpose of comparing unguents with each other, without measuring the friction absolutely, the heating effects of the friction with different unguents are sometimes compared together. The apparatus used is similar to that described under the head of (I.) Static Apparatus; except that there is no reversing-gear, and that the pendulum, or loaded plumber-block, has no index nor graduated arc, and is provided with a thermometer, having its bulb immersed in the passage through which the unguent flows from the grease-box to the journal. Another thermometer, hung on the wall of the room, shows the temperature of the air. The axle is driven at its proper speed, until the temperature shown by the first-mentioned thermometer ceases to rise; and then the elevation of that temperature above the temperature of the air is noted. (See Article 310, page 353.)

In all experiments for the purpose of comparing unguents with each other, care should be taken to remove one sort of unguent completely from the rubbing surfaces, grease-box, and passages, before beginning to test the effect of another sort, lest the mixture of different sorts of unguents should make the experiments inconclusive.

ADDENDUM TO ARTICLE 309, PAGE 348.

Friction of Pistons and Plungers.—From experiments made by Mr. William More and others, it appears that the friction of ordinary pistons and plungers may be estimated at about one-tenth of the amount of the effective pressure exerted by the fluid on the piston.

The friction of a plunger working through a cupped leather collar is equal to the pressure of the fluid upon a ring equal in circumference to the collar, and of a breadth which, according to Mr. More's experiments, is about 0·4 of the depth of bearing-surface of the collar; and according to the experiments of Messrs. Hick and Luthy, from ·01 to ·015 inch (= from ·25 to ·375 millimètres), according to the state of lubrication of the collar.

CHAPTER III.

OF REGULATING APPARATUS.

349. **Regulating Apparatus Classed—Brake—Fly—Governor.**—The effect of all regulating apparatus is to control the speed of machinery. A regulating instrument may act simply by consuming energy, so as to prevent acceleration, or produce retardation, or stop the machine if required; it is then called a *brake;* or it may act by storing surplus energy at one time, and giving it out at another time, when energy is deficient: in this case it is called a *fly;* or it may act by adjusting the power of the prime mover to the work to be done, when it is called a *governor.* The use of a brake involves waste of power. A fly and a governor, on the other hand, promote economy of power and economy of strength.

SECTION I.—*Of Brakes.*

350. **Brakes Defined and Classed.**—The contrivances here comprehended under the general title of *Brakes* are those by means of which friction, whether exerted amongst solid or fluid particles, is purposely opposed to the motion of a machine, in order either to stop it, to retard it, or to employ superfluous energy during uniform motion. The use of a brake involves waste of energy, which is in itself an evil, and is not to be incurred unless it is necessary to convenience or safety.

Brakes may be classed as follows:—

I. *Block-brakes,* in which one solid body is simply pressed against another, on which it rubs.

II. *Flexible brakes,* which embrace the periphery of a drum or pulley (as in Prony's Dynamometer, Article 341, page 383).

III. *Pump-brakes,* in which the resistance employed is the friction amongst the particles of a fluid forced through a narrow passage.

IV. *Fan-brakes,* in which the resistance employed is that of a fluid to a fan rotating in it.

351. **Action of Brakes in General.**—The work disposed of by a brake in a given time is the product of the resistance which it produces into the distance through which that resistance is overcome in a given time.

To *stop* a machine, the brake must employ work to the amount of the whole actual energy of the machine, as already stated in

Article 334. To *retard* a machine, the brake must employ work to an amount equal to the difference between the actual energies of the machine at the greater and less velocities respectively.

To *dispose of surplus energy*, the brake must employ work equal to that energy; that is, the resistance caused by the brake must balance the surplus effort to which the surplus energy is due; so that if n is the ratio which the velocity of rubbing of the brake bears to the velocity of the driving point, P, the *surplus effort* at the driving point, and R the resistance of the brake, we ought to have—

$$R = \frac{P}{n}. \quad\text{..............................(1.)}$$

It is obviously better, when practicable, to store surplus energy, or to prevent its exertion, than to dispose of it by means of a brake.

When the action of a brake composed of solid material is long-continued, a stream of water must be supplied to the rubbing surfaces, to abstract the heat that is produced by the friction, according to the law stated in Article 311, page 354.

352. **Block-Brakes.**—When the motion of a machine is to be controlled by pressing a block of solid material against the rim of a rotating drum, it is advisable, inasmuch as it is easier to renew the rubbing surface of the block than that of the drum, that the drum should be of the harder, and the block of the softer material—the drum, for example, being of iron, and the block of wood. The best kinds of wood for this purpose are those which have considerable strength to resist crushing, such as elm, oak, and beech. The wood forms a facing to a frame of iron, and can be renewed when worn.

When the brake is pressed against the rotating drum, the direction of the pressure between them is obliquely opposed to the motion of the drum, so as to make an angle with the radius of the drum equal to the *angle of repose* of the rubbing surfaces (denoted by φ; see page 349). The component of that oblique pressure in the direction of a tangent to the rim of the drum is the friction (R); the component perpendicular to the rim of the drum is the normal pressure (N) required in order to produce that friction, and is given by the equation

$$N = \frac{R}{f}; \quad\text{..........................(1.)}$$

f being the co-efficient of friction, and the proper value of R being determined by the principles stated in Article 351.

It is in general desirable that the brake should be capable of effecting its purpose when pressed against the drum by means of the strength of one man, pulling or pushing a handle with one hand or one foot. As the required normal pressure N is usually considerably greater than the force which one man can exert, a lever, or screw, or a train of levers, screws, or other convenient mechanism, must be interposed between the brake block and the handle, so that when the block is moved towards the drum, the handle shall move at least through a distance as many times greater than the distance by which the block *directly* approaches the drum, as the required normal pressure is greater than the force which the man can exert.

Although a man may be able occasionally to exert with one hand a force of 100 lbs., or 150 lbs., for a short time, it is desirable that, in working a brake, he should not be required to exert a force greater than he can keep up for a considerable time, and exert repeatedly in the course of a day, without fatigue—that is to say, about 20 lbs. or 25 lbs.

353. The **Brakes of Carriages** are usually of the class just described, and are applied either to the wheels themselves or to drums rotating along with the wheels. Their effect is to stop or to retard the rotation of the wheels, and make them slip, instead of rolling on the road or railway. The resistance to the motion of a carriage which is caused by its brake may be less, but cannot be greater, than the friction of the stopped or retarded wheels on the road or rails under the load which rests on those wheels. The distance which a carriage or train of carriages will run on a level line during the action of the brakes before stopping, is found by dividing the actual energy of the moving mass before the brakes are applied, by the sum of the ordinary resistance and of the additional resistance caused by the brakes; in other words, that distance is as many times greater than the height due to the speed as the weight of the moving mass is greater than the total resistance.

The *skid*, or *slipper-drag*, being placed under a wheel of a carriage, causes a resistance due to the friction of the skid upon the road or rail under the load that rests on the wheel.

354. **Flexible Brakes.** (*A. M.*, 678.)—A flexible brake embraces a greater or less arc of the rim of a drum or pulley whose motion it resists. In some cases it consists of an iron strap, of a radius naturally a little greater than that of the drum; so that when left free, the strap remains out of contact with the drum, and does not resist its motion; but when tension is applied to the ends of the strap, it clasps the drum, and produces the required friction. The rim of the drum may be either of iron or of wood. In other cases the brake consists of a chain, or jointed series of iron bars, usually faced with wooden blocks on the side next the drum. When tension is applied to the ends of the chain, the blocks clasp the drum and produce friction; when that tension is removed, the blocks are

drawn back from the drum by springs to which they are attached, and the friction ceases.

The following formulæ are exact for perfectly flexible continuous bands, and approximate for elastic straps and for chains of blocks. Their demonstration has already been given in Article 310 A, page 351.

In fig. 254, let A B be the drum, and C its axis, and let the direction of rotation of the drum be indicated by the arrow. Let T_1 and T_2 represent the tensions at the two ends of the strap, which embraces the rim of the drum throughout the arc A B. The tension T_1 exceeds the tension T_2 by an amount equal to the friction between the strap and drum, R; that is,

$$R = T_1 - T_2.$$

Let c denote the ratio which the arc of contact, A B, bears to the *circumference* of the drum; f, the co-efficient of friction between the strap and drum; then the ratio $T_1 : T_2$ is *the number whose common logarithm is* $2{\cdot}7288 \, fc$, or

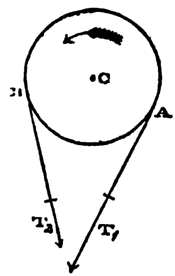

Fig. 254.

$$\frac{T_1}{T_2} = 10^{2\cdot7288 fc} = N; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

which number having been found, is to be used in the following formulæ for finding the tensions, T_1, T_2, required in order to produce a given resistance, R:—

$$\text{Backward or greatest tension, } T_1 = R \cdot \frac{N}{N-1}; \ldots\ldots\ldots\ldots(2.)$$

$$\text{Forward or least tension, } T_2 = R \cdot \frac{1}{N-1}. \ldots\ldots\ldots\ldots(3.)$$

The following cases occur in practice:—

I. When it is desired to produce *a great resistance compared with the force applied to the brake*, the backward end of the brake, where the tension is T_1, is to be fixed to the framework of the machinery, and the forward end moved by means of a lever or other suitable mechanism; when the force to be applied by means of that mechanism will be T_2, which, by making N sufficiently great, may be made small as compared with R.

II. When it is desired that *the resistance shall always be less than a certain given force*, the forward end of the brake is to be fixed, and the backward end pulled with a force not exceeding the given force. This will be T_1; and, as the equation 2 shows, how great soever N may be, R will always be less than T_1. This is the principle of the brake applied by Sir William Thomson to apparatus for paying out submarine telegraph cables, with a view to limiting the resistance within the amount which the cable can safely bear.

In any case in which it is desired to give a great value to the ratio N, the flexible brake may be coiled spirally round the drum, so as to make the arc of contact greater than one circumference.

355. **Pump-Brakes.**—The resistance of a fluid, forced by a pump through a narrow orifice, may be used to dispose of superfluous energy; as in the "cataract," or "dash-pot."

The energy which is expended in forcing a given weight of fluid through an orifice is found by multiplying that weight into the height due to the greatest velocity which its particles acquire in that process, and into a factor greater than unity, which for each kind of orifice is determined experimentally, and whose excess above unity expresses the proportion which the energy expended in overcoming the friction between the fluid and the orifice bears to the energy expended in giving velocity to the fluid.

The following are some of the values of that factor, which will be denoted by 1 + F:—

For an orifice in a thin plate, $1 + \mathrm{F} = 1{\cdot}054$. ....(1.)

For a straight uniform pipe of the length l, and whose *hydraulic mean depth*, that is, the area divided by the circumference of its cross-section, is m,

$$1 + \mathrm{F} = 1{\cdot}505 + \frac{fl}{m}. \qquad (2.)$$

For cylindrical pipes, m is one-fourth of the diameter.

The factor f in the last formula is called the *co-efficient of friction* of the fluid. For *water in iron pipes*, the diameter d being expressed in feet, its value, according to Darcy, is

$$f = 0{\cdot}005\left(1 + \frac{1}{12\,d}\right). \qquad (3.)$$

For *air*, $f = 0{\cdot}006$ nearly. ....(4.)

The greatest velocity of the fluid particles is found by dividing the volume of fluid discharged in a second by the area of the outlet at its most contracted part. When the outlet is a cylindrical pipe, the sectional area of that pipe may be employed in this calculation; but when it is an orifice in a thin plate, there is a *contracted vein* of the issuing stream after passing the orifice, whose area is on an average about 0·62 of the area of the orifice itself; and that contracted area is to be employed in computing the velocity. Its ratio to the area of the orifice in the plate is called the *co-efficient of contraction*.

The computation of the energy expended in forcing a given quantity of a given fluid in a given time through a given outlet, is expressed symbolically as follows:—

Let V be the volume of fluid forced through, in units of volume per second.

D, the heaviness of the fluid (see page 326).

A, the area of the orifice.

c, the co-efficient of contraction.

v, the velocity of outflow.

R, the resistance overcome by the piston of the pump in driving the water.

u, the velocity of that piston.

Then

$$v = \frac{\mathrm{V}}{c\,\mathrm{A}}; \qquad (5.)$$

and

$$\mathrm{R}\,u = \mathrm{D}\,\mathrm{V}\,(1 + \mathrm{F})\,\frac{v^2}{2\,g}; \qquad (6.)$$

the factor 1 + F being computed by means of the formulæ 1, 2, 3, 4.

To find the intensity of the pressure (p) within the pump, it is to be observed, as in Article 302, that if A′ denotes the area of the piston,

$$\mathrm{V} = \mathrm{A}'\,u;\ \mathrm{R} = p\,\mathrm{A}'; \qquad (7.)$$

consequently,

$$p = \frac{\mathrm{R}}{\mathrm{A}'} = \mathrm{D}\,(1 + \mathrm{F}) \cdot \frac{v^2}{2\,g}; \qquad (8.)$$

that is, the *intensity of the pressure is that due to the weight of a vertical column of the fluid, whose height is greater than that due to the velocity of outflow in the ratio* 1 + F : 1.

To allow for the friction of the piston, about *one-tenth* may in general be added to the result given by equation 6. (See page 399.)

The piston and pump have been spoken of as single; and such may be the case when the velocity of the piston is uniform. When a piston, however, is driven by a crank on a shaft rotating at an uniform speed, its velocity varies; and when a pump-brake is to be applied to such a shaft, it is necessary, in order to give a sufficiently near approximation to an uniform velocity of outflow, that there should be at least either three single acting pumps, driven by three cranks making with each other angles of 120°, or a pair of double-acting pumps, driven by a pair of cranks at right angles to each other; and the result will be better if the pumps force the fluid into one common air vessel before it arrives at the resisting orifice.

That orifice may be provided with a valve, by means of which its area can be adjusted so as to cause any required resistance.

A pump-brake of a simple kind is exemplified in the apparatus called the "*cataract*," for regulating the opening of the steam valve in single-acting steam engines. It is fully described in most special treatises on those engines.*

356. **Fan-Brakes.**—A fan, or wheel with vanes, revolving in water, oil, or air, may be used to dispose of surplus energy; and the resistance which it causes may be rendered to a certain extent adjustable at will, by making the vanes so as to be capable of being set at different angles with their direction of motion, or at different distances from their axis.

Fan-brakes are applied to various machines, and are usually adjusted so as to produce the requisite resistance by trial. It is, indeed, by trial only that a final and exact adjustment can be effected; but trouble and expense may be saved by making, in the first place, an approximate adaptation of the fan to its purpose by calculation.

The following formulæ are the results of the experiments of Duchemin, and are approved of by Poncelet in his *Mécanique Industrielle* :—

For a thin flat vane, whose plane traverses its axis of rotation, let

A denote the area of the vane;

l, the distance of its centre of area from the axis of rotation;

s, the distance from the centre of area of the entire vane to the centre of area of that half of it which lies nearest the axis of rotation;

v, the velocity of the centre of area of the vane ($= a\, l$, if a is the angular velocity of rotation);

D, the heaviness of the fluid in which it moves;

R l, the moment of resistance;

k, a co-efficient whose value is given by the formula

$$k = 1{\cdot}254 + 1{\cdot}6244 \frac{\sqrt{A}}{l - s}; \ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

then

$$R\, l = l\, k\, D\, A \cdot \frac{v^2}{2\, g}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

When the vane is oblique to its direction of motion, let i denote

* Pump-brakes have been applied to railway carriages by Mr. Laurence Hill. Hydraulic buffers, which act on the same principle, have been applied to railway carriages by Colonel Clark, R.A.

the acute angle which its surface makes with that direction; then the result of equation 2 is to be multiplied by

$$\frac{2 \sin^2 i}{1 + \sin^2 i} \text{ ....(3.)}$$

It appears that the resistance of a fan with several vanes increases nearly in proportion to the number of vanes, so long as their distances apart are not less at any point than their lengths. Beyond that limit the law is uncertain.

Section II.—*Of Fly-Wheels.*

357. **Periodical Fluctuations of Speed** in a machine (*A. M.*, 689) are caused by the alternate excess and deficiency of the energy exerted above the work performed in overcoming resisting forces, which produce an alternate increase and diminution of actual energy, according to the law explained in Article 330, page 373.

To determine the greatest fluctuation of speed in a machine moving periodically, take A B C, in fig. 255, to represent the motion of the driving point during one period; let the effort P of the prime mover at each instant be represented by the ordinate of the curve D G E I F; and let the sum of the resistances, reduced to the driving point as in Article 305, at each instant, be denoted by R, and represented by the ordinate of the curve D H E K F, which cuts the former curve at the ordinates A D, B E, C F. Then the integral,

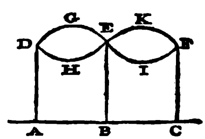

Fig. 255.

$$\int (\mathrm{P} - \mathrm{R})\, d\, s,$$

being taken for any part of the motion, gives the excess or deficiency of energy, according as it is positive or negative. For the entire period A B C, this integral is nothing. For A B, it denotes an *excess of energy received*, represented by the area D G E H; and for B C, an equal *excess of work performed*, represented by the equal area E K F I. Let those equal quantities be each represented by Δ E. Then the actual energy of the machine attains a maximum value at B, and a minimum value at A and C, and Δ E is the difference of those values.

Now let v_0 be the mean velocity, v_1 the greatest velocity, v_2 the least velocity of the driving point, and $\Sigma \cdot n^2$ W the *reduced inertia* of the machine (see Article 315, page 362); then

$$\frac{v_1^2 - v_2^2}{2g} \cdot \Sigma \cdot n^2 \mathrm{W} = \Delta \mathrm{E}; \text{ ....(1.)}$$

which, being divided by the *mean actual energy*,

$$\frac{v_0^2}{2g} \cdot \Sigma \cdot n^2 W = E_0,$$

gives

$$\frac{v_1^2 - v_2^2}{v_0^2} = \frac{\Delta E}{E_0}; \quad \text{(2.)}$$

and observing that $v_0 = (v_1 + v_2) \div 2$, we find

$$\frac{v_1 - v_2}{v_0} = \frac{\Delta \cdot E}{2 E_0} = \frac{g \Delta E}{v_0^2 \Sigma \cdot n^2 W}; \quad \text{(3.)}$$

a ratio which may be called the *co-efficient of fluctuation of speed* or of *unsteadiness*.

The ratio of the periodical excess and deficiency of energy ΔE to the whole energy exerted in one period or revolution, $\int P\, ds$, has been determined by General Morin for steam engines under various circumstances, and found to be from $\frac{1}{10}$ to $\frac{1}{4}$ for single-cylinder engines. For a pair of engines driving the same shaft, with cranks at right angles to each other, the value of this ratio is about one-fourth, and for three engines with cranks at 120°, one-twelfth of its value for single-cylinder engines.

The following table of the ratio, $\Delta E \div \int P\, ds$, for *one revolution* of steam engines of different kinds is extracted and condensed from General Morin's works:—

Non-Expansive Engines.

$\frac{\text{Length of connecting rod}}{\text{Length of crank}}$	=	8	6	5	4
$\Delta E \div \int P\, ds$	=	·105	·118	·125	·132

Expansive Condensing Engines.

Connecting rod = crank × 5.

Fraction of stroke at which steam is cut off,		$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$
$\Delta E \div \int P\, ds$	=	.163	·173	·178	·184	·189	·191

EXPANSIVE NON-CONDENSING ENGINES.

Steam cut off at	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
$\Delta E \div \int P\,ds =$	·160	·186	·209	·232

For double-cylinder expansive engines, the value of the ratio $\Delta E \div \int P\,ds$ may be taken as equal to that for single-cylinder non-expansive engines.

For *tools working at intervals*, such as punching, slotting, and plate-cutting machines, coining presses, &c., ΔE is nearly equal to the whole work performed at each operation.

358. **Fly-Wheels.** (*A. M.*, 690.)—A fly-wheel is a wheel with a heavy rim, whose great moment of inertia being comprehended in the reduced moment of inertia of a machine, reduces the co-efficient of fluctuation of speed to a certain fixed amount, being about $\frac{1}{32}$ for ordinary machinery, and $\frac{1}{50}$ or $\frac{1}{60}$ for machinery for fine purposes.

Let $\frac{1}{m}$ be the intended value of the co-efficient of fluctuation of speed, and ΔE, as before, the fluctuation of energy. If this is to be provided for by the moment of inertia, I, of the fly-wheel alone, let a_0 be its mean angular velocity; then equation 3 of Article 357 is equivalent to the following:—

$$\frac{1}{m} = \frac{g\,\Delta E}{a_0^2\,I}; \qquad (1.)$$

$$I = \frac{m\,g\,\Delta E}{a_0^2}; \qquad (2.)$$

the second of which equations gives the requisite moment of inertia of the fly-wheel.

The fluctuation of energy may arise either from variations in the effort exerted by the prime mover, or from variations in the resistance, or from both those causes combined. When but one fly-wheel is used, it should be placed in as direct connection as possible with that part of the mechanism where the greatest amount of the fluctuation originates; but when it originates at two or more points, it is best to have a fly-wheel in connection with each of those points.

For example, let there be a steam engine which drives a shaft that traverses a workshop, having at intervals upon it pulleys for driving various machine-tools. The steam engine should have a fly-wheel of its own, as near as practicable to its crank, adapted to that value of Δ E which is due to the fluctuations of the effort applied to the crank-pin above and below the mean value of that effort, and which may be computed by the aid of General Morin's tables, quoted in Article 357; and each machine-tool should also have a fly-wheel, adapted to a value of Δ E equal to the whole work performed by the tool at one operation.

As the rim of a fly-wheel is usually heavy in comparison with the arms, it is often sufficiently accurate for practical purposes to take the moment of inertia as simply equal to the weight of the rim multiplied by the square of the mean between its outside and inside radii—a calculation which may be expressed thus:—

$$I = W r^2; \qquad (3.)$$

whence the weight of the rim is given by the formula—

$$W = \frac{m g \Delta E}{a_0^2 r^2} = \frac{m g \Delta E}{v'^2}, \qquad (4.)$$

if v' be the velocity of the rim of the fly-wheel.

In millwork the ordinary values of the product $m g$, the unit of time being the second, lie between 1,000 and 2,000 feet, or approximately between 300 and 600 mètres. In pumping-machinery it is sometimes only about 300 feet, or 90 mètres.

The rim of the fly-wheel of a factory steam engine is very often provided with teeth, or with a belt, in order that it may directly drive the machinery of the factory.

SECTION III.—*Of Governors.*

359. The **Regulator** of a prime mover is some piece of apparatus by which the rate at which it receives energy from the source of energy can be varied; such as the sluice or valve which adjusts the size of the orifice for supplying water to a water-wheel, the apparatus for varying the surface exposed to the wind by windmill sails, the throttle-valve which adjusts the opening of the steam pipe of a steam engine, the damper which controls the supply of air to its furnace, and the expansion gear which regulates the volume of steam admitted into the cylinder at each stroke of the piston.

In prime movers whose speed and power have to be frequently and rapidly varied at will, such as locomotives and winding engines for mines, the regulator is adjusted by hand. In other cases the regulator is adjusted by means of a self-acting instrument driven by the prime mover to be regulated, and called a GOVERNOR.

The special construction of the different kinds of regulators is a subject for a treatise on prime movers. In the present treatise it is sufficient to state that in every governor there is a moving piece which acts on the regulator through a suitable train of mechanism, and which is itself made to move in one direction or in another according as the prime mover is moving too fast or too slow.

The object of a governor, properly so called, is to preserve a certain uniform speed, either exactly or approximately; and such is always the case in millwork. There are other cases, as in marine steam engines, where it may be considered sufficient to prevent sudden variations of speed, without preserving an uniform speed; and in those cases an apparatus may be used possessing only in part the properties of a governor: this may be called a *fly-governor*, to distinguish it from a governor proper.

Governors proper may be distinguished into *position-governors*, *disengagement-governors*, and *differential-governors:* a position-governor being one in which the moving piece that acts on the regulator assumes positions depending on the speed of motion, as in the common steam engine governor, which consists of a pair of revolving pendulums acting directly on a train of mechanism which adjusts the throttle-valve: a disengaging-governor being one which, when the speed deviates above or below its proper value, throws the regulator into gear with one or other of two trains of mechanism which move it in contrary directions so as to diminish or increase the speed, as the case may require, as in water-mill governors; and a differential-governor being one which, by means of an aggregate combination, moves the regulator in one direction or in another with a speed proportional to the difference between the actual speed and the proper speed of the engine.

In almost all governors the action depends on the centrifugal force exerted by two or more masses which revolve round an axis. By another classification, different from that which has already been described, governors may be distinguished into *gravity-governors*, in which gravity is the force that opposes the centrifugal force; and *balanced governors*, in which the actions of gravity on the various moving parts of the governor are mutually balanced, and the centrifugal force is opposed by the elasticity of a spring.

Governors may be further distinguished into those which are truly isochronous—that is to say, which remain without action on the regulator at one speed only; and those which are nearly isochronous—that is to say, which admit of some variation of the permanent or steady speed when the resistance overcome by the engine varies; and lastly, governors may be distinguished into those which are specially adapted to one speed, and those which can be adjusted at will to different speeds.

360. **Pendulum-Governors.**—A pendulum-governor is the simplest kind of gravity-governor. It has a vertical spindle, driven by the engine to be regulated; and from that spindle there hang, at

opposite sides, a pair of revolving pendulums, which, by the positions that they assume at different speeds, act on the regulator.

The relation between the height of a simple revolving pendulum and the number of turns which it makes per second has already been stated in Article 319; but for the sake of convenience it may here be repeated:—Let h denote the height or *altitude* of the pendulum (= O H in fig. 256), and T the number of turns per second; then

$$h = \frac{g}{4\pi^2 T^2} = \frac{\cdot 815 \text{ foot}}{T^2} = \frac{9\cdot 78 \text{ inches}}{T^2} = \frac{0\cdot 248 \text{ mètre}}{T^2}. \quad (1.)$$

If the rods of the revolving pendulums are jointed, as in fig. 257, not to a point in the vertical axis, but to a pair of points,

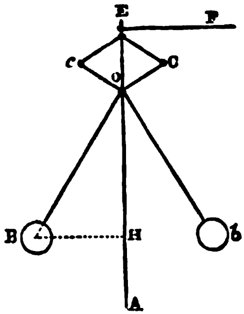

Fig. 256.

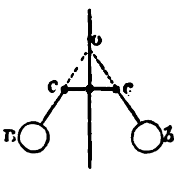

Fig. 257

such as C, c, in arms projecting from that axis, the height is to be measured to the point O, where the lines of tension of the rods cut the axis.

In most cases which occur in practice, the balls are so heavy, as compared with the rods, that the height may be measured without sensible error from the level of the centres of the balls to the point O, where the lines of suspension cut the axis. This amounts to neglecting the effects both of the weight and of the centrifugal force of the rods. These effects may, if required, be taken into account approximately, as follows:—Let B be the weight, and b the radius, of a ball; let R be the weight of a rod, and r the length from O to the centre of B; let h be the height from the centre of B to O, and h' the corrected height; then

$$h' = h\left(1 + \frac{R(r-b)}{2Br}\right) \div \left(1 + \frac{R(r-b)^2}{3Br^2}\right); \ldots\ldots (2.)$$

and the number of revolutions per second will correspond nearly to this corrected height.

The ordinary steam engine governor invented by Watt, which is represented in fig. 256, is a position-governor, and acts on the regulator by means of the variation of its altitude, through a train of levers and linkwork. That train may be very much varied in detail. In the example shown in the figure, the lever O C forms one piece with the ball-rod O B, and the lever O c with the ball-rod O b; so that when the speed falls too low, the balls B, b, by approaching the spindle, cause the point E to rise; and when the speed rises too high, the balls, by receding from the spindle, cause the point E to fall. At the point E there is a collar, held in the forked end of the lever E F, which communicates motion to the regulator.

The ordinary pendulum-governor is not truly isochronous; for when, in order to adapt the opening of the regulator to different loads, it rotates with its revolving pendulums at different angles to the vertical axis, the altitude h assumes different values, corresponding to different speeds.

As in Article 357, let the utmost extent of fluctuation of the speed of the engine between its highest and lowest limits be the fraction $\frac{1}{m}$ of the mean speed; let h be the altitude of the governor corresponding to the mean speed; and let k be the utmost extent of variation of the altitude between its smaller limit, when the regulator is shut, and its greater limit, when the regulator is full open. Then we have the following proportion:—

$$1 : \left(1 + \frac{1}{2m}\right)^2 - \left(1 - \frac{1}{2m}\right)^2 : : h : k;$$

and consequently

$$\frac{k}{h} = \frac{2}{m}. \qquad \text{(3.)}$$

361. **Loaded Pendulum-Governor.**—From the balls of the common governor, whose collective weight is (say) A, let there be, hung by a pair of links of lengths equal to the ball-rods, a load B, capable of sliding up and down the spindle, and having its centre of gravity in the axis of rotation. Then the centrifugal force is that due to A alone; and the effect of gravity is that due to A + 2 B; for when the ball-rods shift their position, the load B moves through twice the vertical distance that the balls move through, and is therefore equivalent to a double load, 2 B, acting directly on the balls. Consequently the altitude for a given speed is greater than that of a simple revolving pendulum, in the ratio $1 + \frac{2B}{A}$; a given *absolute* variation of altitude in moving the regulator produces a proportionate variation of speed smaller than

in the common governor, in the ratio $\frac{A}{A + 2B}$; and the governor is said to be *more sensitive* than a common governor, in the ratio of A : A + 2 B. Such is the construction of Porter's governor.

The links by which the load B is hung may be attached, not to the balls themselves, but to any convenient pair of points in the ball-rods; the links, and the parts of the ball-rods to which they are jointed, always forming a rhombus, or equilateral parallelogram. Let q be the ratio borne by each of the sides of that rhombus to the length on the ball-rods from the centre of a ball to the point where the line of suspension cuts the axis; then in the preceding expressions $2qB$ is to be substituted for 2 B.

In the one case 2 B, and in the other $2qB$, is the weight, applied directly at A, which would be *statically equivalent* to the load B, applied where it is.

362. **Parabolic Pendulum-Governors.**—In fig. 258, let B X be the axis of the spindle, and E the centre of one of the balls, which, as it moves towards or from the spindle, is guided so as to describe a parabolic arc, K E, with the vertex at K. Let E F be a normal to the parabola, cutting the axis in F. The vertical height of F above E is constant, being equal to twice the focal distance of the parabola; hence this governor is absolutely isochronous. That is to say, the balls cannot remain steady in any position except at one particular speed of rotation; being that corresponding to an altitude equal to twice the focal distance of the parabola; and any deviation of the speed above or below that value causes the balls to move continuously outwards and upwards, or inwards and downwards, as the case may be, until their action on the regulator restores the proper speed. The force with which the balls tend to shift their position *vertically*, when a deviation of speed occurs, is expressed very nearly by $\frac{2A\Delta n}{n}$; in which A is the collective weight of the balls, n the proper number of revolutions in a given time, and Δn the deviation from that number. The balls may be guided in various ways, viz:—

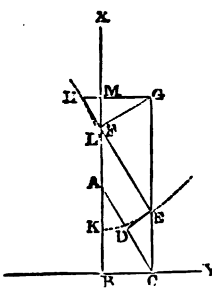

Fig. 258.

I. By hanging each of them by means of a flexible spring from a cheek, L H, of the form of the evolute of the parabola. To find a series of points in the parabola and its evolute, let h be the altitude; then from the vertex K lay off K A = K B = $\frac{1}{2}h$; A will be the focus, and the horizontal line B Y the directrix. Draw A C parallel to an intended position of the ball-rod; bisect it in D;

draw D E perpendicular to A C, and C E parallel to B X; the intersection E will be a point in the parabola, and E D a tangent. Then parallel to C A, draw E F; this will be a normal, and a position of the ball-rod. From F, parallel to D E, draw F G, cutting C E produced in G; and from G, parallel to B Y, draw G, cutting E F produced in H; this will be a point in the evolute. To express this algebraically, let B C = y and C E = z be the co-ordinates of the parabola; and let B M = z' and M H = $-y'$ be those of its evolute. Then we have

$$z = \frac{1}{2}\left(h + \frac{y^2}{h}\right); \; z' = 3z; \; -y' = \frac{y^3}{h^2}.$$

II. Another method of guiding the balls is to support them by means of a pair of properly curved arms, on which they slide or roll. On the top of the balls there rests a horizontal plate or bar, which communicates their vertical movements to the regulator.

III. *Approximate Parabolic Governor.*—In Farcot's governor, the rod E H, in its middle position, is hung from a joint, H, at the end of an arm, M H; this gives approximate isochronism. The co-ordinates of the point H are found by the rules already given.

362A. **Loaded Parabolic Governor.**—When the balls of a parabolic governor are guided in the second manner described in the preceding article, and support above them a plate or bar, to which their vertical movements are communicated, an additional load may be applied to them by means of that plate. Let A be the collective weight of the balls; B, the additional load; then the altitude corresponding to a given speed is greater than in the unloaded governor, in the ratio of A + B : A; and the speed corresponding to a given altitude is greater, in the ratio of $\sqrt{(A + B)} : \sqrt{A}$; and by varying the load, the speed of the governor may be varied at will.

363. **Isochronous Gravity-Governor** (*Rankine's*).—In this form of governor (see fig. 259) the four centrifugal balls marked B are balanced, as regards gravity, about the joint A, on the spindle A M. D, D are sliders on the ball-rods; D C, D C, levers jointed to the sliders, and centred on a point in the spindle at C, and of a length D C = C A; G G, a loaded circular platform hung from the levers C D, C D, by links E F, E F; H, an easy-fitting collar, jointed to the steelyard lever H K, whose fulcrum is at K; L, a weight adjustable on this lever. This governor is truly isochronous; the altitude h of a revolving pendulum of equal speed is given by the equation

$$h = \frac{B \cdot A B^2}{2 D \cdot C D};$$

in which B is the collective weight of the centrifugal masses, and

D the load, suspended directly at D, to which the actual load is statically equivalent. The load D, and consequently the altitude

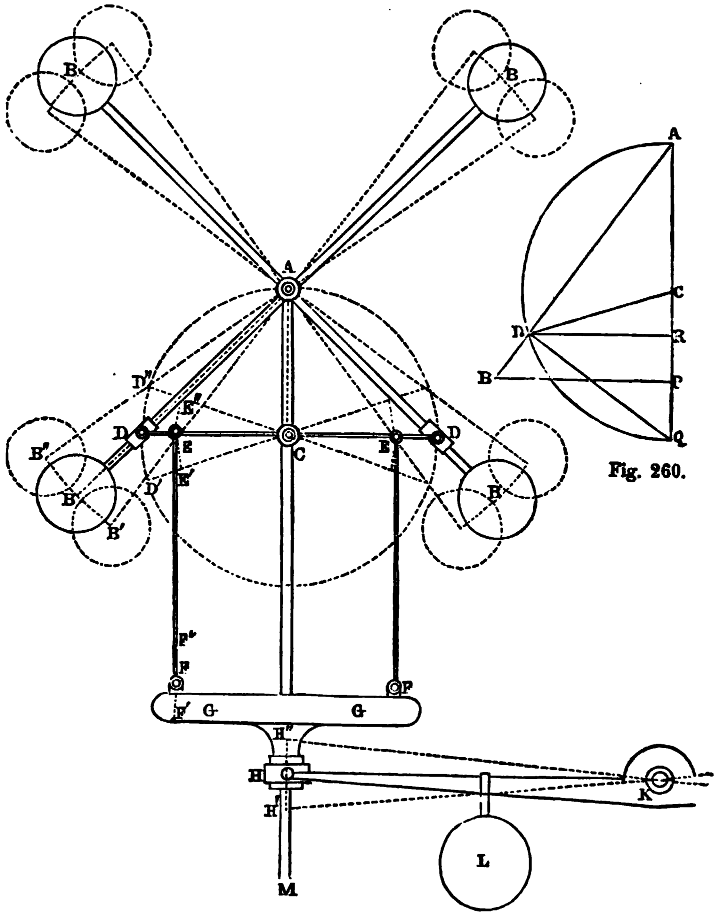

Fig. 260.

Fig. 259.

and the speed, can be varied at will, by shifting the weight L; which can be done either by hand or by the engine itself. The regulator may be acted on by the other end of the lever H K. The levers C D, C D should be horizontal when in their middle position; and then the ball-rods will slope at angles of 45°. Two positions of the parts of the governor when the rods deviate from their middle position, are shown by dotted lines and accented letters. If convenient, the links E F, E F may be hung directly from the slides D, D.

The theory of this governor is illustrated by fig. 260. In any position of the parts, let A C be the axis of rotation; A B, a ball-rod carrying a ball at B; C, the point at which the lever C D = C A is jointed to the spindle; D, the central point of the slider at the end of that lever. About C draw the circle A D Q, cutting the axis of rotation in Q; join D Q; and draw D R and B P perpendicular to A Q.

Then when the position of the parts varies, and the speed is constant, the moment of the centrifugal force of the balls relatively to A varies proportionally to B P · P A, and therefore proportionally to the area of the right-angled triangle A P B; and the moment relatively to A of the load which acts on the point D varies proportionally to D R, and therefore to the area of the right-angled triangle A D Q; but the areas of the triangles A B P and A D Q bear a constant ratio to each other—viz., that of AB^2 to AQ^2; therefore the moment of the centrifugal force at a constant speed, and the moment of load, bear a constant ratio to each other in all positions of the parts of the governor; and if they are equal in one position, they are equal in every position; and if unequal in one position, they are unequal in every position. Therefore the governor is truly isochronous.

To express algebraically the relations between the dimensions, the revolving mass, the load, and the speed; let B be the collective weight of the four balls; D, the total load which is actually or virtually applied at the points D, D; let the length of each ball-rod A B = b; and let the length of each of the levers C D = c. In any position of the governor, let the angle Q A B = θ. Then, because A C D is an isosceles triangle, we have the angle Q C D = 2θ. It is also evident that B P = $b \sin \theta$; A P = $b \cos \theta$; D R = $c \cdot \sin 2\theta = 2c \cdot \cos \theta \sin \theta$.

Let n, as before, be the number of revolutions per second. Then the centrifugal moment of the balls relatively to A is

$$\mathrm{B} \cdot \frac{4\pi^2 n^2}{g} \cdot \mathrm{BP} \cdot \mathrm{PA} = \frac{\mathrm{B}\, b^2 \sin \theta \cos \theta}{h};$$

and the statical moment of the load relatively to A is

$$2\,\mathrm{D} \cdot \mathrm{DR} = 4\,\mathrm{D}\, c \cdot \cos \theta \sin \theta;$$

which two moments, being equated to each other, and common factors struck out, give the following equation:—

$$\frac{\mathrm{B}\, b^2}{h} = 4\,\mathrm{D}\, c;$$

and therefore

$$h = \frac{\mathrm{B}\, b^2}{4\,\mathrm{D}\, c} = \frac{\mathrm{B} \cdot \mathrm{A\,B}^2}{4\,\mathrm{D} \cdot \mathrm{A\,C}};$$

as has already been stated.*

364. **Fluctuations of Isochronous Governors.**—When a truly isochronous governor is rapid in its action on the regulator, and meets with little resistance from friction, it may sometimes happen that the momentum of the moving parts carries them beyond the position suited for producing the proper speed; so that a deviation from the proper speed takes place in the contrary direction to the previous deviation, followed by a change, in the contrary direction, in the position of the governor, which again is carried too far by momentum; and so on; the result being a series of periodical fluctuations in the speed of the engine. When this is found to occur, it may be prevented by the use of a piston working in an oil-cylinder or dash-pot; which will take away the momentum of the moving parts, and cause the regulating action of the governor to take place more slowly, without impairing its accuracy.

365. **Balanced, or Spring Governors,** (*Silver's, Weir's, Hunt's, Sir W. Thomson's,* &c.)—In this class of governors, often called *Marine Governors*, as being specially suited for use on board ship, the action of gravity on the balls is either self-balanced, or made, by rapid rotation, so small compared with the centrifugal force as to be unimportant. The centrifugal force is opposed by springs. To make such a governor isochronous, the springs ought to be so arranged as to make the elastic force exerted by them vary in the simple ratio of the distance from the centres of the balls to the axis.

In order that the action of gravity on the balls may be self-balanced, if there are two balls only, they must move in opposite directions, in a plane perpendicular to the axis of rotation: which axis may have any position, but is usually horizontal. They might be guided by sliding on rods perpendicular to the spindle; but they are more frequently guided by combinations of linkwork, different forms of which are exemplified in Weir's governor and in Hunt's governor. If there are four balls, they are carried by a pair of arms like the letter X, as in fig. 259 (but with the spindle usually horizontal instead of vertical), and such is the arrangement in Silver's Marine Governor. The springs in balanced governors are seldom fitted up with a view to perfect isochronism; but for marine engines this is unimportant, as the principal object of applying governors to them is to prevent changes of speed so great

* It has been pointed out by Mr. Edmund Hunt that this form of governor is virtually a parabolic governor; for the common centre of gravity of the balls and of the load moves in a parabola, of a focal distance equal to half the altitude given by the formula.

and sudden as to be dangerous; such as those which tend to occur when the screw-propeller of a vessel pitching in a heavy sea is alternately lifted out of and plunged into the water.

Rules showing the relation between the deflection of a straight spring, or the extension of a spiral spring, and the elastic force exerted by the spring, have already been given in Article 342, page 386, and Article 345, page 389.

366. **Disengagement-Governors.**—The most complete example of a disengagement-governor is that commonly used for water-wheels, and sometimes also for the steam engine. The peculiar parts of this governor are represented in fig. 261. A A is part of the spindle of a pair of revolving pendulums similar to those in an ordinary governor; B, a cylindrical slider, hung from the ball-rods by links whose lower ends are shown at C, C. D is a tooth or cam projecting from the slider, and sweeping round as the spindle and pendulums rotate. To make the slider rotate truly with the spindle, the part of the spindle on which it slides may either be made square, or may have a projecting longitudinal feather fitting easily a groove in the inside of the slider.

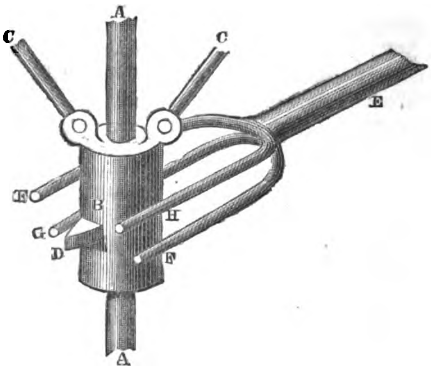

Fig. 261.

E is one end of a lever capable of turning about a vertical axis (not shown), and provided with a fork of four prongs, F, F, G, H. The prongs F, F are just far enough apart to clear the tooth D, as it sweeps round, when the spindle is turning at its proper speed, and the ball-rods and slider in their middle position; and the lever E is then in its middle position also. The prong G is below, and the prong H above, the level of the prongs F, F; and when the lever is in its middle position, the clear distance of G and H from the cylindrical surface of the slider B is one-half of the distance of F, F from that surface. When the spindle begins to fall below its proper speed, the slider moves downwards until the tooth D strikes the prong G, and drives the lever E to one side. Should the spindle begin to turn faster than the proper speed, the slider rises until the tooth D strikes the prong H, and drives the lever E to the contrary side. The lever E acts through any convenient train of mechanism upon the clutch of a set of reversing-gear, like the combination shown in fig. 214, Article 263, page 299. The driving-shaft of that combination is continually driven by the engine. When the lever E is in its middle position, the following shaft is disengaged from the driving-shaft, and remains at rest. When the lever E is shifted to one side or to the other, the reversing-gear drives that following shaft in one direction or in the other; and its motion, being transmitted by a suitable train to the regulator, corrects the deviation of speed. So soon as the spindle resumes its proper speed, the tooth D, by striking one or other of the prongs F, F, replaces the lever E in its middle position, and disengages the regulating train.

367. In **Differential Governors** the regulation of the prime mover is effected by means of the difference between the velocity of a wheel driven by it and that of a wheel regulated by a revolving pendulum. This class of governors is exemplified by fig. 262, representing Siemens's differential governor as applied to prime movers. A is a vertical dead-centre or fixed spindle about which the after-mentioned pieces turn; C is a pulley driven by the prime mover, and fixed to a bevel-wheel, which is seen below it; E is a bevel-wheel similar to the first, and having the same apex to its pitch-cone. To this wheel are hung the revolving masses B, of which there are usually four, although two only are shown. Those masses form sectors of a ring, and are surrounded by a cylindrical casing, F. When the masses revolve with their proper velocity, they are adjusted so as nearly to touch this casing; should they exceed that velocity, they fly outwards and touch the casing, and are retarded by the friction. Their centrifugal force may be opposed either by gravity or by springs. For practical purposes, their angular velocity of revolution about the vertical axis may be considered constant. G, G are horizontal arms projecting from a socket which is capable of rotation about A, and carrying vertical bevel-wheels, which rest on E and support C, and transmit motion from C to E. There are usually four of the arms G, G, with their wheels, though two only are shown. H is one of those arms which projects, and has a rod attached to its extremity to act on the regulator of the prime mover, of what sort soever it may be.

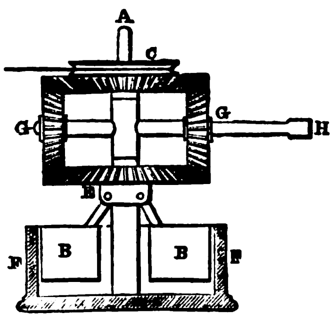

Fig. 262.

When C rotates with an angular velocity equal and contrary to that of E with its revolving pendulums, the arms G, G remain at rest; but should C deviate from that velocity, those arms rotate in one direction or the other, as the case may be, with an angular velocity equal to one-half of the difference between the angular velocity of C and that of E (see Article 234, page 245), and continue in motion until the regulator is adjusted so that the prime mover imparts to C an angular velocity exactly equal to that of the revolving masses B, B.

There are various modifications of the differential governor, but they all act on the same principle.

368. In **Pump-Governors** each stroke of the prime mover to be regulated forces, by means of a small pump, a certain volume of oil into a cylinder fitted with a plunger, like a hydraulic press. The oil is discharged at an uniform rate through an adjustable opening, back into the reservoir which supplies the pump. When the prime mover moves faster or slower than its proper speed, the oil is forced into the cylinder faster or slower, as the case may be, than it is discharged, so as to raise or to lower the plunger; and the plunger communicates its movements to the regulator, so as to correct the deviation of speed.

The **Bellows-Governor** acts on the same principle, using air instead of oil, and a double bellows instead of a pump and a plunger-cylinder

369. In **Fan-Governors** the greater or less resistance of air or of some liquid to the motion of a fan driven by the prime mover, causes the adjustment of the opening of the regulator.

CHAPTER IV.

OF THE EFFICIENCY AND COUNTER-EFFICIENCY OF PIECES, COMBINATIONS, AND TRAINS, IN MECHANISM.

370. **Nature and Division of the Subject.**—The terms *Efficiency* and *Counter-efficiency* have already been explained in Article 335, page 377; and the laws of friction, the most important of the wasteful resistances which cause the efficiency of a machine to be less than unity, have been stated in Articles 309 to 311, pages 348 to 354. In the present Chapter are to be set forth the effects of wasteful resistance, and especially of friction, on the efficiency and counter-efficiency of single pieces, and of combinations and trains of pieces, in Mechanism. In practical calculations the *counter-efficiency* is in general the quantity best adapted for use; because the useful work to be done in an unit of time, or *effective power*, is in general given; and from that quantity, by multiplying it by the counter-efficiency of the machine—that is, by the continued product of the counter-efficiencies of all the successive pieces and combinations by means of which motion is communicated from the driving-point to the useful working-point—is to be deduced the value of the expenditure of energy in an unit of time, or *total power*, required to drive the machine. In symbols, let U be the useful work to be done per second; c, c', c'', &c., the counter-efficiencies of the several parts of the train; T, the total energy to be expended per second; then

$$\mathrm{T} = c \cdot c' \cdot c'' \cdot \&c. \ldots \mathrm{U}. \qquad (1.)$$

When the mean effort required at the driving-point can conveniently be computed by reducing each resistance to the driving-point, and adding together the reduced resistances (as in Article 324, page 369, and Article 338, page 379), the ratio in which the actual effort required at the driving-point is greater than what the required effort would be, in the absence of wasteful resistance, is expressed by the continued product of the counter-efficiencies of the parts of the train, as follows: let P_0 be the effort required, in the absence of wasteful resistance; P, the actual effort required; then

$$\mathrm{P} = c \cdot c' \cdot c'' \cdot \&c. \ldots \mathrm{P}_0; \qquad (2.)$$

and in determining the efficiency or the counter-efficiency of a single piece, the most convenient method of proceeding often consists in comparing together the efforts required to drive that piece, with and without friction, and thus finding the ratios

$$\frac{P_0}{P} = \text{efficiency}; \; \frac{P}{P_0} = \text{counter-efficiency.} \; \ldots\ldots\ldots(3.)$$

In the ensuing sections of this Chapter, the efficiency of single primary pieces is first treated of, and then that of the various modes of connection employed in elementary combinations.

SECTION I.—*Efficiency and Counter-efficiency of Primary Pieces.*

371. **Efficiency of Primary Pieces in General.**—A primary piece in mechanism, moving with an uniform velocity, is balanced under the action of four forces, viz. :—

I. The re-action of the piece which it drives: this may be called the *Useful Resistance*, and denoted by R;

II. The *weight* of the piece itself: this may be denoted by W.

III. The *effort* by which the piece is driven: this may be denoted by P; and its values with and without friction by P_0 and P_1 respectively.

IV. The resultant pressure at the bearings, or *bearing-pressure*, which may be denoted by Q; and which of course is equal and directly opposed to the resultant of the first three forces.

In the absence of friction, the bearing-pressure would be normal to the bearing-surface. The effect of friction is, that the line of action of the bearing-pressure becomes oblique to the bearing-surface, making with the normal to that surface the angle of repose (ϕ), whose tangent ($f = \tan \phi$) is the co-efficient of friction (see Article 309, page 349); and the amount of the friction is expressed by Q sin ϕ, or very nearly by fQ, when the co-efficient of friction is small.

In the class of problems to which this Chapter relates, the first two forces—that is, the useful resistance R, and the weight W—are given in magnitude, position, and direction; and in most cases it is convenient to find their resultant, in magnitude, position, and direction, by the rules of statics: that is to say, if the line of action of R is vertical, by Rule I. of Article 280, page 322; and if inclined, by the Rules given or referred to in Article 278, page 319. In what follows, the resultant of the useful resistance and weight will be called the *given force*, and denoted by R′.

The third force—that is, the effort required in order to drive the piece at an uniform speed—is given in position and direction; for its line of action is the line of connection of the piece under consideration with the piece that drives it. The magnitude of the effort is one of the quantities to be found.

The fourth force—that is, the bearing-pressure—has to be found in position, direction, and magnitude. The general principles according to which it is determined are the following:—

First, That if the lines of action of the given force and the effort are parallel to each other, the line of action of the resultant bearing-pressure must be parallel to them both; and that if they are inclined to each other, the line of action of the resultant bearing-pressure must traverse their point of intersection.

Secondly, That at the centre of pressure, where the line of action of the resultant bearing-pressure cuts the bearing-surfaces, it makes an angle with the common normal of those surfaces equal to their angle of repose, and in such a direction that its tangential component (being the friction) is directly opposed to the relative sliding motion of that pair of surfaces over each other.

Thirdly, That the given force, the effort, and the bearing-pressure, form a system of three forces that balance each other; and are therefore proportional to the three sides of a triangle parallel respectively to their directions.

371 A. **Conditions Assumed to be Fulfilled.**—In all the problems treated of in this section, the following conditions are assumed to be fulfilled:—*First*, that except when otherwise specified, the forces other than bearing-pressures which are applied to the piece under consideration—that is, the useful resistance, the weight, and the effort—act either in parallel directions, or exactly or nearly in one plane, parallel to the planes of motion of the particles of the piece; *secondly*, that the acting parts of the piece do not *overhang* the bearings; and *thirdly*, that the bearing-surfaces fit each other easily without any grasping or pinching. As to the object of the fulfilment of such conditions, and the effects of departure from them, the following explanations have to be made:—

I. The bearing-surface of many primary pieces, and especially of rotating pieces, is in general divided into two parts; for example, an axle is very often supported by two journals. If the forces other than bearing-pressures which are applied to the moving piece, are parallel to each other, the parts of the bearing-pressure will also be parallel to them and to each other; and the sum of the frictional resistances due to the two parts of the bearing-pressure will be simply equal to the frictional resistance due to the whole bearing-pressure treated as one force. The same will be the case when the forces other than bearing-pressures act in one plane, parallel to the planes of motion of the particles of the piece; and will be nearly the case when, although those forces act in different planes, the transverse distance between their planes of action is small compared with the distance between the planes of action of the two components into which the bearing-pressure is divided.

But when that condition is not fulfilled, the friction at the bearings, being proportional to the sum of the two components into which the bearing-pressure is divided, will be greater than the friction due simply to the resultant bearing-pressure considered as one force; and the efficiency of the piece will be diminished.

II. The effect upon the friction, and upon the work lost in overcoming it, produced when the acting parts of a moving piece *overhang* its bearings, may be approximately calculated and allowed for in the following manner:—

Suppose that the bearing-surface of a primary piece, whether sliding or turning, is divided into two parts; and that the transverse distance between the centres of those two parts—that is, the distance in a direction perpendicular to the planes of motion of the particles of the piece—is denoted by c. Let the plane of action of the forces other than bearing-pressures be situated *outside* the space between the two parts of the bearing-surface, and at the transverse distance z from the centre of the nearer of those parts; and consequently at the distance $z + c$ from the centre of the further of them. Let Q be the resultant bearing-pressure. The two components of that resultant pressure, exerted at the two parts of the bearing-surface, will be contrary to each other in direction; and their values will be respectively,

at the nearer part, $\frac{Q(z+c)}{c}$;

and at the further part, $\frac{-Qz}{c}$.

The total friction will be the sum of two components exerted at the two parts of the bearing-surface respectively, and will be proportional to the *arithmetical sum* of the two components of the bearing-pressure; that is, to the force

$$\frac{Q(c + 2z)}{c};$$

whereas, if the plane of action of the resultant of the given force and the effort had not overhung the bearings, the friction would have been simply proportional to Q. Hence the effect of that plane's overhanging the bearings by the distance z, is to increase the friction approximately in the ratio of

$$1 + \frac{2z}{c} : 1.$$

III. As to the condition that the bearing-surfaces should fit each other easily, it is necessary in order that the bearing-pressure may not contain, to any appreciable extent, pairs of components which balance each other, being transverse to the direction of the resultant bearing-pressure; for such components cause an unnecessary addition to the friction. The ratio in which the friction of a *tight-fitting bearing* exceeds that of an easy-fitting bearing of the same dimensions and figure, is very nearly equal to that in which the whole area of the bearing-surface exceeds the area of the projection of that surface on a plane normal to the direction of the resultant bearing-pressure.

When the use of bearing-surfaces in pairs, oblique to the plane of the pressure and motion, is unavoidable (as, for example, in the case of the V-shaped bearings of a planing machine), their effect may be allowed for by increasing the co-efficient of friction in the ratio above-mentioned; which is expressed by the secant of the equal angles which the normals to the bearing-surfaces make with that plane.

372. **Efficiency of a Straight-sliding Piece.**—In fig. 263, let A A be a straight guiding-surface, upon which there slides, in the direction marked by the feathered arrow, the moving piece B.

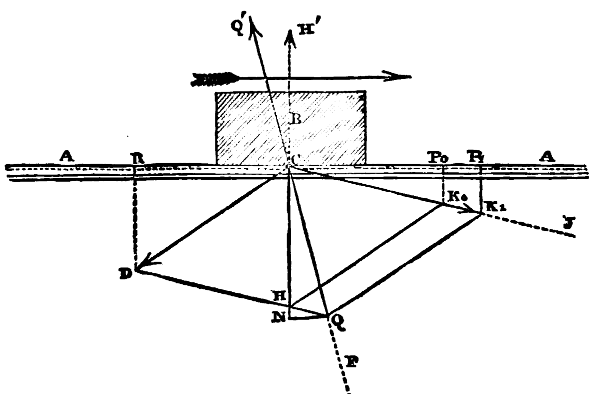

Fig. 263.

tion marked by the feathered arrow, the moving piece B. Let C D represent the *given force*, being the resultant of the useful resistance and of the weight of the piece B. (The figure shows the motion of B as horizontal; but it may be in any direction.) Let C J be the line of action of the effort by which the piece B is driven.

Draw C N perpendicular to A A; and C F making the angle N C F = the angle of repose. Through D, parallel to C J, draw the straight line D H Q, cutting C N in H, and C F in Q; and through H and Q, and parallel to D C, draw H K_0 and Q K_1, cutting C J in K_0 and K_1 respectively. Produce H C to H′, and Q C to Q′, making C H′ = H C, and C Q′ = Q C.

Then, in the absence of friction, C H′ will represent the resultant bearing-pressure exerted upon B by A A; and $CK_0 = DH$ will represent the force in the given direction C J required to drive B at an uniform speed; and when friction is taken into account, C Q′ will represent the resultant bearing-pressure, and CK_1 the actual driving force required; and we shall have

$$\text{the efficiency} = \frac{CK_0}{CK_1}; \text{ and the counter-efficiency} = \frac{CK_1}{CK_0}.$$

If from D, K_0, and K_1 there be let fall upon A A the perpendiculars D R, $K_0 P_0$, and $K_1 P_1$, C R will represent the direct resistance to the advance of B; CP_0, the direct effort in the absence of friction; and CP_1, the direct effort taking friction into account; so that the distance $P_0 P_1$ will represent the friction itself; which is also represented by Q N perpendicular to C N.

To express these results by symbols, let C D = R′ (the given force); let the acute angle A C D be denoted by α, and the acute angle A C J by β; and let φ denote the angle of repose N C Q. Then, in the triangle C D H, we have $\angle DCH = \frac{\pi}{2} - \alpha$, and $CHD = \frac{\pi}{2} - \beta$; and in the triangle C Q D, we have $\angle DCQ = \frac{\pi}{2} - \alpha + \varphi$, and $\angle CQD = \frac{\pi}{2} - \beta - \varphi$; consequently

$$DH = R' \frac{\cos\alpha}{\cos\beta}; \; DQ = R' \cdot \frac{\cos(\alpha - \varphi)}{\cos(\beta + \varphi)};$$

whence it follows that the efficiency and counter-efficiency are given by the following equations:—

$$\text{Efficiency} = \frac{P_0}{P_1} = \frac{DH}{DQ} = \frac{\cos\alpha \cdot \cos(\beta+\varphi)}{\cos\beta \cdot \cos(\alpha-\varphi)} = \frac{1 - f\tan\beta}{1 + f\tan\alpha} \quad (1.)$$

$$\text{Counter-efficiency} = \frac{P_1}{P_0} = \frac{1 + f\tan\alpha}{1 - f\tan\beta}. \quad \ldots\ldots\ldots\ldots \quad (2.)$$

It is to be remarked, that the efficiency diminishes to nothing when $\cot\beta = f$; that is to say, when β is the complement of the angle of repose, φ. In other words, if the oblique effort is applied in the direction C Q, no force, how great soever, will be sufficient to keep the piece B in motion.

373. **Efficiency of an Axle.**—In fig. 264, let the circle A A A represent the trace of the bearing-surface of an axle on a plane perpendicular to its axis of rotation, O—in other words, the transverse section of that surface. Let the arrow near the letter N represent the direction of rotation. Let C D be the given force;

that is, as before, the resultant of the weight of the whole piece that rotates with the axle, and of the useful resistance or re-action exerted on that piece by the piece which it drives; C J, the line of action of the effort by which the rotating piece is driven.

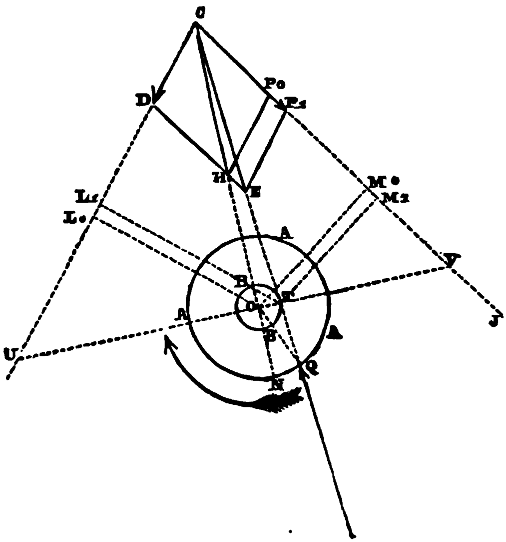

Fig. 264.

In toothed wheel-work the lines of action of the useful resistance and of the effort may be taken as coinciding with the lines of connection of the rotating piece with its follower and with its driver respectively. In pulleys connected with each other by bands, special principles have to be attended to, which will be explained in the ensuing Article.

Let r denote the radius of the bearing-surface.

About O describe the small circle B B, with a radius $= r \sin \phi = f r$, very nearly. Draw the line of action, C T Q, of the resultant bearing-pressure, touching the small circle at that side which will make the bearing-pressure resist the rotation. In the case in which C D and C J intersect each other in a point, C, as shown in the figure, C T Q will traverse that point also; and in the case in which the lines of action of the given force and the effort are parallel to each other, C T Q will be parallel to both. The centre of bearing-pressure is at Q; and O Q T $= \phi$, the angle of repose.

In the former case the efficiency may be found by parallelograms of forces, as follows:—Draw the straight line C O N; this would be the line of action of the resultant bearing-pressure in the absence of friction, and N would be the centre of bearing-pressure. Through D, parallel to C J, draw D H E, cutting C O N in H, and C T Q in E. Through H and E, parallel to D C, draw H P_0 and E P_1. Then, in the absence of friction, H C would represent the bearing-pressure, and C P_0 = D H the effort; the actual bearing-pressure is represented by E C, and the actual effort by C P_1 = D E. Hence the efficiency and counter-efficiency are as follows:—

$$\frac{P_0}{P_1} = \frac{\mathrm{D\,H}}{\mathrm{D\,E}}; \ \frac{P_1}{P_0} = \frac{\mathrm{D\,E}}{\mathrm{D\,H}} \qquad \text{(1.)}$$

Another method, applicable whether the forces are inclined or parallel, is as follows:—From the axis of rotation O, let fall O L_0 and O M_0 perpendicular respectively to the lines of action of the given force and of the effort. Then, by the balance of moments, the effort in the absence of friction is

$$P_0 = R' \cdot \frac{\mathrm{O\,L_0}}{\mathrm{O\,M_0}}.$$

From a convenient point in the actual line of action, C Q, of the bearing-pressure (such, for example, as T, where it touches the small circle B B), let fall T L_1 and T M_1 perpendicular respectively to the same pair of lines of action; then the actual effort will be

$$P_1 = R' \cdot \frac{\mathrm{T\,L_1}}{\mathrm{T\,M_1}}.$$

Hence the efficiency and the counter-efficiency have the following value:—

$$\left.\begin{aligned} \frac{P_0}{P_1} &= \frac{\mathrm{O\,L_0} \cdot \mathrm{T\,M_1}}{\mathrm{O\,M_0} \cdot \mathrm{T\,L_1}}; \\ \frac{P_1}{P_0} &= \frac{\mathrm{O\,M_0} \cdot \mathrm{T\,L_1}}{\mathrm{O\,L_0} \cdot \mathrm{T\,M_1}}. \end{aligned}\right\} \qquad \text{(2.)}$$

The same results are expressed, to a degree of approximation sufficient for practical purposes, by the following trigonometrical formulæ:—Let O $L_0 = l$; O $M_0 = m$; $\angle$ C O $L_0 = \alpha$; $\angle$ C O $M_0 = \beta$. Then we have, very nearly,

$$\frac{P_0}{P_1} = \frac{l}{m} \cdot \frac{m - f r \sin \beta}{l + f r \sin \alpha} = \frac{1 - \frac{f r}{m} \cdot \sin \beta}{1 + \frac{f r}{l} \cdot \sin \alpha}. \qquad \text{(3.)}$$

In making use of the preceding formula, it is to be observed that the *contrary algebraical signs* of sin α and sin β apply to those cases in which the two angles α and β lie at contrary sides of O C. In the cases in which those angles lie at the same side of O C, their algebraical signs are the same; and in the formula they are to be made *both positive* or *both negative*, according as β is *less* or *greater* than α; so that the efficiency may be always expressed by a fraction less than unity. That is to say,

$$\text{If } \beta > \alpha;\ \frac{P_0}{P_1} = \frac{1 - \frac{f r}{m} \sin \beta}{1 - \frac{f r}{l} \sin \alpha}; \text{.............(3 A.)}$$

$$\text{If } \beta < \alpha;\ \frac{P_0}{P_1} = \frac{1 + \frac{f r}{m} \sin \beta}{1 + \frac{f r}{l} \sin \alpha} \text{...............(3 B.)}$$

When the lines of action intersect, let O C be denoted by c; then $l = c \cos \alpha$, and $m = c \cos \beta$; and consequently the three preceding equations take the following form:—

$$\beta \text{ and } \alpha \text{ of contrary signs};\ \frac{P_0}{P_1} = \frac{1 - \frac{f r}{c} \tan \beta}{1 + \frac{f r}{c} \tan \alpha}; \text{.............(4.)}$$

β and α of the same sign;

$$\beta > \alpha;\ \frac{P_0}{P_1} = \frac{1 - \frac{f r}{c} \tan \beta}{1 - \frac{f r}{c} \tan \alpha}; \text{..........(4 A.)}$$

$$\beta < \alpha;\ \frac{P_0}{P_1} = \frac{1 + \frac{f r}{c} \tan \beta}{1 + \frac{f r}{c} \tan \alpha}. \text{...........(4 B.)}$$

When the lines of action of the forces are parallel, we have sin β and sin $\alpha = +1$ or -1, as the case may be; and the formulæ take the following shape:—

When l and m lie at contrary sides of O, the piece is a "lever of the first kind;" and

$$\frac{P_0}{P_1} = \frac{1 - \frac{f \cdot r}{m}}{1 + \frac{f r}{l}}. \text{........................(5.)}$$

When l and m lie at the same side of O;
If $m > l$, the piece is a "lever of the second kind;" and

$$\frac{P_0}{P_1} = \frac{1 - \frac{fr}{m}}{1 - \frac{fr}{l}}. \quad \text{.....................(5 A.)}$$

If $m < l$, the piece is a "lever of the third kind;" and

$$\frac{P_0}{P_1} = \frac{1 + \frac{fr}{m}}{1 + \frac{fr}{l}}. \quad \text{.....................(5 B.)}$$

(As to levers of the first, second, and third kinds, see Article 221, page 233.)

The following method is applicable whether the forces are inclined or parallel; in the former case it is approximate, in the latter exact. Through O, perpendicular to O C, draw U O V, cutting the lines of action of the given force and of the effort in U and V respectively. The point where this transverse line cuts the small circle B B coincides exactly with T when the forces are parallel, and is very near T when they are inclined; and in either case the letter T will be used to denote that point. Then

$$\frac{P_0}{P_1} = \frac{\text{O U}}{\text{O V}} \cdot \frac{\text{T V}}{\text{T U}} \quad \text{.....................(6.)}$$

It is evident that with a given radius and a given co-efficient of friction, the efficiency of an axle is the greater the more nearly the effort and the given force are brought into direct opposition to each other, and also the more distant their lines of action are from the axis of rotation.

374. **Axles of Pulleys connected by Bands.**—When the rotating piece which turns with an axle consists of a pair of pulleys, one receiving motion from a driving pulley, and the other communicating motion to a following pulley, regard must be had to the fact that the useful resistance and the driving effort are each of them the *difference* of a pair of tensions; and that it is upon the *resultant* of each of those pairs of tensions (being their sum, if they act parallel to each other) that the axle-friction depends.

The principles according to which the tensions required at the two sides of a band for transmitting a given effort are determined, have been stated in Article 310 A, pages 351, 352.

The belt which drives the first pulley may be called the *driving belt;* that which is driven by the second pulley, the *following belt.*

The tensions on the two sides of the following belt are given; and the moment of the useful resistance is that of their difference, acting with a leverage equal to the effective radius of the second pulley. Let p be that radius; T_1 and T_2 the two tensions; then the moment of the useful resistance is

$$p\,R = p\,(T_1 - T_2).$$

For the actual useful resistance there is to be substituted a force equal to the resultant of T_1 and T_2, and exerting the same moment. That is to say, let γ denote the angle which the two sides of the band make with each other; then for the actual useful resistance is to be substituted a force,

$$R'' = \sqrt{\{T_1^2 + T_2^2 + 2\,T_1\,T_2 \cos \gamma\}}, \quad \ldots\ldots\ldots\ldots (1.)$$

acting at the following perpendicular distance from the axis of rotation:—

$$k = \frac{p\,(T_1 - T_2)}{R''}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

And this is to be compounded with the weight of the rotating piece, to find the *given force* R′ of the rules in the preceding Article, whose perpendicular distance from the axis will be

$$l = \frac{p\,(T_1 - T_2)}{R'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

The value of k may be expressed in terms of the ratio of the tensions to each other, and independently of their absolute values, as follows:—Let $N = \frac{T_1}{T_2}$ be the ratio of the two tensions found by the rules of Article 310 A, page 351. Then

$$k = \frac{p\,(N - 1)}{\sqrt{\{N^2 + 1 + 2\,N \cos \gamma\}}}. \quad \ldots\ldots\ldots\ldots (4.)$$

In like manner, for the actual line of action of the effort by which the first pulley is driven is to be substituted the line of action of a force exerting the same moment, and equal to the resultant of the tensions of the two sides of the driving-band. The perpendicular distance m of this line of action from the axis of rotation is given by the following formula:—Let p' be the effective radius of the pulley; N', the ratio of the greater to the lesser tension; γ', the angle which the two sides of the band make with each other; then

$$m = \frac{p'\,(N' - 1)}{\sqrt{\{N'^2 + 1 + 2\,N' \cos \gamma'\}}}. \quad \ldots\ldots\ldots\ldots (5.)$$

There are many cases in practice in which the two sides of each of the bands may be treated as sensibly parallel; and then we have simply,

$$\left.\begin{array}{c} R'' = \dfrac{R\,(N+1)}{N-1}; \\ k = \dfrac{p\,(N-1)}{N+1};\ m = \dfrac{p'\,(N'-1)}{N'+1}. \end{array}\right\} \quad \ldots\ldots\ldots(6.)$$

And if, moreover, as frequently happens, the weight of the pulleys and axle is small compared with the tensions, we may neglect it, and make $R' = R''$ and $l = k$, preparatory to applying the rules of the preceding Article to the determination of the efficiency.

375. **Efficiency of a Screw.**—The efficiency of a screw acting as a primary piece is nearly the same with that of a block sliding on a straight guide, which represents the development of a helix situated midway between the outer and inner edges of the screw-thread; the block being acted upon by forces making the same angles with the straight guide that the actual forces do with that helix. As to the development of a helix, see Article 63, page 40; and as to the efficiency of a piece sliding along a straight guide, see Article 372, page 426.

376. **Efficiency of Long Lines of Horizontal Shafting.**—In a line of horizontal shafting for transmitting motive power to long distances in a mill, a great part of the wasted work is spent in overcoming the friction produced simply by the weight of the shaft resting on its bearings; and the efficiency and counter-efficiency as affected by this cause of loss of power can be considered and calculated separately.

For reasons connected with the principles of the strength of materials, to be explained further on, the *cube of the diameter* of a shaft of uniform diameter must be made to bear a certain proportion to the driving moment exerted upon it to keep up its rotation. That is to say, let M_1 denote that moment; h, the diameter of the shaft; then

$$M_1 = A\,h^3; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

A being a co-efficient whose values in practice range, according to circumstances to be explained in the Third Part of this treatise,

for forces in lbs. and dimensions in inches, from 300 to 1,800;

and for forces in kilogrammes and dimensions in millimètres,

from 0·21 to 1·26.

Let w denote the heaviness of iron; f, the co-efficient of friction; then the weight of an unit of length of the shaft is

$$\frac{\pi}{4} w h^2;$$

the friction per unit of length is, very nearly,

$$\frac{\pi}{4} f w h^2;$$

and the *moment of friction* per unit of length is

$$\frac{\pi}{8} f w h^3 = \cdot 3927 f w h^3 \text{ nearly.} \qquad (2.)$$

Let L be the length of a shaft of uniform diameter, such that the whole driving moment is exhausted in overcoming its own friction. This may be called the *exhaustive* length. Then we must have

$$M_1 = A h^3 = \cdot 3927 f w h^3 L; \text{ and therefore}$$

$$L = \frac{A}{\cdot 3927 f w}. \qquad (3.)$$

For lengths in feet, and diameter in inches, we have $w = \frac{10}{3}$; being the weight in pounds of a rod of iron a foot long and an inch square. For lengths in mètres, and diameters in millimètres, we have $w = \cdot 0077$ nearly; being the weight of a rod of iron one mètre long and one millimètre square. Let $f = 0{\cdot}051$; then the following are the values of the exhaustive length L corresponding to different values of A:—

A, British measures,	300	600	1,200	1,800
„ French,	0·21	0·42	0·84	1·26
L, feet	4,500	9,000	18,000	27,000
„ mètres	1,365	2,730	5,460	8,190

It is obvious that the efficiency of a length, l, of shafting of uniform diameter is given by the expression

$$\frac{M_0}{M_1} = 1 - \frac{l}{L}; \qquad (4).$$

M_0 being the driving moment in the absence of friction; M_1, the actual driving moment; and $\frac{l}{L}$, the fraction of that moment expended on friction; also, that the counter-efficiency is

$$\frac{M_1}{M_0} = \frac{L}{L - l}. \qquad (5.)$$

When, besides its own weight, the shaft is loaded with the weights of pulleys and tensions of belts, the effect of such additional load may be allowed for, with a degree of accuracy sufficient for practical purposes, in the following manner:—Find the magnitude of the resultant of the weight of the shaft and additional load; and let m be the ratio which it bears to the weight of the shaft. Then the modified value of the exhaustive length is to be found by putting $m\,w$ instead of w in the denominator of the expression (3.): that is to say

$$L = \frac{A}{\cdot 3927\, f\, m\, w}. \quad \ldots\ldots\ldots\ldots (6.)$$

The waste of work in a long line of shafting may be diminished, and the efficiency increased, by causing it to taper, so that the cube of the diameter shall at each cross-section be proportional to the moment exerted there. The most perfect way of fulfilling that condition is to make the diameter diminish continuously in geometrical progression; the generating line or longitudinal section of the shaft being a logarithmic curve. Let h be the diameter at the driving end, x the distance of a given cross-section from that end, and y the diameter at that cross-section; then

$$y = h\, e^{-\frac{x}{3L}}; \quad \ldots\ldots\ldots\ldots (7.)$$

in which $e^{-\frac{x}{3L}}$ is the reciprocal of the natural number, or antilogarithm, corresponding to the hyperbolic logarithm $\frac{x}{3L}$, and to the common logarithm $\frac{0\cdot 4343\, x}{3\,L}$. Let l be the total length of such a tapering shaft, and M_0 the useful working moment exerted at its smaller end; then we have

$$\left.\begin{array}{ll} \text{Efficiency,} & \dfrac{M_0}{M_1} = e^{-\frac{l}{L}}; \\ \text{Counter-efficiency,} & \dfrac{M_1}{M_0} = e^{\frac{l}{L}}. \end{array}\right\} \quad \ldots\ldots\ldots (8.)$$

This cannot be perfectly realized in practice; but it can be approximated to by making the shaft consist of a series of lengths, or divisions, each of uniform diameter, and increasing in diameter step by step.

Let $\frac{l}{n}$ now denote the length of one of those divisions; the number of divisions being n. The counter-efficiency of each division is expressed by

$$\frac{L}{L - \frac{l}{n}}; \quad \ldots\ldots\ldots\ldots (9.)$$

and consequently, the counter-efficiency of the whole shaft is,

$$\frac{M_1}{M_0} = \left\{ \frac{L}{L - \frac{l}{n}} \right\}^n . \qquad (10.)$$

The diameters of the lengths of shafting, beginning at the driving-end, form a diminishing geometrical progression, of which the common ratio is

$$\left\{ 1 - \frac{l}{n\,L} \right\}^{\frac{1}{3}} . \qquad (11.)$$

SECTION II.—*Efficiency and Counter-efficiency of Modes of Connection in Mechanism.*

377. **Efficiency of Modes of Connection in General.**—In an elementary combination consisting of two pieces, a driver and a follower, there is always some work lost in overcoming wasteful resistance occasioned by the mode of connection; the result being that the work done by the driver at its working-point is greater than the work done upon the follower at its driving-point, in a proportion which is *the counter-efficiency of the connection;* and the reciprocal of that proportion is *the efficiency of the connection.* In calculating the efficiency or the counter-efficiency of a train of mechanism, therefore, the factors to be multiplied together comprise not only the efficiencies, or the counter-efficiencies, of the several primary pieces considered separately, but also those of the several modes of connection by which they communicate motion to each other.

378. **Efficiency of Rolling Contact.**—The work lost when one primary piece drives another by rolling contact is expended in overcoming the *rolling resistance* of the pitch-surfaces, a kind of resistance whose mode of action has been explained in Article 311, page 353; and the value of that work in units of work per second is given by the expression $a\,b\,N$; in which N is the normal pressure exerted by the pitch-surfaces on each other; b, a constant arm, of a length depending on the nature of the surfaces (for example, 0·002 of a foot = 0·6 millimètre for cast iron on cast iron, see page 354); and a, the relative angular velocity of the surfaces.

The useful work per second is expressed by $u\,f\,N$, in which f is the co-efficient of friction of the surfaces, and u the common velocity of the pitch-lines. Hence the *counter-efficiency* is

$$c = 1 + \frac{a\,b}{u\,f} . \qquad (1.)$$

Let p_1 and p_2 be the lengths of two perpendiculars let fall from the two axes of rotation on the common tangent of the two pitch-lines; if the pieces are circular wheels, those perpendiculars will be the radii. Then the absolute angular velocities of the pieces are respectively $\frac{u}{p_1}$ and $-\frac{u}{p_2}$; and their relative angular velocity is therefore

$$a = u\left(\frac{1}{p_1} + \frac{1}{p_2}\right);$$

which value being substituted in equation (1), gives for the counter-efficiency the following value:—

$$c = 1 + \frac{1}{f}\left(\frac{b}{p_1} + \frac{b}{p_2}\right). \quad \text{..................(2.)}$$

It is assumed that the normal pressure is not greater than is necessary in order to give sufficient friction to communicate the motion.

It is evident, from the smallness of b, that the lost work in this case must be almost always a very small fraction of the whole.

379. **Efficiency of Sliding Contact in General.**—In fig. 265, let T be the point of contact of a pair of moving pieces connected by sliding contact. Let the plane of the figure be that containing the directions of motion of the two particles which touch each other at the point T; and let T V be the velocity of the driving-particle, and T W the velocity of the following-particle; whence V W will represent the velocity of sliding, and T U, perpendicular to V W, the common component of the velocities of the two particles along their line of connection R T P. C T C, parallel to V W, and perpendicular to R T P, is a common tangent to the two acting surfaces at the point T; the arrow A represents the direction in which the driver slides relatively to the follower; and the arrow B, the direction in which the follower slides relatively to the driver.

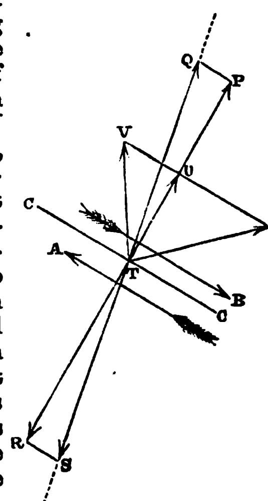

Fig. 265.

Along the line of connection, that is, normal to the acting surfaces at T, lay off T P to represent the effort exerted by the driver on the follower, and T R (= – T P) to represent the equal and opposite useful resistance exerted by the follower against the driver. Draw S T Q, making with R T P an angle equal to the angle of repose of the rubbing surfaces (see Article 309, page 349), and inclined in the proper direction to represent forces opposing the sliding motion; draw P Q and R S parallel to C C. Then T Q will represent the resultant pressure exerted by the driver on the follower, and T S (= – T Q), the equal and opposite resultant pressure exerted by the follower against the driver, and P Q = – R S will represent the friction which is overcome, through the distance V W, in each second; while the useful resistance, T R, is overcome through the distance T U. Hence the useful work per second is T U · T R; the lost work is V W · R S; and the counter-efficiency is

$$c = 1 + \frac{\mathrm{V\,W} \cdot \mathrm{R\,S}}{\mathrm{T\,U} \cdot \mathrm{T\,R}}. \qquad \text{(1.)}$$

Let the angle U T V = α, the angle U T W = β, and let f be the co-efficient of friction. Then we have—

$$\frac{\mathrm{V\,W}}{\mathrm{T\,U}} = \tan \alpha + \tan \beta; \; \frac{\mathrm{R\,S}}{\mathrm{T\,R}} = f;$$

and consequently

$$c = 1 + f(\tan \alpha + \tan \beta). \qquad \text{(2.)}$$

380. **Efficiency of Teeth.**—It has already been shown, in Article 127, page 118, that the relative velocity of sliding of a pair of teeth in outside gearing is expressed at a given instant by

$$(a_1 + a_2)\, t;$$

where t denotes the distance at that instant of the point of contact from the pitch-point. (In inside gearing the angular velocity of the greater wheel is to be taken with the negative sign.)

The distance t is continually varying from a maximum at the beginning and end of the contact, to nothing at the instant of passing the pitch-point. Its *mean value* may be assumed, with sufficient accuracy for practical purposes, to be sensibly equal to *one-half* of its greatest value; and in the formulæ which follow, the symbol t stands for that mean value.

Let P be the mutual pressure exerted by the teeth; f, the co-efficient of friction; then the work lost per second through the friction of the teeth is

$$(a_1 + a_2)\, t f \mathrm{P}.$$

Let u be the common velocity of the two pitch-circles; θ, the *mean obliquity* of the line of connection to the common tangent of the pitch-circles; then $u \cos \theta$ is the mean value of the common component of the velocities of the acting surfaces of the teeth along the line of connection; and the useful work done per second is expressed by

$$P u \cos \theta$$

so that the counter-efficiency is

$$c = 1 + \frac{(a_1 + a_2) t f}{u \cos \theta}. \text{.....................} (1.)$$

Let r_1 and r_2 be the radii of the two pitch-circles; then we have

$$a_1 = \frac{u}{r_1};\ a_2 = \frac{u}{r_2};$$

and consequently

$$c = 1 + f t \sec \theta \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\}. \text{...............} (2.)$$

If two pairs of teeth at least are to be in action at each instant (as in the case of involute teeth, and of some epicycloidal teeth), and if the pitch be denoted by p, we have $t \sec \theta = \frac{p}{2}$; and therefore

$$c = 1 + \frac{f p}{2} \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\} = 1 + \pi f \left\{ \frac{1}{n_1} + \frac{1}{n_2} \right\}; \text{...} (3.)$$

where n_1 and n_2 are the number of teeth in the two wheels.

In many examples of epicycloidal teeth, especially where small pinions are used, the duration of the contact is only $\frac{2}{3}$ or $\frac{3}{4}$ of that assumed in equation (3); and the work lost in friction is less in the same proportion.

The preceding rules have been stated in the form applicable to spur-wheels. In order to make them applicable to bevel-wheels, all that is necessary is to understand that the measurements of radii, distances, and obliquity, are to be made, not on the actual pitch-circles, but on the pitch-circles as shown *on the development of the normal cones;* as to which, see Article 144, page 144.

When there is a *transverse component* in the relative velocity of sliding (as in gearing-screws, Article 154, page 160), the fractional value of the work lost in friction is to be first computed as if for a pair of spur-wheels whose pitch-circles are the osculating circles of the normal screw-lines (see Article 154, pages 161, 162; and Article 155, page 163). Then find in what ratio the velocity of sliding is increased by compounding the transverse component with the direct component $(a_1 + a_2)\,t$; and increase the fraction of work lost through friction in the same proportion.

381. **Efficiency of Bands.**—A band, such as a leather belt or a hempen rope, which is not perfectly elastic, requires the expenditure of a certain quantity of work—first to bend it to the curvature of a pulley, and then to straighten it again; and the quantity of work so lost has been found by experiment to be nearly the same as would be required in order to overcome an additional resistance, varying directly as the sectional area of the band, directly as its tension, and inversely as the radius of the pulley. In the following formulæ for leather belts, the stiffness is given as estimated by Reuleaux (*Constructionslehre für Maschinenbau*, § 307).

Let T be the mean tension of the belt; S, its sectional area; r, the radius of the pulley; b, a constant divisor determined by experiment; R′, the resistance due to stiffness; then

$$R' = \frac{S\,T}{b\,r}. \quad\text{.........................(1.)}$$

b (for leather) = 3·4 inch = 87 millimètres.

To apply this to an endless belt connecting a pair of pulleys of the respective radii r_1 and r_2, let T_1 and T_2 be the tensions of the two sides of the belt, as determined by the rule of Article 310 A, page 351. Then the useful resistance is $T_1 - T_2$; the mean tension is $\frac{T_1 + T_2}{2}$; and the additional resistance due to stiffness is

$$\frac{T_1 + T_2}{2} \cdot \frac{S}{b} \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\};$$

consequently the counter-efficiency is

$$\left.\begin{aligned} c &= 1 + \frac{T_1 + T_2}{2\,(T_1 - T_2)} \cdot \frac{S}{b} \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\}; \\ &= 1 + \frac{N + 1}{2\,(N - 1)} \cdot \frac{S}{b} \left\{ \frac{1}{r_1} + \frac{1}{r_2} \right\}; \end{aligned}\right\} \quad\text{............(2.)}$$

N denoting $\frac{T_1}{T_2}$, as in Article 374, page 432. The sectional area, S, of a leather belt is given by the formula

$$S = \frac{T_1}{p}; \quad\text{..............................(3.)}$$

where p denotes the safe working tension of leather belts, in units of weight per unit of area; its value being, according to Morin,

0·2 kilogramme on the square millimètre, or
285 lbs. on the square inch.

The ordinary thickness of the leather of which belts are made is about 0·16 of an inch, or 4 millimètres; and from this and from the area the breadth may be calculated. A double belt is of double thickness, and gives the same area with half the breadth of a single belt.

When a band runs at a high velocity, the *centrifugal tension*, or tension produced by centrifugal force, must be added to the tension required for producing friction on the pulleys, in order to find the total tension at either side of the band, with a view to determining its sectional area and its stiffness. The centrifugal tension is given by the following expression :—

$$\frac{w\,S\,v^2}{g}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

in which w is the heaviness (being, for leather belts, nearly equal to that of water); S, the sectional area; v, the velocity; and g, gravity (= 32·2 feet, or 9·81 metres per second).

When centrifugal force is taken into consideration, the following formula is to be used for calculating the sectional area; T_1 being the tension at the driving-side of the belt, as calculated by the rules of Article 310 A, page 351, *exclusive of centrifugal tension*:—

$$S = \frac{T_1}{p - \frac{w\,v^2}{g}}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

and the following formula for the counter-efficiency :—

$$c = 1 + \frac{T_1 + T_2 + \frac{2\,w\,v^2}{g}}{2\,(T_1 - T_2)} \cdot \frac{S}{b} \cdot \left\{\frac{1}{r_1} + \frac{1}{r_2}\right\} \ldots\ldots(6.)$$

The questions of areas of bands and centrifugal tension will be further considered in the part of this treatise relating to the strength of machinery.

For calculating the efficiency of hempen ropes used as bands, it is unnecessary in such questions as that of the present Article to use a more complex formula than that of Eytelwein—viz.,

$$R' = \frac{D^2\,T}{b'\,r}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.)$$

where D is the diameter of the rope, and b' = 54 millimètres = 2·125 inches.

In all the formulæ, $\frac{D^2}{b'}$ is to be substituted for $\frac{S}{b}$. The proper value of D^2 is given by the formula

$$D^2 = \frac{T_1}{p'}; \text{ .... } (8.)$$

where $p' = 1000$ for measures in inches and lbs.; and
$p' = 0{\cdot}7$ for measures in millimètres and kilogrammes.

382. **Efficiency of Linkwork.**—In fig. 266, let $C_1 T_1$, $C_2 T_2$ be two levers, turning about parallel axes at C_1 and C_2, and connected with each other by the link $T_1 T_2$; T_1 and T_2 being the connected points.

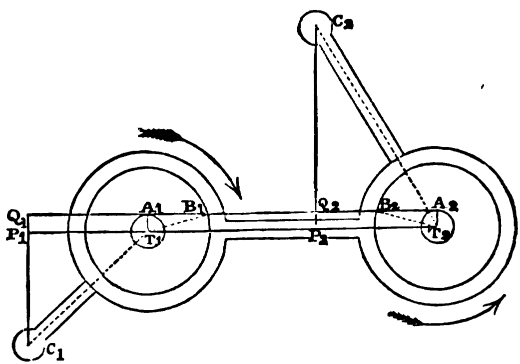

Fig. 266.

The pins, which are connected with each other by means of the link, are exaggerated in diameter, for the sake of distinctness. Let $C_1 T_1$ be the driver, and $C_2 T_2$ the follower, the motion being as shown by the arrows. From the axes let fall the perpendiculars $C_1 P_1$, $C_2 P_2$, upon the line of connection. Then the angular velocities of the driver and follower are inversely as those perpendiculars; and, in the absence of friction, the driving moment of the first lever and the working moment of the second are directly as those perpendiculars; the driving pressure being exerted along the line of connection $T_1 T_2$. Let M_2 be the working moment; and let M_0 be the driving moment in the absence of friction; then we have

$$M_0 = \frac{M_2 \cdot C_1 P_1}{C_2 P_2}.$$

To allow for the friction of the pins, multiply the radius of each pin by the sine of the angle of repose; that is, very nearly by the co-efficient of friction; and with the small radii thus computed, $T_1 A_1$ and $T_2 A_2$, draw small circles about the connected points. Then draw a straight line, $Q_1 A_1 B_1 Q_2 B_2 A_2$, touching both the small circles, and in such a position as to represent the line of action of a force that resists the motion of both pins in the eyes of the link. This will be the line of action of the resultant force exerted through the link. Let fall upon it the perpendiculars $C_1 Q_1$, $C_2 Q_2$; these will be proportional to the actual driving moment and working moment respectively; that is to say, let M_1 be the driving moment, including friction; then

$$M_1 = \frac{M_2 \cdot C_1 Q_1}{C_2 Q_2}.$$

Comparing this with the value of the driving moment without friction, we find for the counter-efficiency

$$c = \frac{M_1}{M_0} = \frac{C_1 Q_1 \cdot C_2 P_2}{C_2 Q_2 \cdot C_1 P_1}; \qquad (1.)$$

and for the efficiency

$$\frac{1}{c} = \frac{M_0}{M_1} = \frac{C_2 Q_2 \cdot C_1 P_1}{C_1 Q_1 \cdot C_2 P_2}. \qquad (2.)$$

(See page 449.)

383. **Efficiency of Blocks and Tackle.** (See Articles 200, 201, pages 214 to 216.)—In a tackle composed of a fixed and a running block containing sheaves connected together by means of a rope, let the number of plies of rope by which the blocks are connected with each other be n. This is also the collective number of sheaves in the two blocks taken together, and is the number expressing the *purchase*, when friction is neglected.

Let c denote the counter-efficiency of a single sheave, as depending on its friction on the pin, according to the principles of Article 373, page 427. Let c' denote the counter-efficiency of the rope, when passing over a single sheave, determined by the principles of Article 381, the tension being taken as nearly equal to $\frac{R}{n}$; where R is the useful load, or resistance opposed to the motion of the running block. $R \div n$ is also the effort to be exerted on the hauling part of the rope, in the absence of friction. Then the counter-efficiency of the tackle will be expressed approximately by

$$(c\, c')^n; \qquad (1.)$$

so that the actual or effective purchase, instead of being expressed by n, will be expressed by

$$n\,(c\, c')^{-n}. \qquad (2.)$$

384. **Efficiency of Connection by means of a Fluid.**—When motion is communicated from one piston to another by means of an intervening mass of fluid, as described in Articles 207 to 210, pages 221 to 224, the efficiencies and counter-efficiencies of the two pistons have in the first place to be taken into account; which quantities are to be determined by means of the principles stated at page 399; that is to say, with ordinary workmanship and packing, the efficiency of each piston may be taken at 0·9 nearly; while with a carefully made cupped leather collar the counter-efficiency of a plunger may be taken at the following value:—

$$1 - \frac{4\,b}{d}; \qquad \text{(1.)}$$

in which d is the diameter of the plunger; and b a constant, whose value is from 0·01 to 0·015 of an inch, or from 0·25 to 0·38 of a millimètre. For if c be the circumference of the plunger, and p the effective pressure of the liquid, the whole amount of the pressure on the plunger is $\frac{p\,c\,d}{4}$; and the pressure required to overcome the friction is $p\,c\,b$.

The efficiency and counter-efficiency of the intervening mass of fluid remain to be considered; and if that fluid is a liquid, and may therefore be regarded as sensibly incompressible, these quantities depend on the work which is lost in overcoming the resistance of the passage which the liquid has to traverse.

To prevent unnecessary loss of work, that passage should be as wide as possible, and as nearly as possible of uniform transverse section; and it should be free from sudden enlargements and contractions, and from sharp bends, all necessary enlargements and contractions which may be required being made by means of gradually tapering conoidal parts of the passage, and all bends by means of gentle curves. When those conditions are fulfilled, let Q be the volume of liquid which is forced through the passage in a second; S, the sectional area of the passage; then,

$$v = \frac{Q}{S}, \qquad \text{(2.)}$$

is the velocity of the stream of fluid. Let b denote the wetted border or circumference of the passage; then,

$$m = \frac{S}{b}, \qquad \text{(3.)}$$

is what is called the *hydraulic mean depth* of the passage. In a cylindrical pipe, $m = \frac{1}{4}$ diameter. Let l be the length of the passage, and w the heaviness of the liquid. Then the *loss of pressure* in overcoming the friction of the passage is

$$p' = \frac{f\,l}{m} \cdot \frac{w\,v^2}{2\,g};\text{.......................}(4.)$$

in which g denotes gravity, and f a co-efficient of friction whose value, for water in cylindrical cast-iron pipes, according to the experiments of Darcy, is

$$f = 0{\cdot}005\left(1 + \frac{1}{12\,d}\right);^{*}\text{.................}(5.)$$

d being the diameter of the pipe in feet.

Let p be the pressure on the driven or following piston; then the pressure on the driving piston is $p + p'$; and the *counter-efficiency of the fluid* is

$$1 + \frac{p'}{p};\text{...........................}(6.)$$

which, being multiplied by the product of the counter-efficiencies of the two pistons, gives the *counter-efficiency of the intervening liquid.*

When the intervening fluid is *air*, there is a loss of work through friction of the passage, depending on principles similar to those of the friction of liquids; and there is a further loss through the escape by conduction of the heat produced by the compression of the air.

The friction which has to be overcome by the air, and which causes a certain loss of pressure between the compressing pumps and the working machinery, consists of two parts, one occasioned by the resistance of the valves, and the other by the friction along the internal surface of pipes.

To overcome the resistance of valves, about *five per cent.* of the effective pressure may be allowed.

The friction in the pipes depends on their length and diameter, and on the velocity of the current of air through them. It is nearly proportional to the square of the velocity of the air.

A velocity of about *forty feet per second* for the air in its compressed state has been found to answer in practice. The diameter of pipe required in order to give that velocity can easily be computed, when the dimensions of the cylinders of the machinery to be driven, and the number of strokes per minute, are given.

When the diameter of a pipe is so adjusted that the velocity of the air is 40 feet per second, the pressure expended in overcoming

* When the diameter is expressed in millimètres, for $\frac{1}{12\,d}$ substitute $\frac{25{\cdot}4}{d}$.

its friction may be estimated at *one per cent. of the total or absolute pressure of the air, for every five hundred diameters of the pipe that its length contains.*

Although the abstraction from the air of the heat produced by the compression involves a certain sacrifice of motive power (say from 30 to 35 per cent.) still the effects of the heated air are so inconvenient in practice, that it is desirable to cool it to a certain extent during or immediately after the compression. This may be effected by injecting water in the form of spray into the compressing pumps; and for that purpose a small forcing pump of about $\frac{1}{100}$th of the capacity of the compressing pumps has been found to answer in practice. The air may be thus cooled down to about 104° Fahr. or 40° Cent.

The factor in the counter-efficiency due to the loss of heat expresses the ratio in which the volume of air as discharged from the compressing pump at a high temperature is greater than the volume of the same air when it reaches the working machinery at a reduced temperature; which ratio may be calculated approximately by taking *two-sevenths of the logarithm of the absolute working pressure of the compressed air in atmospheres, and finding the corresponding natural number.* That is to say, let p_0 denote one atmosphere (= at the level of the sea 14·7 lbs. on the square inch, or 10333 kilogrammes on the square mètre); let p_1 be the absolute working pressure of the air, so that $p_1 - p_0$ is the effective pressure; then the counter-efficiency due to the escape of heat is,

$$c = \left(\frac{p_1}{p_0}\right)^{\frac{2}{7}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.)$$

From examples of the practical working of compressed air, when used to transmit motive power to long distances, it appears that in order to provide for leakage and various other imperfections in working, the capacity of the compressing pumps should be very nearly double of the net volume of uncompressed air required; and it has also been found necessary, in working the compressing pumps, to provide from three to four times the power of the machinery driven by the compressed air.

Addendum to Article 343, Page 386.

Rotatory Dynamometers—Epicyclic-Train Dynamometer.—The term of "epicyclic-train dynamometers" may be applied to those instruments in which the power to be measured is transmitted through an epicyclic train, and the effort exerted is measured by means of the force required to *hold the train-arm at rest.* In King's dynamometer, for example, there is a train of wheel-work of which the principle (though not the details) is sufficiently well represented by fig. 176, page 245. The bevel-wheel B is driven by the prime mover; and through the bevel-wheels (or bevel-wheel, there being usually only one) carried by the arm A, it drives the bevel-wheel C, which drives the working machinery. The train-arm A is kept steady by a weight, or by a spring; and it is obvious that the moment of that force relatively to the common axis of rotation of B, C, and A, must be *double* of the moment transmitted from B to C; which latter moment—that is, half the moment of the weight or spring that holds A steady, being multiplied by $2\pi\times$ the number of turns in a given time, gives the work done in that time. This apparatus may be made to record its results on a travelling strip of paper, like other kinds of dynamometers.

Addenda to Article 381, Page 440.

I. **Use of Relieving Rollers between Pulleys.**—When a pair of pulleys connected with each other by means of a band are near together, the bearings of their shafts may be relieved from the pressure due to the tension of the band by placing between the pulleys a smooth idle wheel or roller, turning in rolling contact with them both. The axis of rotation of the roller should be in the same plane with those of the pulleys; and two out of the three shafts should have their bearings so fitted up as to be capable of a small extent of motion in a direction perpendicular to the axes of rotation, in order that the distances of those axes from each other may adjust themselves when the band is tightened, and that the tension of the band and the pressure transmitted through the roller may balance each other without the aid of pressures at the bearings.

II. **Efficiency of Telodynamic Transmission.**—The phrase "Telodynamic Transmission" is used to denote Mr. C. F. Hirn's method of transmitting motive power to long distances by means of an endless wire rope, connecting a pair of large pulleys, and moving at a high speed. The pulleys are made of cast iron; and each of them has at the bottom of its groove a dovetail-shaped recess filled with gutta-percha, which is driven in and rammed tight by means of a mallet; the wire rope bears against the gutta-percha bottom of the groove; and this is found both to transmit an effort better, and to ensure greater durability of the rope and pulleys, than when the rope bears against a cast-iron surface.

The ordinary speed of the rope is from 50 to 80 feet per second; and with wrought-iron pulleys, it is considered that it might be increased to 100 feet per second. The effort to be transmitted is calculated from the power to be transmitted, by expressing that power in units of work per second, and dividing by the speed. The available tensions at the driving and returning sides of the rope are calculated by the rules of Article 310 A, page 351; in practice it is considered sufficiently accurate to make the former *twice*, and the latter *once*, the effort to be transmitted. To each of those tensions is to be added the centrifugal tension (see Article 381, page 441) in order to obtain the total tensions. The transverse dimensions of the rope are adapted to the *total tension at the driving side of the rope*, by the application of rules to be given in the Part of this Treatise relating to strength.

In order that the rope may not be overstrained by the bending of the wires of which it consists, in passing round the driving and following pulleys, the diameter of each of those pulleys should not be less than 140 times the diameter of the rope, and is sometimes as much as 260 times.

The distance between the driving and following pulleys is not made less than about 100 feet; for at less distances shafting is more efficient; nor is it made more than 500 feet in one span, because of the great depth of the catenary curves in which the rope hangs. When the distance between the driving and following pulleys exceeds 500 feet, the rope is supported at intermediate points by pairs of bearing pulleys, so as to divide the whole distance into intervals of 500 feet or less.

The bearing pulleys are constructed in the same way with the driving and following pulleys, and of about half the diameter.

The loss of work due to the stiffness of the rope may be regarded as insensible; because when the diameters of the pulleys are sufficient, the wires of which the rope is made straighten themselves by their own elasticity after having been bent.

It has been found by practical experience that the losses of power in this apparatus are nearly as follows, in fractions of the whole power transmitted:—

Overcoming the axle-friction of the driving and following pulleys, about $\frac{1}{40}$, or	0·0250
Overcoming the axle-friction of each pair of bearing pulleys, about $\frac{1}{900}$, or	0·0011

Hence the efficiency of telodynamic transmission may be estimated at

$$0{\cdot}975 - \frac{N}{900};$$

N being the number of pairs of intermediate bearing pulleys.*

* For detailed information on the subject of Telodynamic Transmission, see the following authorities:—*Notice sur la Transmission Telodynamique,*

Addendum to Article 382, Page 442.

Effect of Obliquity of a Connecting-Rod on Friction.—The alternate thrust and tension along the connecting-rod is almost always an important component, and sometimes the most important component, of the force which is balanced by the pressure of the bearings of a crank-shaft; and the lateral component of that alternate thrust and tension is the cause of the friction of the guides by which the head of the piston-rod is made to move in a straight line, when there is no parallel motion.

The direction of the connecting-rod is continually changing between certain limits; and this causes a continual change in the ratio borne by the whole force exerted along that rod, and by its lateral component, to its direct component.

Let r be the crank-arm, c the length of the connecting-rod; then the *mean* value of the ratio which the lateral component bears to the direct component is very nearly as follows:—

$$\frac{Q}{P} = \frac{0{\cdot}7854\, r}{\sqrt{(c^2 - 0{\cdot}617\, r^2)}}; \qquad (1.)$$

and if f be the co-efficient of friction of the guides, the counter-efficiency of the piston-rod head will be nearly

$$c = 1 + \frac{fQ}{P}. \qquad (2.)$$

The mean ratio borne by the total force (T) exerted along the connecting-rod to its direct component (P) is nearly as follows:—

$$\frac{T}{P} = \frac{c}{\sqrt{(c^2 - 0{\cdot}617\, r^2)}}; \qquad (3.)$$

and the axle-friction of the crank-shaft is increased nearly in that ratio, beyond what it would be if the obliquity of the connecting-rod were insensible.*

par C. F. Hirn (Colmar, 1862). Reuleaux, *Constructionslehre für Maschinenbau* (Braunschweig, 1854 to 1862), §§ 324 to 342.

* The exact solution of these questions is given by the aid of elliptic functions; but for practical purposes the approximate solution in the text is sufficient.

PART III.

MATERIALS, CONSTRUCTION, AND STRENGTH OF MACHINERY.

CHAPTER I.

OF MATERIALS USED IN MACHINERY.

385. **General Explanations.**—The materials used in machinery are of two principal kinds—inorganic and organic.

The inorganic materials consist almost wholly of metals; for although stony and earthy materials occur in the foundations of fixed machines, and in houses which contain machinery, they are little used in machinery itself.

The organic materials consist chiefly of wood, of vegetable and animal fibre in the form of ropes and bands, and of indian rubber and gutta percha, and a few miscellaneous substances.

The present chapter gives a summary of those properties of materials upon which their use in machinery and millwork depends; and it is necessarily to a great extent identical with those parts of *A Manual of Civil Engineering* which treat of the same materials.

SECTION I.—*Of Iron and Steel.*

386. **Kinds of Iron and Steel.**—The metallic products of the iron manufacture are of three principal kinds—*malleable iron*, *cast iron*, and *steel*. Malleable iron is pure or nearly pure iron. Cast iron is a granular and crystalline compound of iron and carbon, more or less mixed with uncombined carbon in the form of plumbago. It is harder then pure iron, more brittle, and less tough. Steel is a compound of iron with less carbon than there is in cast iron; it is harder than cast iron, and tougher than wrought iron, though less ductile; and it is the strongest of all known substances *for its dimensions*. It is also the strongest of all metals *for its weight;* but in the comparison of tenacity with weight, steel and all metals are exceeded by many kinds of organic fibre. There are many intermediate gradations between pure iron and the hardest steel, some of which are known by such names as "steely iron" and "semi-steel."

387. **Impurities of Iron.**—The strength and other good qualities of iron and steel depend mainly on the absence of impurities, and especially of sulphur, phosphorus, silicon, calcium, and magnesium.

Sulphur and calcium, and probably also magnesium, make iron "*red-short*," that is, brittle at a red heat; phosphorus and silicon make it "*cold-short*," that is, brittle at low temperatures. These are both serious defects; but the latter is the worse.

Sulphur comes in general from coal or coke used as fuel. Its pernicious effects can be avoided altogether by using fuel which contains no sulphur; and hence the strongest and toughest of all iron is that which is smelted, reduced, and puddled either with charcoal, or with coke that is free from sulphur.

Phosphorus comes in most cases from phosphate of iron in the ore, or from phosphate of lime in the ore, the fuel, or the flux. The ores which contain most phosphorus are those found in strata where animal remains abound.

Calcium and *Silicon* are derived respectively from the decomposition of lime and of silica by the chemical affinity of carbon for their oxygen. The only iron which is entirely free from those impurities is that which is made by the reduction of ores that contain neither silica nor lime: such as pure magnetic iron ore, pure hæmatite, and pure sparry iron ore.

388. **Cast Iron** is the product of the process of *smelting* iron ores. In that process the ore in fragments, mixed with fuel and with flux (that is to say, with a substance such as lime, which tends to combine with the earthy constituents of the ore), is subjected to an intense heat in a blast-furnace, and the products are *slag*, or glassy matter formed by the combination of the flux with the earthy ingredients of the ore, and *pig iron*, which is a compound of iron and carbon, either unmixed, or mixed with a small quantity of uncombined carbon in a state of plumbago.

The ore is often *roasted* or calcined before being smelted, in order to expel carbonic acid and water.

The total quantity of carbon in pig iron ranges from two to five per cent. of its weight.

Different kinds of pig iron are produced from the same ore in the same furnace under different circumstances as to temperature and quantity of fuel. A high temperature and a large quantity of fuel produce *gray cast iron*, which is further distinguished into No. 1, No. 2, No. 3, and so on; No. 1 being that produced at the highest temperature. A low temperature and a deficiency of fuel produce *white cast iron*. Gray cast iron is of different shades of bluish-gray in colour, granular in texture, softer and more easily fusible than white cast iron. White cast iron is silvery white, either granular or crystalline, comparatively difficult to melt, brittle, and excessively hard.

It appears that the differences between those kinds of iron depend not so much on the total quantities of carbon which they contain as on the proportions of that carbon which are respectively in the conditions of mixture and of chemical combination with the iron. Thus, gray cast iron contains *one* per cent., and sometimes less, of carbon in chemical combination with the iron, and from *one* to *three* or *four* per cent. of carbon in the state of plumbago in mechanical mixture; while white cast iron is a homogeneous chemical compound of iron with from two to four per cent. of carbon. Of the different kinds of gray cast iron, No. 1 contains the greatest proportion of plumbago, No. 2 the next, and so on.

There are two kinds of white cast iron, the *granular* and the *crystalline*. The granular kind can be converted into gray cast iron by fusion and slow cooling; and gray cast iron can be converted into granular white cast iron by fusion and sudden cooling. This takes place most readily in the best iron. Crystalline white cast iron is harder and more brittle than granular, and is not capable of conversion into gray cast iron by fusion and slow cooling. Gray cast iron, No. 1, is the most easily fusible, and produces the finest and most accurate castings; but it is deficient in hardness and strength; and therefore, although it is the best for castings of moderate size, in which accuracy is of more importance than strength and stiffness, it is inferior to the harder and stronger kinds, No. 2 and No. 3, for pieces requiring great strength and stiffness.

The presence of plumbago renders cast iron comparatively weak and pliable, so that the order of strength and stiffness among different kinds of cast iron from the same ore and fuel is as follows:—

Granular white cast iron.
Gray cast iron, No. 3.
" " No. 2.
" " No. 1.

Crystalline white cast iron is not introduced into this classification because its extreme brittleness makes it unfit for use in machinery.

Granular white cast iron, also, although stronger and harder than gray cast iron, is too brittle to be a safe material for the entire mass of any piece in a machine that is exposed to shocks; but it is used to form a hard and impenetrable *skin* to a piece of gray cast iron by the process called *chilling*. This consists in lining the portion of the mould, where a hardened surface is required, with suitably-shaped pieces of iron. The melted metal, on being run in, is cooled and solidified suddenly where it touches the cold iron; and for a certain depth from the chilled surface, varying from ⅛th to ½-inch in different kinds of iron, it takes the white granular condition, while the remainder of the casting takes the gray condition.

Even in castings which are not chilled by an iron lining to the mould, the outermost layer, being cooled more rapidly than the interior, approaches more nearly to the white condition, and forms a *skin*, harder and stronger than the rest of the casting.

A strong kind of cast iron called *toughened cast iron*, is produced by the process invented by Mr. Morries Stirling, of adding to the cast iron, and melting amongst it, from one-fourth to one-seventh of its weight of wrought-iron scrap.

Malleable Cast Iron is made by the following process:—The castings to be made malleable are imbedded in the powder of red hæmatite (which consists almost wholly of peroxide of iron); they are then raised to a bright red heat (which occupies about 24 hours), maintained at that heat for a period varying from three to five days, according to the size of the casting, and allowed to cool (which occupies about 24 hours more). The oxygen of the hæmatite extracts part of the carbon from the cast iron, which is thus converted into a sort of soft steel; and its tenacity (according to experiments by Messrs. A. More & Son) becomes about three times that of the original cast iron.

389. The **Strength of Cast Iron** of every kind, like that of granular substances in general, is marked by two properties: the smallness of the tenacity (which is on an average about 16,000 or 18,000 lbs. on the square inch) as compared with the resistance to crushing (which ranges from 80,000 to 110,000), and the different values of the stress immediately before rupture of the same kind of iron in bars torn directly asunder, and in beams of different forms when broken across.

For the results of experiments on the strength of various kinds of cast iron, see the tables of the following chapter.

The strength of cast iron to resist cross breaking was found by Mr. Fairbairn to be increased by *repeated meltings* up to the *twelfth*, when it was greater than at first in the ratio of 7 to 5 nearly. After the twelfth melting that sort of strength rapidly fell off.

The resistance to crushing went on increasing after each successive melting; and after the eighteenth melting it was double of its original amount, the iron becoming silvery white and intensely hard.

The transverse strength of No. 3 cast iron was found by Mr. Fairbairn not to be diminished by raising its temperature to 600° Fahr. (being about the temperature of melting lead). At a red heat its strength fell to two-thirds.

390. **Castings for Machinery.**—The best course for an engineer to take, in order to obtain cast iron of a certain strength, is not to

specify to the founder any particular kind or mixture of pig iron, but to specify a certain minimum strength which the iron should show when tested by experiment.

As to the appearance of good iron for castings, it should have on the outer surface a smooth, clear, and continuous skin, with regular faces and sharp angles. When broken, the surface of fracture should be of a light bluish-gray colour and close-grained texture, with considerable metallic lustre; both colour and texture should be uniform, except that near the skin the colour may be somewhat lighter and the grain closer; if the fractured surface is *mottled*, either with patches of darker or lighter iron, or with crystalline spots, the casting will be unsafe; and it will be still more unsafe if it contains air-bubbles. The iron should be soft enough to be slightly indented by a blow of a hammer on an edge of the casting. When cut by tools of different kinds, the iron should show a smooth, compact, and bright surface, free from bubbles and other irregularities, of an uniform colour, and capable of taking a good polish.

Castings are tested for air-bubbles by ringing them with a hammer all over the surface.

Cast iron, like many other substances, when at or near the temperature of fusion, is a little more bulky for the same weight in the solid than in the liquid state, as is shown by the solid iron floating on the melted iron. This causes the iron as it solidifies to fill all parts of the mould completely, and to take a sharp and accurate figure. The solid iron contracts in cooling from the melting point down to the temperature of the atmosphere, by about one per cent. in each of its linear dimensions, or *one-eighth of an inch in a foot nearly;* and therefore patterns for castings are made larger in that proportion than the intended pieces of cast iron which they represent.

The rate of linear expansion of cast iron between the freezing and boiling points of water is about ·00111.

A convenient instrument in making patterns for castings is a *contraction-rule;* that is, a rule on which each division is longer in the proportion already mentioned than the true length to which it corresponds.

In designing patterns for castings, care must be taken to avoid all abrupt variations in the thickness of metal, lest parts of the casting near each other should be caused to cool and contract with unequal rapidity, and so to split asunder or overstrain the iron. It is advantageous also that castings, especially those for moving pieces in machinery, such as wheels, should be of symmetrical figures, or as nearly so as is consistent with their purposes, in order that they may have no tendency to become distorted while cooling.

Iron becomes more compact and sound by being cast under pressure; and hence cast-iron cylinders, pipes, columns, shafts, and the like, are stronger when cast in a vertical than in a horizontal position, and stronger still when provided with a *head*, or additional column of iron, whose weight serves to compress the mass of iron in the mould below it. The air-bubbles ascend and collect in the head, which is broken off when the casting is cool.

Care should be taken not to cut or remove the skin of a piece of cast iron more than is absolutely necessary, at those points where the stress is intense. In order that this rule may be carried out in pieces (such as toothed wheels) which are shaped to an accurate figure by cutting or abrading tools, care should be taken to make them as nearly as practicable of the true figure by casting alone, so that the depth of skin to be cut away may be as small as possible.

391. **Wrought or Malleable Iron** in its perfect condition is simply pure iron. It falls short of that perfect condition to a greater or less extent owing to the presence of impurities, of which the most common and injurious have been mentioned, and their effects stated, in Article 387, page 451; and its strength is in general greater or less according to the greater or less purity of the ore and fuel employed in its manufacture.

Malleable iron may be made either by direct reduction of the ore or by the abstraction of the carbon and various impurities from pig iron, mainly by means of oxygen. The latter is the more common process; and the ordinary method of carrying it on, by stirring the iron in a reverberatory furnace, is called *puddling*. The oxygen which carries off the carbon in the process of puddling comes partly from the air and partly from a bed of cinder and oxide of iron, called the *fettling*, with which the bottom of the furnace is covered. The *bloom*, or lump of iron drawn from the puddling furnace, is hammered, to drive out the *cinder* with which it is mixed—a compound of silica and protoxide of iron; it is then rolled into bars, which are cut into lengths, fagotted into bundles, re-heated, and re-rolled, until bars are obtained of the required dimensions. The *fibrous* structure of bar iron is owing to the process of fagotting and rolling, by which it is made. In Mr. Bessemer's process a blast of air is blown through the molten iron, in a large vessel or retort, until the carbon and silicon are oxidized and removed.

Strength and toughness in bar iron are indicated by a fine, close, and uniform fibrous structure, free from all appearance of crystallization, with a clear, bluish-gray colour and silky lustre on a torn surface where the fibres are shown.

Plate iron of the best kind consists of alternate layers of fibres crossing each other. It should have a hard, smooth skin, somewhat glossy, and when broken, should show perfect uniformity of structure, and be free from all tendency to split into layers.

To examine the internal structure of iron, whether in bars or in plates, a short piece may be notched on one side, near the middle, and bent double. During this process the uncut part should not break; and if the iron gives way at all, it should do so by splitting along the fibres near the bottom of the notch. The fitness of bar iron for structures, machines, and smithwork of different kinds, is tested by bending and punching it cold, and by punching and forging it hot, so as to ascertain whether it shows any signs of brittleness either when cold or when hot (called "cold-short" and "red-short").*

Malleable iron is distinguished by the property of *welding:* two pieces, if raised nearly to a white heat, and pressed or hammered firmly together, adhering so as to form one piece. In all operations of which welding forms a part, such as rolling and forging, it is essential that the surfaces to be welded should be brought into close contact, and should be perfectly clean and free from oxide of iron, cinder, and all foreign matter.

In all cases in which several bars are to be fagotted and hammered, or rolled into one, attention should be paid to the manner in which they are "piled" or built together, so that the pressure exerted by the hammer or the rollers may be transmitted through the whole mass. If this be neglected, the finished bar, plate, or other piece, may show flaws, marking the divisions between the bars of the pile.

Wrought iron, although it is at first made more compact and strong by *reheating* and hammering, or otherwise working it, soon reaches a state of maximum strength; after which all reheating and working rapidly make it weaker. Good bar iron has in general attained its maximum strength; and, therefore, in all operations of forging it, whether on a great or small scale, by the steam-hammer or by that in the hand of the blacksmith, the desired size and figure ought to be given with the least possible amount of reheating and working.

It is of great importance to the strength of all pieces of forged iron that the *continuity of the fibres* near the surface should be as little interrupted as possible; in other words, that the fibres near the surface should lie in layers parallel to the surface.†

* For full information as to the tests to which iron and steel are subjected by the Admiralty regulations, reference may be made to Chapter xviii. of the *Treatise on Shipbuilding*, by E. J. Reed, Esq., C.B., Chief Constructor of the Royal Navy.

† On this subject, see a Paper by the Author of this work, in the *Proceedings of the Institution of Civil Engineers* for 1843. See also the *Transactions of the Institution of Engineers in Scotland* for 1862-63, pages 37, 41, 43.

Another important principle in designing pieces of forged iron which are to sustain shocks and vibrations, is to avoid as much as possible abrupt variations of dimensions, and angular figures, especially those with re-entering angles; for at the points where such abrupt variations and angles occur, fractures are apt to commence. If two parts of a shaft, for example, or of a beam exposed to shocks and vibrations, are to be of different thicknesses, they should be connected by means of curved surfaces, so that the change of thickness may take place gradually, and without re-entering angles.

392. **Steel and Steely Iron.**—Steel is a compound of iron with from 0·5 to 1·5 per cent. of its weight of carbon. These, according to most authorities, are the only essential constituents of steel.

The term "steely iron" or "semi-steel" may be applied to compounds of iron with less than 0·5 per cent. of carbon. They are intermediate in hardness and other properties between steel and malleable iron.

In general such compounds are the harder and the stronger, and also the more easily fusible, the more carbon they contain. Those kinds which contain less carbon, though weaker, are more easily welded and forged, and from their greater pliability, are the fitter for pieces that are exposed to shocks.

Impurities of different kinds affect steel injuriously in the same way with iron.

There are certain foreign substances which have a beneficial effect on steel. One 2,000th part of its weight of silicon causes molten steel to cool and solidify without bubbling or agitation; but a larger proportion is not to be used, as it would make the steel brittle. The presence of manganese in the iron, or its introduction into the crucible or vessel in which steel is made, improves the steel by increasing its toughness and making it easier to weld and forge.

Steel is distinguished by the property of *tempering;* that is to say, it can be hardened by sudden cooling from a high temperature, and softened by gradual cooling; and its degree of hardness or softness can be regulated with precision by suitably fixing that temperature. The ordinary practice is, to bring all articles of steel to a high degree of hardness by sudden cooling, and then to soften them more or less by raising them to a temperature which is the higher the softer the articles are to be made, and letting them cool very gradually. The elevation of temperature previous to the *annealing* or gradual cooling is produced by plunging the articles into a bath of a fusible metallic alloy. The temperaure of the bath ranges from 430° to 560° Fahr.

According to the experiments of Mr. Kirkaldy, a great increase of strength is produced by hardening steel in oil.

Steel is made by various processes, which have of late become very numerous. They may all be classed under two heads—viz., adding carbon to malleable iron, and abstracting carbon from cast iron. The former class of processes, though the more complex, laborious, and expensive, is preferred for making steel for cutting tools and other fine purposes, because of its being easier to obtain malleable iron than cast iron in a high state of purity. The latter class of processes is well adapted for making great masses of steel and steely iron rapidly and at moderate expense. The following are some of the different kinds of steel, and the processes by which they are made:—

Blister Steel is made by a process called "*cementation*," which consists in imbedding bars of the purest wrought iron (such as that manufactured by charcoal from magnetic iron ore) in a layer of charcoal, and subjecting them for several days to a high temperature. Each bar absorbs carbon, and its surface becomes converted into steel, while the interior is in a condition intermediate between steel and iron. Cementation may also be performed by exposing the surface of the iron to a current of carburetted hydrogen gas at a high temperature. Cementation is sometimes applied to the surfaces of articles of malleable iron, in order to give them a skin or coating of steel, and is called "*case-hardening*."

Shear Steel is made by breaking bars of blister steel into lengths, making them into bundles or fagots, and rolling them out at a welding heat, and repeating the process until a near approach to uniformity of composition and texture has been obtained. It is used for various tools and cutting implements.

Cast Steel is made by melting bars of blister steel in a crucible, along with a small additional quantity of carbon (usually in the form of coal-tar) and some manganese. It is the purest, most uniform, and strongest steel, and is used for the finest cutting implements.

Another process for making cast steel, but one requiring a higher temperature than the preceding, is to melt bars of the purest malleable iron with manganese, and with the whole quantity of carbon required in order to form steel. The quality of the steel, as to hardness, is regulated by the proportion of carbon. A sort of semi-steel, or steely iron, made by this process, and containing a small proportion of carbon only, is known as *homogeneous metal*.

The making of large masses of steel by adding the proper ingredients to liquid malleable iron has been much facilitated by the use of Siemens's regenerative furnace, which enables a very high temperature to be kept up, with an ease and economy unknown before.

Steel made by the air-blast is produced from molten pig iron by Mr. Bessemer's process. In the first place, the carbon is removed by the air-blast, so that the vessel is full of pure malleable iron in the melted state; and then carbon is added in the proper proportion, along with manganese and silicon. The usual way of adding the carbon is by running into the vessel a sufficient quantity of a compound called "spiegeleisen," consisting of highly carbonized cast iron and manganese. The steel thus produced is run into large ingots, which are hammered and rolled like blooms of wrought iron.

Puddled Steel is made by puddling pig iron, and stopping the process at the instant when the proper quantity of carbon remains. The bloom is shingled and rolled like bar iron.

The broken surface of a piece of steel shows a mass of very small crystalline grains, finer than those of cast iron. Uniformity in the size and colour of the grains is a mark of good steel; and the smaller they are, the finer and the harder is its quality. In fine cast steel the grains are so small as not to be separately distinguishable by the naked eye; and the fracture presents a smooth but dull surface, of an uniform slate-gray colour.

As to expansion by heat, see page 326.

393. **Strength of Wrought Iron and Steel.**—The numerical results of experiments on the strength of wrought iron and steel will be found in the tables between this chapter and the next.

Wrought iron, like fibrous substances in general, is more tenacious along than across the fibres; and its tenacity, or resistance to tearing asunder, is greater than its resistance to crushing, except when in the form of blocks whose lengths are less than, or but little greater than, their diameters.

The ductility of wrought iron often causes it to yield by degrees to a load, so that it is difficult to determine its strength with precision.

Wrought iron has its longitudinal tenacity considerably increased by rolling and wire-drawing; so that the smaller sizes of bars are on the whole more tenacious than the larger; and iron wire is more tenacious still, as is shown in the Tables.

Wrought iron is weakened by too frequent reheating and forging; so that, even in the best of large forgings, the tenacity is only about *three-fourths* of that of the bars from which the forgings were made, and sometimes even less.

The strength both of iron and steel is injured by the action of tools which overstrain the particles in the neighbourhood of the portion of material which they remove, and especially by punching. In the case of steel, the strength lost through punching is partially, but not wholly, restored by annealing. The drilling of holes has no such weakening effect.

Plate iron is somewhat less tenacious crosswise than lengthwise; but the difference ought not to exceed about one-tenth.

For details as to co-efficients of strength in iron and steel, reference must be made to the tables of the next chapter; but the following short table gives a condensed view of the values of the *ultimate tenacity* which ought to be shown by really good bars and plates of iron and steel, fit to be used as materials in making machinery:—

	Lbs. on the Square Inch.	Kilogrammes on the Square Millimètre.
Iron, large forgings,................from	40,000	28
to	50,000	35
Iron Plates, lengthwise,...........from	50,000	35
to	60,000	42
Do. crosswise, at least 90 per cent. of tenacity lengthwise.		
Iron Bars and Rods,...............from	55,000	39
to	65,000	46
Do., rivet iron, at least,...........	60,000	42
Iron Wire,.................................	90,000	63
Mild Steel,...........................from	70,000	49
to	90,000	63
Hard Steel,...........................from	90,000	63
to	110,000	77
Hardest Cast Steel,......................	130,000	91

It is highly important also that the iron and steel of which pieces exposed to shocks and vibrations are to be made should possess *toughness;* and this may be tested by observing *in what proportion the length of the piece is increased at the instant before breaking.* The ultimate elongation of really good and tough specimens of iron and steel, as ascertained in Mr. Kirkaldy's experiments, was nearly as follows, in fractions of the original length:—

Bar Iron, from	0·15	to 0·30
Plate Iron, lengthwise, from	0·04	to 0·17
Do. crosswise, from	0·015	to 0·11
Steel Bars, from	0·05	to 0·19
Steel Plates, from	0·03	to 0·19

394. **Preservation of Iron.**—Continual motion, especially of a vibratory kind, tends to prevent the rusting of iron and steel;* hence most of the moving pieces in machinery have little or no

* See Mallet, "On the Corrosion of Iron," in the *Reports of the British Association* for 1843 and 1849.

need of any special means of protection, except shelter from the weather and proper care in keeping them clean. But the framework of machines may often require some protection against corrosion. The corrosion of iron is a sort of slow combustion, during which the iron combines with oxygen, and produces rust. The ordinary methods of preserving iron consist principally in preventing the access of oxygen to the metal.

Cast iron will often last for a long time without rusting, if care be taken not to injure its skin, which is usually coated with a film of silicate of the protoxide of iron, produced by the action of the sand of the mould on the iron. Chilled surfaces of castings are without that protection, and therefore rust more rapidly.

The corrosion of iron is more rapid when partly wet and partly dry, than when wholly immersed in water or wholly exposed to the air. It is accelerated by impurities in water, and especially by the presence of decomposing organic matter or of free acids. It is also accelerated by the contact of iron with any metal which is electro-negative relatively to the iron, or, in other words, has less affinity for oxygen (such as copper), or with the rust of the iron itself. If two portions of a mass of iron are in different conditions, so that one has less affinity for oxygen than the other, the contact of the former makes the latter oxidate more rapidly. In general, hard and crystalline iron is less rapidly oxidable than ductile and fibrous iron. Cast iron and steel decompose rapidly in warm or impure sea-water.

The following are amongst the ordinary methods of preserving iron :—

I. Boiling in coal-tar, especially if the pieces of iron have first been heated to the temperature of melting lead.

II. Heating the pieces of iron to the temperature of melting lead, and smearing their surfaces, while hot, with cold linseed oil, which dries and forms a sort of varnish.

III. Painting with oil paint, which must be renewed from time to time. The linseed oil process is a good preparation for painting.

IV. Coating with zinc, commonly called "galvanizing." This is efficient, provided it is not exposed to acids capable of dissolving the zinc; but it is destroyed by sulphuric acid in the atmosphere of places where much coal is burned. It lasts well at sea.

V. Coating with tin, applied to thin sheet iron.

Section II.—*Of Various Metals and Alloys.*

395. **Zinc—Tin—Lead—Copper.**—These are the metals which, next to iron, occur most frequently in machinery; but, owing to their softness, none of them are suited, in a pure state, for frame-

work or for moving pieces. It is by compounding them, so as to form alloys, that sufficient hardness is obtained. Zinc, lead, and copper are used for vessels to hold liquids which would corrode iron, and for flexible tubes; zinc and tin, as already mentioned, are used for coating iron, to preserve it; copper, having great tenacity when rolled and hammered, is used for making boilers into which substances are to be introduced which would be injurious to iron, or be injured by it; also for making rivets for leather driving-belts.

As to the heaviness of those metals, see pages 327, 328; as to their expansion by heat, see pages 326, 327; as to their strength, see the tables of the next chapter. Their melting points are as follows:—

	Fahrenheit.	Centigrade.
Tin,..................	426°	219°
Lead,................	630	332
Zinc,.................	about 700	about 370
Copper, ...	about 2550	about 1400

396. **Bronze and Brass** are the names given to alloys of copper with tin and with zinc. The name *brass*, in common language, is applied to such alloys indiscriminately; but, strictly speaking, *bronze* is the proper name of the alloys of copper with tin; *brass*, that of the alloys of copper with zinc.

Bronze is at least equal to copper in tenacity, and is considerably superior in hardness and resistance to crushing. Brass is inferior to copper in strength. Both bronze and brass make good castings, which quality is not possessed by copper.

These properties render bronze and brass (and especially bronze, where strength is required) suitable both for framework and for moving pieces in machinery.

Bronze is used, in particular, for the bushes or bearings of rotating shafts, because it has the hardness requisite for durability, and at the same time is not so hard and durable as iron. This latter quality ensures that the shaft shall not be worn by the bearing, but the bearing by the shaft.

As zinc is cheaper than tin, alloys of copper with zinc are preferable to those of copper with tin in those cases in which strength and durability are of secondary importance.

The following general principle should be observed in the manufacture of all alloys whatsoever, as being essential to the soundness, strength, and durability of the compound metal:—*The quantities of the constituents should bear definite atomic proportions to each other.*

For example, the chemical equivalents of copper, tin, zinc, and lead bear to each other the following proportions:—

	Copper.		Tin.		Zinc.		Lead.
	31·5	......	59	......	32·5	......	103·5
or	63	......	118	......	65	......	207

and the proportions in which they are combined in any alloy should be expressed by multiples of those numbers.

When this rule is not observed, the metal produced is not a homogeneous compound, but a mixture of two or more different compounds in irregular masses, shown by a mottled appearance when broken; and those masses being different in expansibility and elasticity, tend to separate from each other; and being different in chemical composition, they produce electric circuits and promote corrosion.

The following is a list of the most useful alloys of copper with tin and zinc:—

Composition. By Equivalents.		Composition. By Weight.		Alloys of Copper with Tin.
Copper.	Tin.	Copper.	Tin.	
12	1	378	59	Very hard bronze.
14	1	441	59	Hard bronze for machinery bearings.
16	1	504	59	Bronze, or gun-metal: contracts in cooling from its melting point, $\frac{1}{130}$.
18	1	567	59	Bronze somewhat softer.
20	1	630	59	Soft bronze for toothed wheels, &c.

Composition. By Equivalents.		Composition. By Weight.		Alloys of Copper with Zinc.
Copper.	Zinc.	Copper.	Zinc.	
4	1	126	32·5	Malleable brass.
2	1	63	32·5	Ordinary brass: melting point, 1869° Fahr.: contracts in cooling, $\frac{1}{60}$.
3	2	94·5	65	Yellow metal for sheathing and fastenings of ships.
4	3	126	97·5	Spelter-solder, for brazing copper and iron.

Various alloys of copper, tin, and zinc are used in machinery, and may be regarded as modifications of true bronze, produced by substituting one or two equivalents of zinc for tin. They are less expensive than true bronze, but not so tough.

397. **Other Alloys.**—The strongest of all alloys yet known is *Aluminium Bronze*, as a reference to the tables of the strength of metals will show. Different sorts contain from 5 to 10 per cent. of aluminium, and from 95 to 90 per cent. of copper; and if 31·5 be taken as the equivalent of copper, and 13·7 as that of aluminium, their atomic constitution is probably from 8 to 4 equivalents of copper to 1 equivalent of aluminium.

Alloys of copper with lead, called *pot-metal*, are used for cocks and valves where strength is unimportant; but they are weak and brittle; and in bronze for bearings, lead is an adulteration.

Soft Metal, or *Babbitt's Metal*, consists of 50 parts of tin, 1 of copper, and 5 of antimony. It may be considered as a sort of metallic grease. It is used to make bearings for heavily loaded shafts, in the following way:—A bronze or cast-iron bush is prepared, with a recess about a quarter of an inch deep in its bearing-surface, bounded at the ends by ledges, to prevent the soft metal from escaping; the soft metal in a melted state is run into that recess, either round a core of the shape and size of the journal or round the journal itself.

Soft Solder, used for soldering tin-plate, when of the best quality, is a compound of 4 equivalents of tin to 1 of lead; or by weight, very nearly 2 parts of tin to 1 of lead. It melts at 360° Fahr. Its ultimate tenacity is about 7,500 lbs. on the square inch.

Section III.—*Of some Stony Materials.*

398. **Stone Bearings for Shafts** have occasionally been used. The natural stones fit for this purpose are those which are wholly free from grittiness, and are somewhat inferior in hardness to iron; such as gypsum, pure clay slate, pure compact limestone and marble, and silicate of magnesia, or soapstone, the last being the best. Stones containing crystals of quartz, such as sandstone, sandy limestones and slates, &c., are not suitable. A material called *adamas* is sometimes used for bearings: it consists of silicate of magnesia ground, calcined, moulded by hydraulic pressure into blocks of suitable figures, and baked. The advantage of silicate of magnesia consists in its combining a certain greasiness of surface with a degree of hardness sufficient for durability.

Section IV.—*Of Wood and other Organic Materials.*

399. **Structure of Wood.**—Wood is the material of trees belonging almost exclusively to that class of the vegetable kingdom in which the stem grows by the formation of successive layers of wood all over its external surface, and is therefore said by botanists to be *exogenous.*

The tissues of which wood consists are distinguished into two kinds—*cellular tissue,* consisting of clusters of minute cells; and *vascular tissue,* or *woody fibre,* consisting of bundles of slender tubes, the latter being distinguished from the former by its fibrous appearance. The difference, however, between those two kinds of tissue, although very distinct both to the eye and to the touch, is really one of degree rather than of kind; for the fibres or tubes of vascular tissue are simply very much elongated cells, tapering to points at the ends, and breaking joint with each other.

The tenacity of wood when strained along the grain depends on the tenacity of the walls of those tubes or fibres; the tenacity of wood when strained across the grain depends on the adhesion of the sides of the tubes and cells to each other. Examples of the difference of strength in those different directions are given in the tables.

When a woody stem is cut across, the cellular and vascular tissues are seen to be arranged in the following manner :—

In the middle of the stem is the *pith*, composed of cellular tissue, inclosed in the *medullary sheath*, which consists of vascular tissue of a particular kind. From the pith there extend, radiating outwards to the bark, thin partitions of cellular tissue, called *medullary rays;* between these, additional medullary rays extend inwards from the bark, to a greater or less distance, but without penetrating to the pith.

When the medullary rays are large and distinct, as in oak, they are called "*silver grain*."

Between the medullary rays lie bundles of vascular tissue, forming the woody fibre, arranged in nearly concentric rings or layers round the pith. In most cases each ring is the result of a year's growth of the tree. These rings are traversed radially by the medullary rays. The boundary between two successive rings is marked more or less distinctly by a greater degree of porosity, and by a difference of hardness and colour.

The rings are usually thicker at that side of the tree which has had most air and sunshine, so that the pith is not exactly in the centre.

The wood of the entire stem may be distinguished into two parts—the outer and younger portion, called "*sap-wood*," being softer, weaker, and less compact, and sometimes lighter in colour than the inner and older portion, called "*heart-wood*." The heart-wood is alone to be employed in those structures and machines in which strength and durability are required.

The number of rings of sap-wood ranges from five to forty and upwards in different sorts of wood, and is greatest in trees of the pine and fir kind.

The structure of a *branch* is similar to that of the trunk from which it springs, except as regards the difference in the number of annual rings, corresponding to the difference of age. A branch becomes partially imbedded in those layers of the trunk which are formed after the time of its first sprouting; it causes a perforation in those layers, accompanied by distortion of the fibres, and constitutes what is called a *knot*. (On various matters mentioned in this Article, see Balfour's *Manual of Botany*, Part I., chapters i. and ii.)

400. **Classification of Wood.**—For mechanical purposes, trees may be classed according to the structure of the wood; and upon a comparison of that structure in different kinds of trees, a division into two great classes at once suggests itself, which exactly corresponds with a botanical division, viz.:—

PINE-WOOD, comprising all timber trees belonging to the coniferous order; and

LEAF-WOOD, comprising all other timber trees.

Beyond this primary division, the place of a tree in the botanical system has little or no connection with the structure of its timber.

In the following table those two great classes are subdivided according to a system proposed by Tredgold, founded, in the first place, on the greater or less distinctness of the medullary rays:—

CLASS I.—Pine-Wood. (Natural order *Coniferæ.*)
Examples: Pine, Fir, Larch, Cowrie, Yew, Cedar, Juniper, Cypress, &c.

CLASS II.—Leaf-Wood. (Non-coniferous trees.)

Division 1.—With distinct large medullary rays.
(The trees in this division form part of the natural order *Amentaceæ.*)

Subdivision 1.—Annual rings distinct.
Example: Oak.

Subdivision 2.—Annual rings indistinct.
Examples: Beech, Plane, Sycamore, &c.

Division 2.—Without distinct large medullary rays.

Subdivision 1.—Annual rings distinct.
Examples: Chestnut, Ash, Elm, &c.

Subdivision 2.—Annual rings indistinct.
Examples: Mahogany, Walnut, Box, Teak, Greenheart, Mora, Lignum-vitæ, &c.

The chief practical bearings of the foregoing classification are as follows:—

Pine-wood, or coniferous timber, in most cases contains turpentine. It is distinguished by straightness in the fibre and regularity in the figure of the trees; qualities favourable to its use for long pieces in framework. At the same time, the lateral adhesion of the fibres is small, so that it is much more easily shorn and split along the grain, or torn asunder across the grain, than leaf-wood; and is therefore less fitted to resist thrust or shearing stress, or any kind of stress that does not act along the fibres. Even the toughest kinds of pine-wood are easily wrought; and this quality, combined with lightness and stiffness, makes certain kinds, such as deal, specially well suited for making patterns for large castings. A peculiar characteristic of pine-wood (but one which requires the microscope to make it visible) is that of having the vascular tissue "*punctated*;" that is to say, there are small lenticular hollows in the sides of the tubular fibres. This structure is probably connected with the smallness of the lateral adhesion of those fibres to each other. Pine-wood is, on the whole, inferior to leaf-wood for works of carpentry and machinery in exposed situations; because the strong kinds (as pine and fir) are deficient in durability; and the durable kinds (as cedar and cypress) are deficient in strength.

In *Leaf-wood*, or non-coniferous timber, there is no turpentine. The degree of distinctness with which the structure is seen, whether as regards medullary rays or annual rings, depends on the degree of difference of texture of different parts of the wood. Such difference tends to produce unequal shrinking in drying; and consequently those kinds of wood in which the medullary rays and the annual rings are distinctly marked, are more liable to warp than those in which the texture is more uniform. At the same time, the former kinds of wood are, on the whole, the more flexible, and in many cases are very tough and strong, which qualities make them suitable for pieces that have to bear shocks.

401. **Appearance of Good Timber.**—There are certain appearances which are characteristic of strong and durable wood, to what class soever it belongs. In the same species of wood, that specimen will in general be the strongest and the most durable which has grown the slowest, as shown by the narrowness of the annual rings.

The cellular tissue, as seen in the medullary rays (when visible), should be hard and compact.

The vascular or fibrous tissue should adhere firmly together, and should show no woolliness at a freshly-cut surface; nor should it clog the teeth of the saw with loose fibres.

If the wood is coloured, darkness of colour is in general a sign of strength and durability.

The freshly-cut surface of the wood should be firm and shining, and should have somewhat of a translucent appearance. A dull, chalky appearance is a sign of bad timber.

In wood of a given species, the heavier specimens are in general the stronger and the more lasting.

Amongst resinous woods, those which have least resin in their pores, and, amongst non-resinous woods, those which have least sap or gum in them, are in general the strongest and most lasting.

Timber should be free from such blemishes as "clefts," or cracks radiating from the centre; "cup-shakes," or cracks which partially separate one annual layer from another; "upsets," where the fibres have been crippled by compression; "rind-galls," or wounds in a layer of the wood, which have been covered and concealed by the growth of subsequent layers over them; and hollows, or spongy places, in the centre or elsewhere, indicating the commencement of decay.

402. **Examples of Pine-Wood.**—The following are a few examples of timber of this class:—

I. Pine timber is the wood of various species of the genus *Pinus*, the best being that of the Red Pine, or Scottish Fir (*Pinus sylvestris*), grown in the north of Europe. This wood is stiff, strong, and straight-grained, and well suited for large framing.

Pine timber is also obtained from various other species, chiefly North American, of which the best are the Yellow Pine (*Pinus variabilis*) and White Pine (*Pinus Strobus*). It is softer and less durable than the Red Pine of the north of Europe, but lighter, and can be had in larger logs.

Timber similar in its properties to the best kinds of pine is produced by the Kauri or Cowrie of New Zealand (*Dammara Australis*).

II. White Fir, or Deal timber of the best kind, is the wood of the Spruce Fir (*Abies excelsa*), grown in the north of Europe.

This is an excellent kind of timber for light framing and joiners' work, and is specially well suited for making patterns of machinery.

Amongst other kinds of spruce fir applied to the same purposes are the North American White Spruce (*Abies alba*), and Black Spruce (*Abies nigra*).

403. **Examples of Leaf-Wood with Large Rays.**—I. Oak timber belongs to the first subdivision of Tredgold's system. It is the strongest, toughest, and most lasting of those grown in temperate climates, and is well suited for framing in which strength, toughness, and durability are required; but it has in general the defect, which is a serious one as regards machinery, of being subject to warp. It is obtained from various species or varieties of the botanical genus *Quercus*.

The wood of the oak contains gallic acid, which contributes to the durability of the timber, but corrodes iron. Metal fastenings for oak should therefore be of copper, or its alloys; or, if of iron, they should be well coated with zinc.

The following are examples of trees belonging to Tredgold's second subdivision:—

II. Beech (*Fagus sylvatica*), common in Europe.

III. American Plane (*Platanus occidentalis*), common in North America.

IV. Sycamore (*Acer pseudo-platanus*), also called Great Mapble, and in Scotland and the north of England, Plane; common in Western Europe.

All these afford compact wood of uniform texture. They are valuable for blocks which have to resist a crushing force. They last well when constantly wet (especially beech), but when alternately wet and dry they decay rapidly.

404. **Examples of Leaf-Wood without Large Rays.**—The examples of timber in this Article belong to the first subdivision of the second division according to Tredgold's system, having no large distinct medullary rays, and having the divisions between the annual rings distinctly marked by a more porous structure. They are in general strong, but flexible; and therefore, in machinery, they are suitable for pieces in which the power of bearing shocks is of more importance than rigidity.

I. The ASH (*Fraxinus excelsior*) furnishes timber whose toughness and flexibility render it superior to that of all other European trees for making handles of tools, shafts of carriages, spokes of wooden wheels, and the like; but which is not sufficiently stiff and durable to be used in framing.

II. The common ELM (*Ulmus campestris*) and smooth-leaved ELM (*Ulmus glabra*) yield timber which is very durable when constantly wet, but not when alternately wet and dry. Its strength across the grain, and its resistance to crushing, are comparatively great; and these properties render it useful for some parts of mechanism, such as cogs of wheels and shells of ships' blocks. There are other European species of elm, such as the Wych Elm (*Ulmus montana*); but their timber is inferior to that of the two species named.

A North American species, the Rock Elm, is said to be not only durable under water, but straight-grained and tough, so as to be well suited for framing.

405. **Examples of Leaf-Wood without Large Rays continued.**—The kinds of timber mentioned in this Article are examples of the second subdivision of Tredgold's second division, having no large distinct medullary rays, and no distinct difference of compactness in the rings. This uniformity of structure is accompanied by comparative freedom from warping; and hence this subdivision contains various sorts of wood which are specially well adapted both for framing and for moving pieces in machinery, where accuracy and constancy of form are required.

I. MAHOGANY (*Swietenia Mahagoni*) is produced in Central America and the West India Islands—that of the former region being commonly known as "Bay Mahogany;" that of the latter, as "Spanish Mahogany." When of good quality, it is very straight-grained; very strong in all directions (though easily split along the grain), very durable, and preserves its shape under varying circumstances as to heat and moisture, better than any other kind of timber which can be procured in equal abundance. Mahogany varies much in quality; bay mahogany being in general superior to Spanish mahogany in strength, stiffness, and durability, and in the size of the logs, which are from 24 to 48 inches square. Bay mahogany of good quality is probably the best of all timber for the framing of machinery. Spanish mahogany is the more highly valued for ornamental purposes. Spanish mahogany is distinguished by having a white chalky substance in its pores, those of bay mahogany being empty.

II. Lignum-vitæ (*Guaiacum officinale*) is produced in the West India Islands. It is remarkable for heaviness, compactness, toughness, and hardness, and for the property of resisting a crushing force with nearly equal strength across and along the grain—a property which makes it specially useful for rollers, sheaves, and other moving pieces in mechanism. In converting logs into sheaves, the direction of the fibre of the timber is parallel to the axis of the sheave. The heart-wood is yellowish-green, the sap-wood greenish-yellow; and it is considered advisable, in cutting it into pieces suitable for sheaves, to leave a ring of sap-wood all round the heart-wood, which is thus protected against too rapid drying, and prevented from splitting.

Properties similar to those of Lignum-vitæ are possessed by Box-wood (*Buxus sempervirens*), Ebony (*Brya ebenus*, and other genera and species), Ironwood (*Mesua Nagaha*), and various other woods, chiefly tropical.

The same subdivision embraces various kinds of timber grown in tropical climates, which are highly valued for shipbuilding purposes, and which would be suitable also for the framing of machines —such as the Teak (*Tectona grandis*) and Saul (*Shorea robusta*) of India, and the Greenheart (*Nectandra Rodiæi*), Mora (*Mora excelsa*), and Sabicu (*Acacia proxima*) of South America and the West Indies.

406. **Seasoning.**—Seasoning timber consists in expelling, as far as possible, the moisture which is contained in its pores.

Natural Seasoning is performed simply by exposing the timber freely to the air in a dry place, sheltered, if possible, from sunshine and high winds. The seasoning yard should be paved and well drained, and the timber supported on stone or cast-iron bearers, and piled so as to admit of the free circulation of air over all the surfaces of the pieces.

Natural seasoning to fit timber for carpenters' work, usually occupies about two years; for joiners' work and machinery, about four years; but much longer periods are sometimes employed.

To steep timber in water for a fortnight after felling it, extracts part of the sap, and makes the drying process more rapid.

Artificial Seasoning consists in drying the timber in an oven by means of a current of hot air. It occupies from seven to nine days for each inch of the thickness of the piece of timber.

In the course of drying, timber loses weight and shrinks in its transverse dimensions. The loss of weight ranges in different examples from 6 per cent. to 40 per cent.; and the transverse shrinking from 2 per cent. to 8 per cent., the most common rate being 3 per cent. The sorts of wood which shrink most in drying are the most subject to warp.

407. **Durability, Decay, and Preservation of Wood.**—All kinds of timber are more lasting when kept constantly dry, and at the same time freely ventilated.

Timber kept constantly wet is softened and weakened; but it does not necessarily decay. Various kinds of timber, some of which have been already mentioned, such as greenheart, elm and beech, possess great durability in that condition.

The situation which is least favourable to the duration of timber is that of alternate wetness and dryness, or of a slight degree of moisture, especially if accompanied by heat and confined air.

Timber exposed to confined air alone, without the presence of any considerable quantity of moisture, decays by "*dry rot*," which is accompanied by the growth of a fungus, and finally converts the wood into a fine powder.

Amongst the most efficient means of preserving wood, are good seasoning and the free circulation of air.

Protection against moisture is afforded by oil paint, provided that the timber is perfectly dry when first painted, and that the paint is renewed from time to time. A coating of pitch or tar may be used for the same purpose.

Protection against the dry rot may be obtained by saturating the timber with solutions of metallic salts, such as sulphate of iron, sulphate of copper, bichloride of mercury, and chloride of zinc.

Timber is protected against wet rot, dry rot, and white ants, by saturation with the liquid called commercially "*creosote*," which is a kind of pitch oil.

408. **Strength of Timber.**—Amongst different specimens of timber of the same species, those which are most dense in the dry state are in general also the strongest.

Tables of the results of experiments on the strength of different kinds of timber, strained in various ways, are given in the next chapter.

The following are some general remarks as to the different ways in which the strength of timber is exerted:—

I. The *Tenacity along the grain*, depending, as it does, on the tenacity of the fibres of the vascular tissue, is on the whole greatest in those kinds and pieces of wood in which those fibres are straightest and most distinctly marked. It is not materially affected by temporary wetness of the timber, but is diminished by long-continued saturation with water, and by steaming and boiling.

The *Tenacity across the grain*, depending chiefly on the lateral adhesion of the fibres, is always considerably less than the tenacity along the grain, and is diminished by wetness and increased by dryness. Very few exact experiments have been made upon it. Its smallness in pine-wood as compared with leaf-wood forms a marked distinction between those two classes of timber, the proportion which it bears to the tenacity along the grain having been found to be, by some experiments—

In pine-wood, from 1-20th to 1-10th.
In leaf-wood, from 1-6th to 1-4th and upwards.

II. The *Resistance to Shearing*, by sliding of the fibres on each other, is the same, or nearly the same, with the tenacity across the grain.

III. The *Resistance to Crushing* along the grain, depending, as it does, on the resistance of the fibres to being crippled, or "upset," and split asunder, is greatest when their lateral adhesion is greatest, and was found by Mr. Hodgkinson to be nearly twice as great for dry timber as for the same timber in the green state. In most kinds of timber, when dry, it ranges from one-half to two-thirds of the tenacity.

Experiments have been made on the crushing of timber across the grain, which takes place by a sort of shearing; but they have not led to any precise result, except that timber in general is both more compressible and weaker against a transverse than against a longitudinal pressure; and consequently, that intense transverse compression of pieces of timber ought to be avoided. Certain special kinds of timber are valued for the property of resisting compression across the grain well. Of these the most generally used is lignum-vitæ, already mentioned in Article 405, page 470.

IV. The *Modulus of Rupture* of timber, which expresses its resistance to cross-breaking, is usually somewhat less than its tenacity; but seldom much less.

409. **Use of Wood in Machinery.**—The following tabular arrangement of the more ordinary kinds of wood, according to the purposes in machinery to which they are applicable, is principally based on a similar table given by Holtzapffel in his treatise on *Mechanical Manipulation*.

FRAMEWORK.

Strong, stiff, durable, and free from warping.
Mahogany.

Strong longitudinally, stiff, and straight-grained.
Pine, Deal.

Strong, tough, and durable.
Oak, Teak, Saul.

Tough and pliable.
Ash.

Strong against pressure.
Elm (durable when wet), Beech.

LEVERS AND CONNECTING-RODS.

Strong and stiff.
Pine, Deal, Mahogany.

Strong and tough.
Oak, Teak.

Tough and pliable.
Ash, Hazel, Hickory, Lancewood.

PULLEYS, SHEAVES, ROLLERS.

Lignum-vitæ, Box, Mahogany.

BEARINGS FOR SHAFTS.

Box, Beech, Holly, Lignum-vitæ, Elm.

When wood is used for bearings, the ends of the fibres should be exposed to the pressure.

COGS.

Crabtree, Hornbeam, Locust, Beech.

PATTERNS.

Deal, Mahogany, Pine, Alder.

In machinery whose speed is liable to be suddenly changed or checked, it is often useful to make some of the parts which transmit the motion of wood, although the whole of the remainder may be of iron; the object being that the wood, by yielding to a shock, may prevent it from damaging the iron; and also that in the event of breakage occurring, it may take place in the wooden parts, which can be replaced more easily and at less cost than the iron parts.

For example, the great spur fly-wheel by means of which a steam engine or a water-wheel drives the machinery of a mill is very generally a *mortise-wheel;* that is to say, a cast-iron wheel with rectangular sockets called *mortises* in its rim, into which are fitted wooden teeth called cogs. The pinion which those teeth drive is wholly of cast iron. Wooden cogs are made double the thickness of cast-iron teeth that have to bear the same pressure.

Another instance of the application of the same principle is when, in a steam engine that drives an iron rolling mill, the middle part of the thickness of the connecting-rod, which transmits thrust, is made of wood, the tension being transmitted by means of a wrought-iron strap.

409 A. **Pasteboard,** composed of layers of paper perpendicular to the pressure, is sometimes used for bearings of shafts.

410. **Organic Materials for Bands, Leather, Gutta-Percha, Indian Rubber, Cotton, Flax, and Hemp.**—I. *Leather Belts.*—The ordinary material for driving-belts in machinery is ox-leather from the back of the animal. It is of a nearly uniform thickness, ranging from $\frac{1}{6}$ to $\frac{1}{4}$ of an inch (from 4 to 6 millimètres). It is to be had in pieces up to $4\frac{1}{2}$ or 5 feet long, and about 8 inches broad. The several lengths of leather of which a belt is made are spliced and cemented together, and fastened to each other by means of pins or rivets of copper or of soft brass. The two ends of the belt are connected with each other by a lacing of thongs, or by copper or brass pins.

A belt is said to be *single* or *double* according as it is made of one or of two thicknesses of leather.

The inside of the leather is rougher than the outside, and is placed next the pulleys; crossed belts being twisted so as to bring the same side of the leather in contact with both pulleys (fig. 123, page 182).

Leather belts, when new, are not quite of the heaviness of water—say about 60 lbs. per cubic foot; but after having been for some time in use, they become thinner and denser by compression, and are then about as heavy as water. The weight of single belting may be approximately estimated as follows:—

Per foot length and inch breadth, ...	0·068 lb.
Per square mètre of surface,...............	4 kilogrammes.

The following table shows the results of experiments by Mr. Henry R. Towne on the ultimate tenacity of belts, compared with the practical rule of General Morin as to their safe working tension. The tensions in lengths of belt are calculated from the above estimate of the heaviness.

Ultimate Tenacity.	Lbs. per In. wide.	Kilos. per Mm. wide.	Feet of Belt.	Mètres of Belt.
The solid leather,..................	675	12	10,000	3,000
At the rivet-holes of the splices,	382	6·8	5,600	1,700
At the lacing, ...	210	3·75	3,100	940
Safe working tension (see page 441), ...	45	0·8	660	200

II. *Raw Hide Belts.*—The process of tanning, which makes leather durable, impairs its strength; the tenacity of raw hide being about once and a half that of tanned leather. When raw hide is used for belts or for ropes, it is soaked with grease to keep it pliable and protect it against the action of air and moisture.

III. *Gutta-Percha* is sometimes used for flat belts. They are made of the same dimensions with leather belts for transmitting the same force, and are nearly of the same weight.

IV. *Woven Belts* are made of a flaxen or cotton fabric; a sufficient number of plies being used to give a thickness equal to that of leather belts, and cemented together with indian rubber. When made of flax, they are said to be about three times more tenacious than tanned leather belts of the same transverse dimensions.

V. *Ropes and Cords,* when of organic materials, are made of leather, raw hide, and catgut, and of flax, hemp, and other vegetable fibre. Round cords of leather and of hide are made by twisting strips of those materials into round strands, and spinning or plaiting those strands into ropes. The ultimate tenacity, when the material is of the best kind, may be taken, for leather, as given by the table in the first division of this Article; and for raw hide, as one and a half that of leather. Assuming the heaviness, when well twisted, to be equal to that of water, this will give the following results:—

ULTIMATE TENACITY.	Feet of Rope.	Mètres of Rope.	Lbs. on the circular Inch.	K. on the circular Mm.
Leather, ...	10,000	3,000	3,360	2·36
Raw Hide, ...	15,000	4,500	5,040	3·54
Working Tension— factor of safety, 6.				
Leather, ...	1,667	500	560	0·39
Raw Hide, ...	2,500	750	840	0·59

Hemp, as used in ropes, is spun, or "*laid up,*" with a right-handed twist into *yarns.* Yarns are laid up left-handed into *strands.* Three strands laid up right-handed make a *hawser;* three hawsers laid up left-handed make a *cable.* Hempen ropes are classed according to the number and arrangement of their strands. The following are the commonest kinds:—

Hawser-laid rope, ... 3 strands.
Cable-laid rope = 3 hawsers twisted together, ... 9 strands.
Shroud-laid rope = core or heart surrounded by ...... 4 strands.

The *girth squared* is the dimension commonly employed in calculating the weight and strength of hempen ropes. The proof, or testing load, is given by multiplying the girth squared by one of the factors in the following table; the breaking load is from two to three times the proof load; the working load is about *one-fourth* of the proof load: that is, about one-tenth of the breaking load.

	Multiplier for Proof Load. Lbs.	Multiplier for Weight. Lb. per 100 Fathoms.	Proof Strength in Length of Rope. Feet.	Metres.
Hawser-laid rope,	420	23·1	10,920	3,300
Shroud-laid ,,	336	22·4	9,000	2,740
Cable-laid ,,	269	21·5	7,500	2,290

Tarred Ropes have about three-fourths of the strength of white ropes of the same size.

Wire ropes contain a certain proportion of organic material in the hempen cores round which the wires are spun, but it does not sensibly contribute to their strength, which will be more fully considered further on.

GENERAL TABLES OF THE STRENGTH OF MATERIALS.

I.

TABLE OF THE RESISTANCE OF MATERIALS TO STRETCHING AND TEARING BY A DIRECT PULL, *in pounds avoirdupois per square inch.*

MATERIALS.	Tenacity, or Resistance to Tearing.	Modulus of Elasticity, or Resistance to Stretching.
STONES, NATURAL AND ARTIFICIAL:		
Brick, Cement,	280 to 300	
Glass,	9,400	8,000,000
Slate,	9,600 to 12,800	13,000,000 to 16,000,000
Mortar, ordinary,	50	
METALS:		
Brass, cast,	18,000	9,170,000
" wire,	49,000	14,230,000
Bronze (Copper 8, Tin 1),	36,000	9,900,000
" Aluminium,	73,000	
Copper, cast,	19,000	
" sheet,	30,000	
" bolts,	36,000	
" wire,	60,000	17,000,000
Iron, cast, various qualities,	13,400 to 29,000	14,000,000 to 22,900,000
" average,	16,500	17,000,000
Iron, wrought, plates,	51,000	
" joints, double rivetted,	35,700	
" " single rivetted,	28,600	
" bars and bolts,	60,000 to 70,000	29,000,000
" hoop, best-best,	64,000	
" wire,	70,000 to 100,000	25,300,000
" wire-ropes,	90,000	15,000,000
Lead, sheet,	3,300	720,000
Steel bars,	100,000 to 130,000	29,000,000 to 42,000,000
Steel plates, average,	80,000	
Tin, cast,	4,600	
Zinc,	7,000 to 8,000	

Materials.	Tenacity, or Resistance to Tearing.	Modulus of Elasticity, or Resistance to Stretching.
Timber and other Organic Fibre:		
Acacia, false. See "Locust."		
Ash (*Fraxinus excelsior*), ...	17,000	1,600,000
Bamboo (*Bambusa arundinacea*),	6,300	
Beech (*Fagus sylvatica*), ...	11,500	1,350,000
Birch (*Betula alba*), ...	15,000	1,645,000
Box (*Buxus sempervirens*), ...	20,000	
Cedar of Lebanon (*Cedrus Libani*),	11,400	486,000
Chestnut (*Castanea Vesca*), ...	10,000 to 13,000	1,140,000
Elm (*Ulmus campestris*), ...	14,000	700,000 to 1,340,000
Fir: Red Pine (*Pinus sylvestris*),	12,000 to 14,000	1,460,000 to 1,900,000
" Spruce (*Abies excelsa*), ...	12,400	1,400,000 to 1,800,000
" Larch (*Larix Europæa*), ...	9,000 to 10,000	900,000 to 1,360,000
Flaxen Yarn, ...	about 25,000	
Hazel (*Corylus Avellana*), ...	18,000	
Hempen Ropes, ...from 12,000 to	16,000	
Hide, Ox, undressed, ...	6,300	
Hornbeam (*Carpinus Betulus*),	20,000	
Lancewood (*Guatteria virgata*),	23,400	
Leather, Ox, ...	4,200	24,300
Lignum-Vitæ (*Guaiacum officinale*), ...	11,800	
Locust (*Robinia Pseudo-Acacia*),	16,000	
Mahogany (*Swietenia Mahagoni*),	8,000 to 21,800	1,255,000
Maple (*Acer campestris*), ...	10,600	
Oak, European (*Quercus sessiliflora* and *Quercus pedunculata*),	10,000 to 19,800	1,200,000 to 1,750,000
" American Red (*Quercus rubra*), ...	10,250	2,150,000
Silk fibre, ...	52,000	1,300,000
Sycamore (*Acer Pseudo-Platanus*),	13,000	1,040,000
Teak, Indian (*Tectona grandis*),	15,000	2,400,000
" African, (?)	21,000	2,300,000
Whalebone, ...	7,700	
Yew (*Taxus baccata*), ...	8,000	

II.

TABLE OF THE RESISTANCE OF MATERIALS TO SHEARING AND DISTORTION, *in pounds avoirdupois per square inch.*

MATERIALS.	Resistance to Shearing.	Transverse Elasticity, or Resistance to Distortion.
METALS:		
Brass, wire-drawn,		5,330,000
Copper,		6,200,000
Iron, cast,	27,700	2,850,000
„ wrought,	50,000	8,500,000 to 10,000,000
TIMBER:		
Fir: Red Pine,	500 to 800	62,000 to 116,000
„ Spruce,	600	......
„ Larch,	970 to 1,700	......
Oak,	2,300	82,000
Ash and Elm,	1,400	76,000

III.

TABLE OF THE RESISTANCE OF MATERIALS TO CRUSHING BY A DIRECT THRUST, *in pounds avoirdupois per square inch.*

MATERIALS.	Resistance to Crushing.
STONES, NATURAL AND ARTIFICIAL:	
Brick, weak red,	550 to 800
„ strong red,	1,100
„ fire,	1,700
Chalk,	330
Granite,	5,500 to 11,000
Limestone, marble,	5,500
„ granular,	4,000 to 4,500
Sandstone, strong,	5,500
„ ordinary,	3,300 to 4,400
Rubble masonry, about four-tenths of cut stone.	
METALS:	
Brass, cast,	10,300
Bronze, Aluminium,	132,000
Iron, cast, various qualities,	82,000 to 145,000
„ „ average,	112,000
„ wrought,	about 36,000 to 40,000

Materials.	Resistance to Crushing.
Timber,* Dry, crushed along the grain:	
Ash,	9,000
Beech,	9,360
Birch,	6,400
Blue-Gum (*Eucalyptus Globulus*),	8,800
Box,	10,300
Bullet-tree (*Achras Sideroxylon*),	14,000
Cabacalli,	9,900
Cedar of Lebanon,	5,860
Ebony, West Indian (*Brya Ebenus*),	19,000
Elm,	10,300
Fir: Red Pine,	5,375 to 6,200
„ American Yellow Pine (*Pinus variabilis*),	5,400
„ Larch,	5,570
Hornbeam,	7,300
Lignum-Vitæ,	9,900
Mahogany,	8,200
Mora (*Mora excelsa*),	9,900
Oak, British,	10,000
„ Dantzic,	7,700
„ American Red,	6,000
Teak, Indian,	12,000
Water-Gum (*Tristania nerifolia*),	11,000

IV.

Table of the Resistance of Materials to Breaking Across, *in pounds avoirdupois per square inch.*

Materials.	Resistance to Breaking, or Modulus of Rupture.†
Stones:	
Sandstone,	1,100 to 2,360
Slate,	5,000

* The resistances stated are for *dry* timber. Green timber is much weaker, having sometimes only half the strength of dry timber against crushing.

† The modulus of rupture is eighteen times the load which is required to break a bar of one inch square, supported at two points one foot apart, and loaded in the middle between the points of support.

MATERIALS.	Resistance to Breaking, or Modulus of Rupture.
METALS:	
Iron, cast, open-work beams, average, ...	17,000
„ „ solid rectangular bars, var. qualities,	33,000 to 43,500
„ „ „ „ average,......	40,000
„ wrought, ...	40,000 to 54,000
TIMBER:	
Ash,..	12,000 to 14,000
Beech,..	9,000 to 12,000
Birch, ...	11,700
Blue-Gum,...	16,000 to 20,000
Bullet-tree, ...	15,900 to 22,000
Cabacalli,...	15,000 to 16,000
Cedar of Lebanon,.............................	7,400
Chestnut,..	10,660
Cowrie (*Dammara australis*), ...	11,000
Ebony, West Indian, ...	27,000
Elm,...	6,000 to 9,700
Fir: Red Pine, ...	7,100 to 9,540
„ Spruce,..	9,900 to 12,300
„ Larch,...	5,000 to 10,000
Greenheart (*Nectandra Rodiæi*),..................	16,500 to 27,500
Lancewood, ...	17,350
Lignum-Vitæ,......................................	12,000
Locust, ...	11,200
Mahogany, Honduras,..........................	11,500
„ Spanish, ...	7,600
Mora,..	22,000
Oak, British and Russian,....................	10,000 to 13,600
„ Dantzic, ...	8,700
„ American Red,................................	10,600
Poon,...	13,300
Saul,...	16,300 to 20,700
Sycamore, ...	9,600
Teak, Indian,......................................	12,000 to 19,000
„ African,..	14,980
Tonka (*Dipteryx odorata*), ...	22,000
Water-Gum, ...	17,460
Willow (*Salix*, various species),.................	6,600

V.—Miscellaneous Supplementary Table.

Material.	Dimensions.	Tearing Load, lbs.	Length of 1 lb. weight, in feet.	Tenacity in feet of the Material.
Cast steel bar,............	1 in. × 1 in.	130,000	0·297	38,610
Charcoal iron wire,......	area 1 sq. in.	100,000	0·3	30,000
Iron wire rope,...........	girth 1·27 in.	4,480	6·0	26,880
Iron bar, strong,.........	1 in. × 1 in.	60,000	0·3	18,000
Boiler plate, strong,.....	area 1 sq. in.	50,000	0·3	15,000
Teak wood,...............	1 in. × 1 in.	15,000	3·0	54,000
Deal,........................	1 in. × 1 in.	12,000	4·0	48,000
Hempen hawser,.........	girth 1 in.	1,050	26·0	27,300
Hempenrope,cable-laid,	girth 10 in.	67,200	0·279	18,750

VI.—Supplementary Table for Wrought Iron and Steel.

Description of Material.	Tenacity in lbs. per Square Inch. Lengthwise.		Crosswise.		Ultimate Extension.	
Malleable Iron.						
Wire—very strong, charcoal,.............	114,000	Mo.				
Wire—average,...........	86,000	T.				
Wire—weak, ...	71,000	Mo.				
Yorkshire(Lowmoor),...	64,200	F.	52,490	F.		
" from	66,390	N.			0·20	
to	60,075				0·26	
Yorkshire (Lowmoor) and Staffordshire rivet iron,............	59,740	F.			0·2 to 0·25	
Charcoal bar,..............	63,620	F.			0·2	
Staffordshire bar,...from	62,231	N.			·302	
to	56,715				·186	
Yorkshire bridgeiron,...	49,930	F.	43,940	F.	·04;	·029
Staffordshire bridge iron,	47,600	F.	44,385		·04;	·036
Lanarkshire bar,... from	64,795	N.			·158	
to	51,327				·238	
Lancashire bar,......from	60,110	N.			·169	
to	53,775				·216	
Swedish bar,.........from	48,933	N.			·264	
to	41,251				·278	
Russian bar,.........from	59,096	N.			·153	
to	49,564				·133	
Bushelled iron from turnings,............	55,878	N.			·166	
Hammered scrap,........	53,420	N.			·248	
Angle-iron from various districts, from	61,260	N.				
to	50,056					

TABLE—*continued.*

Description of Material.	Tenacity in lbs. per Square Inch. Lengthwise.		Crosswise.		Ultimate Extension.
Straps from various districts, ... from	55,937	N.			·108
to	41,386				·048
Bessemer's iron, cast ingot,..................	41,242	W.			
Bessemer's iron, hammered or rolled,....	72,643	W.			
Bessemer's iron, boiler plate,..................	68,319	W.			
Yorkshire plates,...from	58,487	N.	55,033	N.	·109; ·059
to	52,000		46,221		·170; ·113
Staffordshire plates, from	56,996	N.	51,251	N.	·04; ·034
to	46,404		44,764		·13; ·059
Staffordshire plates, best-best, charcoal,	45,010	F.	41,420	F.	·05; ·045
Staffordshire plates, best-best, from	59,820	F.	54,820	F.	·05; ·038
to	49,945	F.	46,470	F.	·067; ·04
Staffordshire plates, best,	61,280	F.	53,820	F.	·077; ·045
Staffordshire plates, common,.............	50,820	F.	52,825	F.	·05; ·043
Lancashire plates,........	48,865	F.	45,015	F.	·043; ·028
Lanarkshire plates, from	53,849	N.	48,848	N.	·033; ·014
to	43,433		39,544		·093; ·046
Durham plates, ...	51,245	N.	46,712		·089; ·064

Effects of Reheating and Rolling.

Puddled bar, ...	43,904	C.
The same iron five times piled, reheated and rolled,............	61,824	
The same iron eleven times piled, reheated and rolled,............	43,904	

Strength of Large Forgings.

Bars cut out of large forgings, from	47,582	N.	44,578	·231; ·168
to	43,759		36,824	·205; ·064
Bars cut out of large forgings,..............	33,600	M.		

TABLE—*continued.*

Description of Material.	Tenacity in lbs. per Square Inch. Lengthwise.		Crosswise.		Ultimate Extension.
STEEL AND STEELY IRON.					
Cast steel bars, rolled and forged, from	132,909	N.			·052
to	92,015				·153
Cast steel bars, rolled and forged, ...	130,000	R.			
Blistered steel bars, rolled and forged....,	104,298	N.			·097
Shear steel bars, rolled and forged, ...	118,468	N.			·135
Bessemer's steel bars, rolled and forged,...	111,460	N.			·055
Bessemer's steel bars, cast ingots, ...	63,024	W.			
Bessemer's steel bars, hammered or rolled,	152,912	W.			
Spring steel bars, hammered or rolled,....	72,529	N.			·180
Homogeneous metal bars, rolled, ...	90,647	N.			·137
Homogeneous metal bars, rolled,.........	93,000	F.			
Homogeneous metal bars, forged,.........	89,724	N.			·119
Puddled steel bars, rolled and forged,.......... from	71,484	N.			·191
to	62,768				·091
Puddled steel bars, rolled and forged,...	90,000	F.			
Puddled steel bars, rolled and forged,...	94,752	M.			
Mushet's gun-metal,.....	103,400	F.			0·034
Cast steel plates,....from	96,289	N.	97,308	N.	·057; ·096
to	75,594		69,082		·198; ·196
Cast steel plates,...hard,	102,900	F.			·031
soft,	85,400				·031
Homogeneous metal plates, first quality,	96,280	N.	97,150	N.	·086; ·144
Homogeneous metal plates, second quality,	72,408		73,580		·059; ·032
Puddled steel plates,............ from	102,593	N.	85,365	N.	·028; ·013
to	71,532		67,686		·082; ·057
Puddled steel plates,....	93,600	F.			0·125

TABLE—*continued.*

Description of Material.	Tenacity in lbs. per Square Inch. Lengthwise.	Crosswise.	Ultimate Extension.
Coleford Gun-metal.			
Weakest, ...	108,970	F.	·190
Strongest, ...	160,540	F.	·030
Mean of ten sorts, ...	137,340	F.	·072

In the preceding table the following abbreviations are used for the names of authorities:—

C., Clay; F., Fairbairn; H., Hodgkinson; M., Mallet; Mo., Morin; N.,* Napier & Sons; R., Rennie; T., Telford; W., Wilmot.

The column headed "Ultimate Extension" gives the ratio of the elongation of the piece, at the instant of breaking, to its original length. It furnishes an index (but a somewhat vague one) to the ductility of the metal, and its consequent safety as a material for resisting shocks.

When two numbers separated by a semicolon appear in the column of ultimate extension (thus ·082; ·057), the first denotes the ultimate extension lengthwise, and the second crosswise.

VII.—Resilience of Iron and Steel.

Metal under Tension.	Ultimate Tenacity.	Working Tenacity.	Modulus of Elasticity.	Modulus of Resilience.
Cast iron—Weak, ...	13,400	4,467	14,000,000	1·425
" Average, ...	16,500	5,500	17,000,000	1·78
" Strong, ...	29,000	9,667	22,900,000	4·08
Bar iron—Good average, ..	60,000	20,000	29,000,000	13·79
Plate iron—Good average,	50,000	16,667	24,000,000?	11·57?
Iron wire—Good average,	90,000	30,000	25,300,000	35·57
Steel—Soft, ...	90,000	30,000	29,000,000	31·03
" Hard, ...	132,000	44,000	42,000,000	46·10

In the above Table of Resilience the working tenacity is for a "dead" or steady load. The modulus of resilience is calculated by dividing the square of that working tenacity by the modulus of elasticity.

* The experiments whose extreme results are marked N. were conducted for Messrs. R. Napier & Sons by Mr. Kirkaldy. For details, see *Transactions of the Institution of Engineers in Scotland*, 1858-59; also Kirkaldy *On the Strength of Iron and Steel.*

VIII.—Supplementary Table for Cast Iron.

Kinds of Iron.		Direct Tenacity.	Resistance to Direct Crushing.	Modulus of Rupture of Square Bars.	Modulus of Elasticity.
No. 1. Cold blast,......	from	12,694	56,455	36,693	14,000,000
	to	17,466	80,561	39,771	15,380,000
No. 1. Hot blast, ...	from	13,434	72,193	29,889	11,539,000
	to	16,125	88,741	35,316	15,510,000
No. 2. Cold blast,......	from	13,348	68,532	33,453	12,586,000
	to	18,855	102,408	39,609	17,036,000
No. 2. Hot blast, ...	from	13,505	82,734	28,917	12,259,000
	to	17,807	102,030	38,394	16,301,000
No. 3. Cold blast,......	from	14,200	76,900	35,881	14,281,000
	to	15,508	115,400	47,061	22,908,000
No. 3. Hot blast, ...	from	15,278	101,831	35,640	15,852,000
	to	23,468	104,881	43,497	22,733,000
No. 4. Smelted by coke without sulphur,.........		—	—	41,715	—
Toughened cast iron,	from	23,461	129,876	—	—
	to	25,764	119,457	—	—
No. 3. Hot blast after first melting, ...		—	98,560	39,690	—
No. 3. Hot blast after twelfth melting,.........		—	163,744	56,060	—
No. 3. Hot blast after eighteenth melting, ...		—	197,120	25,350	—
Malleable cast iron,............		48,000	—	—	—

It is to be understood that the numbers in one line of the preceding table do not necessarily belong to the *same specimen* of iron, each number being an *extreme* result for the kind of iron specified in the first column.

CHAPTER II.

PRINCIPLES AND RULES RELATING TO STRENGTH AND STIFFNESS.

411. The **Object of this Chapter** is to give a summary of the principles, and of the general rules of calculation, which are applicable to problems of strength and stiffness, whatsoever the particular material may be. It is to a certain extent identical with a similar summary which appeared in *A Manual of Civil Engineering*, but modified to adapt its principles to the problems which occur in machinery. Various special problems relating to machinery will be considered in the third Chapter.

SECTION I.—*Of Strength and Stiffness in General.*

412. **Load, Stress, Strain, Strength.**—The *load*, or combination of external forces, which is applied to any piece, moving or fixed, in a machine, produces *stress* amongst the particles of that piece, being the combination of forces which they exert in resisting the tendency of the load to disfigure and break the piece, accompanied by *strain*, or alteration of the volumes and figures of the whole piece, and of each of its particles.

If the load is continually increased, it at length produces either *fracture* or (if the material is very tough and ductile) such a disfigurement as is practically equivalent to fracture, by rendering the piece useless.

The *Ultimate Strength* of a body is the load required to produce fracture in some specified way. The *Proof Strength* is the load required to produce the greatest strain of a specific kind consistent with safety; that is, with the retention of the strength of the material unimpaired. A load exceeding the proof strength of the body, although it may not produce instant fracture, produces fracture eventually by long-continued application and frequent repetition.

The *Working Load* on each piece of a machine is made less than the ultimate strength, and less than the proof strength, in certain ratios determined partly by experiment and partly by practical experience, in order to provide for unforeseen contingencies.

Each solid has as many different kinds of strength as there are different ways in which it can be strained or broken, as shown in the following classification:—

	Strain.	Fracture.
Elementary......	Extension	......Tearing.
	Compression	......Crushing.
Compound.......	Distortion	......Shearing.
	Twisting	......Wrenching.
	Bending	......Breaking across.

413. **Co-efficients or Moduli of Strength** are quantities expressing the *intensity* of the stress under which a piece of a given material gives way when strained in a given manner; such intensity being expressed in units of weight for each unit of sectional area of the layer of particles at which the body first begins to yield. In Britain, the ordinary unit of intensity employed in expressing the strength of materials is the *pound avoirdupois on the square inch.* As to other units, see Article 302, page 342.

Co-efficients of strength are of as many different kinds as there are different ways of breaking a body. Their use will be explained in the sequel. Tables of their values are given at the end of the volume.

Co-efficients of strength, when of the same kind, may still vary according to the direction in which the stress is applied to the body. Thus the tenacity, or resistance to tearing, of most kinds of wood is much greater against tension exerted along than across the grain.

414. **Factors of Safety.**—A factor of safety, in the ordinary sense, is the ratio in which the load that is just sufficient to overcome instantly the strength of a piece of material is greater than the greatest safe ordinary working load.

The proper value for the factor of safety depends on the nature of the material; it also depends upon how the load is applied. The load upon any piece in a structure or in a machine is distinguished into *dead load* and *live load.* A *dead load* is a load which is put on by imperceptible degrees, and which remains steady; such as the weight of a structure, or of the fixed framing in a machine. A *live load* is one that is or may be put on suddenly, or accompanied with vibration; like a swift train travelling over a railway bridge; or like most of the forces exerted by and upon the moving pieces in a machine.

It can be shown that in most cases which occur in practice a live load produces, or is liable to produce, *twice,* or very nearly twice, the effect, in the shape of stress and strain, which an equal dead load would produce. The *mean* intensity of the stress produced by a suddenly applied load is no greater than that produced by the same load acting steadily; but in the case of the suddenly applied load, the stress begins by being insensible, increases to double its mean intensity, and then goes through a series of fluctuations, alternately below and above the mean, accompanied by vibration of the strained body. Hence the ordinary practice is to make the factor of safety for a live load *double* of the factor of safety for a dead load.

A distinction is to be drawn between *real* and *apparent* factors of safety. A real factor of safety is the ratio in which the ultimate or breaking stress is greater than the real working stress at the time when the straining action of the load is greatest. The apparent factor of safety has to be made greater than the real factor of safety in those cases in which the calculation of strength is based, not upon the greatest straining action of the load, but upon a mean straining action, which is exceeded by the greatest straining action in a certain proportion. In such cases the apparent factor of safety is the product obtained by multiplying the real factor of safety by the ratio in which the greatest straining action exceeds the mean.

Another class of cases in which the apparent exceeds the real factor of safety is when there are additional straining actions besides that due to the transmission of motive power, and when those additional actions, instead of being taken into account in detail, are allowed for in a rough way by means of an increase of the factor of safety. A third class of cases is when there is a possibility of an increased load coming by accident to act upon the piece under consideration. For example, a steam engine may drive two lines of shafting, exerting half its power on each; one may suddenly break down, or be thrown out of gear, and the engine may for a short time exert its whole power on the other.

The following table shows the ordinary values of real factors of safety:—

	Real Factors of Safety. Dead Load.	Real Factors of Safety. Live Load.
Perfect materials and workmanship, ...	2	4
Ordinary materials and workmanship—		
Metals, ...	3	6
Wood, Hempen Ropes, ...from	3 to 5	10
Masonry and Brickwork, ...	4	8

The following are examples of apparent factors of safety:—

Real Factor of Safety, 6.	Ratio in which Greatest Effort exceeds Mean Effort, nearly.	Apparent Factor of Safety.
Steam engines acting against a constant resistance—		
Single engine, ...	1·6	9·6
Pair of engines driving cranks at right angles, ...	1·1	6·6
Three engines driving equiangular cranks, ...	1·05	6·3

Ordinary cases of varying effort and resistance, ...	2·0	12·0
Lines of shafting in millwork; apparent factor of safety for twisting stress due to motive power, to cover allowances for bending actions, accidental extra load, &c., ...	from 18 to 36	

Almost all the experiments hitherto made on the strength of materials give co-efficients or moduli of *ultimate strength;* that is, co-efficients expressing the intensity of the stress exerted by the most severely strained particles of the material just before it gives way. In calculations for the purpose of designing framework or machinery to bear a given working load, there are two ways of using the factor of safety,—one is, to multiply the working load by the factor of safety, so as to determine the breaking load, and use this load in the calculation, along with the modulus of ultimate strength: the other is, to divide the modulus of ultimate strength by the factor of safety, and thus to find a modulus or co-efficient of *working stress*, which is to be used in the calculation, along with the *working load*. It is obvious that the two methods are mathematically equivalent, and must lead to the same result; but the latter is on the whole the more convenient in designing machines.

415. The **Proof** or **Testing** by experiment of the strength of a piece of material is conducted in two different ways, according to the object in view.

I. If the piece is to be *afterwards used*, the testing load must be so limited that there shall be no possibility of its impairing the strength of the piece; that is, it must not exceed the *proof strength*, being from one-third to one-half of the ultimate strength. About double or treble of the working load is in general sufficient. Care should be taken to avoid vibrations and shocks when the testing load approaches near to the proof strength.

II. If the piece is to be *sacrificed* for the sake of ascertaining the strength of the material, the load is to be increased by degrees until the piece breaks, care being taken, especially when the breaking point is approached, to increase the load by small quantities at a time, so as to get a sufficiently precise result.

The *proof strength* requires much more time and trouble for its determination than the ultimate strength. One mode of approximating to the proof strength of a piece is to apply a moderate load and remove it, apply the same load again and remove it, two or three times in succession, observing at each time of application of the load the *strain* or alteration of figure of the piece when loaded, by stretching, compression, bending, distortion, or twisting, as the case may be. If that alteration does *not sensibly increase* by repeated applications of the same load, the load is within the limit of proof strength. The effects of a greater and a greater load being successively tested in the same way, a load will at length be reached whose successive applications produce increasing disfigurements of the piece; and this load will be greater than the proof strength, which will lie between the last load and the last load but one in the series of experiments.

It was formerly supposed that the production of a *set*—that is, a disfigurement which continues after the removal of the load—was a test of the proof strength being exceeded; but Mr. Hodgkinson showed that supposition to be erroneous, by proving that in most materials a set is produced by almost any load, how small soever.

The strength of bars and beams to resist breaking across, and of axles to resist twisting, can be tested by the application of known weights either directly or through a lever.

To test the tenacity of rods, chains, and ropes, and the resistance of pillars to crushing, more powerful and complex mechanism is required. The apparatus most commonly employed is the hydraulic press. In computing the stress which it produces, no reliance ought to be placed on the load on the safety valve, or on a weight hung to the pump handle, as indicating the intensity of the pressure, which should be ascertained by means of a pressure gauge. This remark applies also to the proving of boilers by water pressure. From experiments by Messrs. Hick and Lüthy it appears that, in calculating the stress produced on a bar by means of a hydraulic press, the friction of the collar may be allowed for by deducting a force equivalent to the pressure of the water upon an area of a length equal to the circumference of the collar, and one-eightieth of an inch broad. (See page 444.)

For the exact determination of general laws, although the load may be applied at one end of the piece to be tested by means of a hydraulic press, it ought to be resisted and measured at the other end by means of a combination of levers.

416. **Stiffness or Rigidity, Pliability, their Moduli or Co-efficients.**—Rigidity or stiffness is the property which a solid body possesses of resisting forces tending to change its figure. It may be expressed as a quantity, called a *modulus or co-efficient of stiffness*, by taking the ratio of the intensity of a given stress of a given kind to the strain, or alteration of figure, with which that stress is accompanied—that strain being expressed as a quantity by dividing the alteration of some dimension of the body by the original length of that dimension. In most materials which are used in machinery, the moduli of stiffness, though not exactly constant, are nearly constant for stresses not exceeding the proof strength.

The reciprocal of a modulus of stiffness may be called a "*modulus of pliability;*" that is to say,

$$\text{Modulus of Stiffness} = \frac{\text{Intensity of Stress}}{\text{Strain}};$$

$$\text{Modulus of Pliability} = \frac{\text{Strain}}{\text{Intensity of Stress}}.$$

The use of specific moduli of stiffness will be explained in the sequel. Values of them are given in the tables prefixed to this chapter.

417. The **Elasticity of a Solid** consists of stiffness, or resistance to change of figure, combined with the power of recovering the original figure when the straining force is withdrawn. If that recovery is complete and immediate, the body is *perfectly elastic;* if there is a *set*, or permanent change of figure, after the removal of the straining force, the body is *imperfectly elastic.* The elasticity of no solid substance is absolutely perfect, but that of many substances is nearly perfect when the stress does not exceed the proof strength, and may be made sensibly perfect by restricting the stress within small enough limits.

Moduli or *Co-efficients of Elasticity* are the values of moduli of stiffness when the stress is so limited that the value of each of those moduli is sensibly constant, and the elasticity of the body sensibly perfect.

418. **Resilience** or **Spring** is the quantity of *mechanical work* required to produce the proof stress on a given piece of material, and is equal to the product of the *proof strain*, or alteration of figure, into the mean load which acts during the production of that strain; that is to say, in general, very nearly one-half of the proof load.

419. **Heights or Lengths of Moduli of Stiffness and Strength.**—The term *height* or *length*, as applied to a modulus or co-efficient of strength or of stiffness, means the length of an imaginary vertical column of the material to which the modulus belongs, whose weight would cause a pressure on its base equal in intensity to the stress expressed by the given modulus. Hence

Height of a modulus in feet

$$= \frac{\text{Modulus in lbs. on the square foot}}{\text{Heaviness of material in lbs. to the cubic foot}}$$

$$= \frac{\text{Modulus in lbs. on the square inch}}{\text{Weight of 12 cubic inches of the material}};$$

Height of a modulus in inches

$$= \frac{\text{Modulus in lbs. on the square inch}}{\text{Heaviness of material in lbs. to the cubic inch}};$$

Height of a modulus in mètres

$$= \frac{\text{Modulus in kilogrammes on the square mètre}}{\text{Heaviness of material in kilogrammes to the cubic mètre}}.$$

Several examples of this mode of stating the intensity of stress have already been given; as at pages 474, 475; and in the Tables, page 482.

SECTION II.—*Of Resistance to Direct Tension.*

420. **Strength, Stiffness, and Resilience of a Tie.**—The word *tie* is here used to denote any piece in framing or in mechanism, such as a rod, bar, band, cord, or chain, which is under the action of a pair of equal and opposite longitudinal forces tending to stretch it, and to tear it asunder. The common magnitude of those two forces is the load; and it is equal to the product of the sectional area of the piece into the intensity of the tensile stress. The values of that inteusity, corresponding to the immediate breaking load, the proof load, and the working load, are called respectively the moduli or co-efficients of *ultimate tenacity*, of *proof tension*, and of *working tension.*

In symbols, let P be the load, S the sectional area, and p the intensity of the tensile stress; then

$$P = p\,S. \qquad \text{(1.)}$$

If the sectional area varies at different points, the *least* area is to be taken into account in calculations of strength.

The elongation of a tie produced by any load, P, not exceeding the proof load, is found as follows, provided the sectional area is uniform.

Let x denote the original length of the tie, Δx the elongation, and $\alpha = \frac{\Delta x}{x}$ the extension; that is, the *proportion* which that elongation bears to the original length of the bar, being the numerical measure of the strain.

Let E denote the *modulus of direct elasticity*, or resistance to stretching, for examples of which, see the Tables. Then

$$\alpha = \frac{p}{E}; \ \Delta x = \alpha x = \frac{p}{E} x. \qquad \text{(2.)}$$

Let f' denote the *proof tension* of the material, so that f' S is the proof load of the tie; then the *proof extension* is $f' \div E$.

The **Resilience** or **Spring** of the tie, or the work done in stretching it to the limit of proof strain, is computed as follows. The length, as before, being x, the elongation of the tie produced by the proof load is $f' x \div E$. The force which acts through this space has for its least value 0, for its greatest value $P = f' S$, and for its mean value $f' S \div 2$; so that the work done in stretching the tie to the proof strain is

$$\frac{f' S}{2} \cdot \frac{f' x}{E} = \frac{f'^2}{E} \cdot \frac{S x}{2}. \quad\text{.....................(3.)}$$

The co-efficient $f'^2 \div E$, by which one-half of the volume of the tie is multiplied in the above formula, is called the MODULUS OF RESILIENCE. For examples of its value, see the Tables, page 485.

A *sudden pull* of $f' S \div 2$, or one-half of the proof load, being applied to the bar, will produce the entire proof strain of $f' \div E$, which is produced by the *gradual* application of the proof load itself; for the work performed by the action of the constant force $f' S \div 2$, through a given space, is the same with the work performed by the action, through the same space, of a force increasing at an uniform rate from 0 up to $f' S$. Hence a tie, to resist with safety the sudden application of a given pull, requires to have twice the strength that is necessary to resist the gradual application and steady action of the same pull. This is an illustration of the principle, that the factor of safety for a live load is twice that for a dead load.

421. **Thin Cylindrical and Spherical Shells.**—Let r denote the radius of a thin hollow cylinder, such as the shell of a high-pressure boiler;

t, the thickness of the shell;

f, the ultimate tenacity of the material, in pounds per square inch;

p, the intensity of the pressure, in pounds per square inch, required to burst the shell. This ought to be taken at SIX TIMES the effective working pressure—*effective pressure* meaning the excess of the pressure from within above the pressure from without, which last is usually the atmospheric pressure, of 14·7 lbs. on the square inch or thereabouts.

Then

$$p = \frac{f t}{r}; \quad\text{.........................(1.)}$$

and the proper proportion of thickness to radius is given by the formula,—

$$\frac{t}{r} = \frac{p}{f}. \quad\text{.........................(2.)}$$

Thin spherical shells are *twice as strong* as cylindrical shells of the same radius and thickness.

The tenacity of good wrought-iron boiler-plates is about 50,000 lbs.

per square inch. That of a double-rivetted joint, *per square inch of the iron left between the rivet holes* (if drilled, and not punched), is the same; that of a single-rivetted joint somewhat less, as the tension is not uniformly distributed. It is convenient in practice to state the tenacity of rivetted joints in lbs. *per square inch of the entire plate;* and it is so stated in the annexed table, in which the results for rivetted joints are from the experiments of Mr. Fairbairn, and that for a welded joint from an experiment by Mr. Dunn. The joints of plate-iron boilers are single rivetted; but from the manner in which the plates break joint, the ultimate tenacity of such boilers is considered to approach more nearly to that of a double-rivetted joint than to that of a single-rivetted joint.

Wrought-iron plate joints, double-rivetted, the diameter of each hole being $\frac{3}{10}$ of the pitch, or distance from centre to centre of holes, ...	35,000
Wrought-iron plate joints, single rivetted, ...	28,000
Wrought-iron boiler shells, with single-rivetted joints properly crossed, ...	34,000
Wrought-iron retort, with a welded joint, ...	30,750
Cast-iron boilers, cylinders, and pipes (average), ...	16,500
Malleable cast-iron cylinders, ...	48,000

422. **Thick Hollow Cylinders and Spheres.**—The assumption that the tension in a hollow cylinder or sphere is uniformly distributed throughout the thickness of the shell is approximately true only when the thickness is small as compared with the radius.

Let R represent the external and r the internal radius of a thick hollow cylinder, such as a hydraulic press, the tenacity of whose material is f, and whose bursting pressure is p. Then we must have

$$\frac{R^2 - r^2}{R^2 + r^2} = \frac{p}{f}; \text{ ....(1.)}$$

and, consequently,

$$\frac{R}{r} = \sqrt{\left(\frac{f + p}{f - p}\right)}; \text{ ....(2.)}$$

by means of which formula, when r, f, and p are given, R may be computed.

In the case of a hollow sphere the following formulæ give the ratios of the bursting pressure to the tenacity, and of the external to the internal radius:—

$$\frac{p}{f} = \frac{2 R^3 - 2 r^3}{R^3 + 2 r^3}; \text{ ....(3.)}$$

$$\frac{\mathrm{R}}{r} = \sqrt[3]{\left(\frac{2f + 2p}{2f - p}\right)} \dots\dots\dots\dots\dots\dots\dots (4.)$$

Section III.—*Of Resistance to Distortion and Shearing.*

423. **Distortion and Shearing Stress in General.**—In framework and mechanism many cases occur in which the principal pieces, such as plates, links, bars, or beams, being themselves subjected to tension, pressure, twisting, or bending, are connected with each other at their joints by rivets, bolts, pins, keys, or screws, which are under the action of a shearing force, tending to make them give way by the sliding of one part over another.

Every shearing stress is equivalent to a pair of direct stresses of the same intensity, one tensile and the other compressive, exerted in directions making angles of 45° with the shearing stress. Hence it follows that a body may give way to a shearing stress either by actual shearing, at a plane parallel to the direction of the shearing force, or by tearing, in a direction making an angle of 45° with that force. The manner of breaking depends on the structure of the material, hard and brittle materials giving way by tension, and soft and tough materials by shearing.

When a shearing force does not exceed the limit within which moduli of stiffness are sensibly constant, it produces distortion of the body on which it acts. Let q denote the intensity of shearing stress applied to the four lateral faces of an originally square prismatic particle, so as to distort it; and let ν be the *distortion*, expressed by the *tangent of the difference between each of the distorted angles of the prism and a right angle;* then

$$\frac{q}{\nu} = \mathrm{C}, \dots\dots\dots\dots\dots\dots\dots\dots\dots (1.)$$

is the *modulus of transverse elasticity*, or *resistance to distortion;* of which examples are given in the tables, page 479.

One mode of expressing the distortion of an originally square prism is as follows:—Let α denote the proportionate elongation of one of the diagonals of its end, and $-\alpha$ the proportionate shortening of the other; then the distortion is

$$\nu = 2\alpha.$$

The ratio $\frac{\mathrm{C}}{\mathrm{E}}$ of the modulus of transverse elasticity to the modulus of direct elasticity defined in Article 420, page 493, has different values for different materials, ranging from $\dot{0}$ to $\frac{1}{2}$. For wrought iron and steel it is about $\frac{1}{3}$.

The *ultimate shearing strength*, or modulus of resistance to shearing,—in other words, the intensity of the greatest shearing stress when the body is on the point of giving way,—is, in wrought iron and steel, and most other metals, equal, or nearly equal, to the tenacity: in cast iron it is about once and a half greater than the tenacity; in timber it is nearly equal to the tenacity across the grain. (See the Tables, page 479.)

424. **Strength of Fastenings and Joint-Pins.**—The connecting pieces already referred to as being exposed to the action of a shearing force may be distinguished into *fastenings*, such as rivets, keys, wedges, gibs and cottars, and screws, by which two pieces are secured together so as to act as one piece; and *joint-pins*, by which two pieces are so connected as to be free to turn about the joint. It is obvious that the figure of a joint-pin, as well as that of the hole or socket in which it works, must be that of a surface of revolution, such as a circular cylinder; and that the fit, though accurate, must be easy, like that of an axle in its bearings. Most fastenings and joint-pins are exposed to a bending as well as to a shearing action, and in some cases the most severe stress is that arising from the bending action; but in other cases the most severe stress is that produced by the shearing load. These latter cases are as follows:—All rivets, keys, and other fastenings which are tightly jammed in their holes; all cylindrical joint-pins, fixed at one end, in which the length of the loaded part is less than one-third of the diameter; and all cylindrical joint-pins, fixed at both ends, in which the length of the loaded part is less than two-thirds of the diameter.

In order that the shearing stress on a connecting piece may be uniformly distributed over the cross-section, it is necessary that the fastening should be held so tight in its hole or socket that the friction at its surface may be at least of equal intensity to the shearing stress; and then the intensity of that stress is represented simply by $P \div A$; P being the shearing load, and A the area which resists it.

But when the connecting piece fits easily, as must always be the case with joint-pins, the *greatest intensity* of the stress, to which the strength of the connecting piece must be adapted, exceeds the *mean intensity* $P \div A$, in a ratio which depends on the figure of the cross-section; and whose values, for the ordinary figures, are

$$\text{for rectangular cross-sections, } \frac{3}{2};$$

$$\text{for circular and elliptic cross-sections, } \frac{4}{3};$$

and the sectional area must accordingly be made greater in that ratio than the area which would have been sufficient had the connecting piece fitted tightly.

The chief kinds of connecting pieces, to which these principles have to be applied, will now be considered separately.

425. **Rivets** are made of the most tough and ductile metal. (See, for example, "Rivet Iron," in pages 460 and 482.)

The ordinary dimensions of rivets in practice are as follows:—

Diameter of a rivet for plates less than half an inch thick, about double the thickness of the plate.

For plates of half an inch thick and upwards, about once and a-half the thickness of the plate.

Length of a rivet before clenching, measuring from the head = sum of the thicknesses of the plates to be connected + $2\frac{1}{2}$ × diameter of the rivet.

The longitudinal compression to which a rivet is subjected during the operation of clenching, whether by hand or by machinery, tends to make it fit its hole tightly, and thus to produce uniform distribution of the stress; but as such uniformity cannot be expected to be always realized, it is usual to assume, in practice, that there is a deviation from uniformity of shearing stress sufficient to neutralize the greater toughness of the metal in the rivets than in the plates which they connect; and, therefore, the distance apart of the rivets used to connect two pieces of metal plate together is regulated by the rule, that *the joint sectional area of the rivets shall be equal to the sectional area of plate left after punching the rivet holes.* This rule leads to the following algebraical formula:—

Let t denote the thickness of the plates;
d, the diameter of a rivet;
n, the number of ranks of rivets;

it being understood that the rivets which form a rank stand in a line perpendicular to the direction of the tension which tends to pull the plates asunder.

c, the *pitch*, or distance from centre to centre of the adjoining rivets in one rank; then

$$c = d + \frac{\cdot 7854\, n\, d^2}{t}. \qquad \text{(1.)}$$

Each plate is weakened by the rivet holes in the ratio

$$\frac{c-d}{c} = \frac{\cdot 7854\, n\, d}{t + \cdot 7854\, n\, d}; \qquad \text{(2.)}$$

In "single-rivetted" joints, $n = 1$; in "double-rivetted" joints, $n = 2$, and the two ranks of rivets form a zig-zag; in "chain-rivetted" joints, n may have any value greater than 1. A single-rivetted joint is weakened by unequal distribution of the tension on the plate in the ratio of 4 : 5.

Suppose that in a chain-rivetted joint the pitch c from centre to centre of the rivets is fixed, so as not to weaken the plates below a given limit; then in order to find how many ranks of rivets there should be,—in other words, how many rivets there should be in each file,—the following formula may be used :—

$$n = \frac{(c - d)\, t}{\cdot 7854\, d^2} \quad \text{.......................(3.)}$$

426. **Pins, Keys, Wedges, Gibs, and Cottars.**—These fastenings are, like rivets, themselves exposed to a shearing load, while they serve to transmit a pull or thrust from one piece in framework or mechanism to another; and the rule for determining their proper sectional area is the same, with this modification only, that it is safest in most, if not in all cases, to allow for the possibility of an easy fit, according to the rule stated at the end of Article 424, page 497.

In order that a wedge, key, or cottar may be safe against slipping out of its seat, its angle of obliquity ought not to exceed the angle of repose of metal upon metal, which, to provide for the contingency of the surfaces being greasy, may be taken at about 4°. (Article 309, page 349.)

427. **Bolts and Screws.**—If a bolt has to withstand a shearing stress, its diameter is to be determined like that of a cylindrical pin. If it has to withstand tension, its diameter is to be determined by having regard to its tenacity. In either case the effective diameter of the bolt is its least diameter; that is, if it has a screw on it, the diameter of the spindle inside the thread. It is to be observed, however, that in order to provide for possible irregularities in the distribution of the stress, it is customary to use for screws a very large factor of safety, ranging from 12 to 15; the mean intensity of the working stress on wrought-iron screws being only about 4,000 lbs. on the square inch, or 2·8 kilogrammes on the square millimètre.

The ordinary form of section of the thread of a fastening screw is an isosceles triangle with the angles rounded; and according to the proportions recommended by Mr. Whitworth, the angle at the summit is 55°, making the height of the triangle = 0·96 of its base. One-sixth of that height is taken away by the rounding of the edge of the thread, and another sixth by the rounding of the bottom of the groove, leaving two-thirds, or 0·64 of the base; and as the base of the triangle is the pitch of the screw, the projection of the thread is 0·64 of the pitch.

The pitch should not in general be greater than *one-fifth of the effective diameter*, and may be considerably less: for example, one-tenth and one-twelfth are ordinary proportions.

In order that the resistance of a screw or screw-bolt to rupture by stripping a triangular thread may be at least equal to its resistance to direct tearing asunder, the length of the nut should be *at least one-half* of the effective diameter of the screw; and it is often in practice considerably greater; for example, once and a half that diameter.

The head of a bolt is usually about twice the diameter of the spindle, and of a thickness which is usually greater than five-eighths of that diameter.

Section IV.—*Of Resistance to Twisting and Wrenching.*

428. **Twisting or Torsion in General.**—Torsion is the condition of strain into which a cylindrical or prismatic body is put when a pair of couples of equal and opposite moment, tending to make it rotate about its axis in contrary directions, are applied to its two ends. Such is the condition of shafts which transmit motive power. The moment is called the *twisting moment*, and at each cross-section of the bar it is resisted by an equal and opposite moment of stress. Each particle of the shaft is in a state of distortion, and exerts shearing stress.

In British measures, twisting moments are expressed in *inch-lbs.*

429. **Strength of a Cylindrical Shaft.** (*A. M.*, 321.)—A cylindrical shaft, A B, fig. 267, being subjected to the twisting moment of a pair of equal and opposite couples applied to the cross-sections A and B, it is required to find the condition of stress and strain at any intermediate cross-section, such as S, and also the angular displacement of any cross-section relatively to any other.

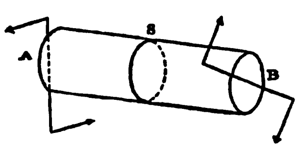

Fig. 267.

From the uniformity of the figure of the bar, and the uniformity of the twisting moment, it is evident that the condition of stress and strain of all cross-sections is the same; also, because of the circular figure of each cross-section, the condition of stress and strain of all particles at the same distance from the axis of the cylinder must be alike.

Suppose a circular layer to be included between the cross-section S, and another cross-section at the longitudinal distance $d\,x$ from it. The twisting moment causes one of those cross-sections to rotate relatively to the other, about the axis of the cylinder, through an angle which may be denoted by $d\,\theta$. Then if there be two points at the same distance, r, from the axis of the cylinder, one in the one cross-section and the other in the other, which points were originally in one straight line parallel to the axis of the cylinder, the twisting moment shifts one of those points laterally, relatively to the other, through the distance $r\,d\,\theta$. Consequently the part of the layer which lies between those points is in a condition of *distortion*, in a plane perpendicular to the radius r; and the distortion is expressed by the ratio

$$\nu = r \cdot \frac{d\,\theta}{d\,x}; \text{..........................(1.)}$$

which varies *proportionally to the distance from the axis.* There is therefore a *shearing stress* at each point of the cross-section, whose direction is perpendicular to the radius drawn from the axis to that point, and whose intensity is *proportional to that radius,* being represented by

$$q = \mathrm{C}\,\nu = \mathrm{C}\,r \cdot \frac{d\,\theta}{d\,x}. \text{ ....(2.)}$$

The STRENGTH of the shaft is determined in the following manner:—Let q_1 be the limit of the shearing stress to which the material is to be exposed, being the *ultimate* resistance to wrenching if it is to be broken, the *proof* resistance if it is to be tested, and the *working* resistance if the working moment of torsion is to be determined. Let r_1 be the external radius of the axle. Then q_1 is the value of q at the distance r_1 from the axis; and at any other distance, r, the intensity of the shearing stress is

$$q = \frac{q_1\,r}{r_1}. \text{ ....(3.)}$$

Conceive the cross-section to be divided into narrow concentric rings, each of the breadth $d\,r$. Let r be the *mean radius* of one of these rings. Then its area is $2\,\pi\,r\,d\,r$; the intensity of the shearing stress on it is that given by equation (3), and the leverage of that stress relatively to the axis of the cylinder is r; consequently the moment of the shearing stress of the ring in question, being the product of those three quantities, is

$$\frac{2\,\pi\,q_1}{r_1} \cdot r^3\,d\,r;$$

which being integrated for all the rings from the centre to the circumference of the cross-section, gives for the moment of torsion, and of resistance to torsion,

$$\mathrm{M} = \frac{\pi}{2}\,q_1\,r_1^3 = \frac{\pi}{16}\,q_1\,h_1^3; \text{ ....(4.)}$$

if $h = 2 r_1$ be the diameter of the shaft,

$$\left(\frac{\pi}{2} = 1·5708; \frac{\pi}{16} = 0·196 \text{ nearly}\right).$$

If the axle is *hollow*, h_0 being the diameter of the hollow, the moment of torsion becomes

$$M = \frac{\pi}{16} \cdot q_1 \frac{h_1^4 - h_0^4}{h_1}. \qquad \text{(5.)}$$

The following formulæ serve to calculate the diameters of shafts when the twisting moment and stress are given; solid shafts:—

$$h_1 = \left(\frac{5·1 \text{ M}}{q_1}\right)^{\frac{1}{3}}; \qquad \text{(6.)}$$

hollow shafts—

$$h_1 = \left\{\frac{5·1 \text{ M}}{q_1\left(1 - \frac{h_0^4}{h_1^4}\right)}\right\}^{\frac{1}{3}}. \qquad \text{(7.)}$$

which last formula serves to compute the diameter of a hollow axle, when the *ratio* $h_0 : h_1$ of its internal and external diameter has been fixed.

Values of the ultimate shearing strength of various substances are given in the Tables. As for the *working stress*, a long series of practical trials has shown that wrought-iron axles bear a stress of 9,000 lbs. per square inch, or 6·3 kilogrammes on the square millimètre, for any length of time, if well manufactured of good material, the factor of safety being about 6. If the ultimate shearing stress of cast iron, 27,000 lbs. on the square inch, is divided by the same factor, the modulus of working stress is found to be 4,500 lbs. on the square inch, or nearly 3·2 kilogrammes on the square millimètre.

It is chiefly in the shafting of mills that those large apparent factors of safety are met with, referred to in Article 414, page 490.

430. **Angle of Torsion.**—Suppose a pair of diameters, originally parallel, to be drawn across the two circular ends, A and B, fig. 267, page 500, of a cylindrical shaft, solid or hollow; it is proposed to find the angle which the directions of those lines make with each other when the shaft is twisted, either by the working moment of torsion, or by any other moment.

This question is solved by means of equation (2) of Article 429, page 501, which gives for the *angle of torsion per unit of length*,

$$\frac{d\,\theta}{dx} = \frac{q}{\text{C}\,r}$$

The condition of the shaft being uniform at all points of its length, the above quantity is constant; and if x be the length of the shaft, and θ the angle of torsion sought, expressed in length of arc to radius 1, we have $\frac{\theta}{x} = \frac{d\,\theta}{d\,x}$, and therefore,

$$\theta = \frac{x\,q}{C\,r}. \qquad \text{(1.)}$$

I. Let the moment of torsion be the *working moment*, for which

$$\frac{q}{r} = \frac{q_1}{r_1} = \frac{2\,q_1}{h_1};$$

the value taken for the modulus, q_1, being the *safe working stress.*

Then the angle of *working torsion* is

$$\theta = \frac{2\,q_1\,x}{C\,h_1}; \qquad \text{(2.)}$$

and is the same whether the shaft is solid or hollow. This formula gives the angle θ in *circular measure;* that is, in arc to radius unity; so that if at each end of the shaft there is an arm of the length y, the displacement of the end of one of those arms relatively to the other will be $y\,\theta$.

Values of C, the co-efficient of transverse elasticity, are given in the tables. In calculating the *working torsion* of wrought-iron shafts, we may make

$$\frac{q_1}{C} = \text{from } \frac{1}{1{,}000} \text{ to } \frac{1}{1{,}200}. \qquad \text{(3.)}$$

II. The *proof torsion,* to which a shaft may be twisted by a gradually applied load when testing it, may be made double the working torsion.

III. Let the moment of torsion have any amount, M, consistent with safety. Then for $\frac{q}{r}$ we have to put its value in terms of M and h_1; and the results are as follows:—

For solid shafts, $\frac{q}{r} = \frac{2\,M}{\pi\,r_1^4}$; and

$$\theta = \frac{32\,M\,x}{\pi\,C\,h_1^4} = 10{\cdot}2\,\frac{M\,x}{C\,h_1^4} \text{ nearly}; \qquad \text{(4.)}$$

For hollow shafts, $\frac{q}{r} = \frac{2\,M}{\pi\,(r_1^4 - r_0^4)}$; and

$$\theta = \frac{32\ \mathrm{M}\ x}{\pi\ \mathrm{C}\ (h_1^4 - h_0^4)} = 10{\cdot}2\ \frac{\mathrm{M}\ x}{\mathrm{C}\ (h^4 - h_0^4)}\ \text{nearly.}\ \ldots\ldots\ldots(5.)$$

An example of the application of equation (4) has already been given in Article 344.

431. The **Resilience of a Cylindrical Shaft** is the product of one-half of the moment of proof torsion into the corresponding anglo of torsion; and it is given by the following equation:—

$$\left.\begin{array}{l} \dfrac{\mathrm{M}\ \theta}{2} = \dfrac{\pi}{16} \cdot \dfrac{q_1^2\ h_1^2\ x}{\mathrm{C}}\ \text{for a solid shaft; or} \\ \dfrac{\mathrm{M}\ \theta}{2} = \dfrac{\pi}{16} \cdot \dfrac{q_1^2\ (h_1^4 - h_0^4)\ x}{\mathrm{C}\ h_1^2}\ \text{for a hollow shaft.} \end{array}\right\}\ \ldots\ldots(1.)$$

432. **Shafts not Circular in Section.**—When the cross-section of a shaft is not circular, it is certain that the ratio $\frac{q}{r}$ of the shearing stress at a given point to the distance of that point from the axis of the shaft is not a constant quantity at different points of the cross-section, and that in many cases it is not even approximately constant; so that formulæ founded on the assumption of its being constant are erroneous. The mathematical investigations of M. de St. Venant have shown how the intensity of the shearing stress is distributed in certain cases.

The most important case in practice to which M. de St. Venant's method has been applied is that of a square shaft; and it appears that its moment of torsion is given by the formula,

$$\mathrm{M} = 0{\cdot}281\ q_1\ h^3\ \text{nearly};$$

in which h is one side of the square cross-section.

SECTION V.—*Of Resistance to Bending and Cross-Breaking.*

433. **Resistance to Bending in General.**—In explaining the principles of the resistance which bodies oppose to bending and cross-breaking, it is convenient to use the word *beam* as a general term to denote the body under consideration; but those principles are applicable not only to beams for supporting weights, but to levers, cross-heads, cross-tails, shafts, journals, cranks, and all pieces in machinery or framework to which forces are applied tending to bend them and to break them across; that is to say, forces transverse to the axis of the piece.

Conceive a beam which is acted upon by a combination of parallel transverse forces that balance each other, to be divided into two parts by an imaginary transverse section; and consider separately the conditions of equilibrium of one of those parts. The external transverse forces which act on that part, and constitute the load on it, do not necessarily balance each other. Their resultant may be found by Rule IV. of Article 280, page 324. That resultant is called the *Shearing Load* at the cross-section under consideration, and it is balanced by the *Shearing Stress* exerted by the particles which that cross-section traverses. The resultant moment of the same set of forces, relatively to the same cross-section, may be found by the same rule; it is called the *Bending Moment* at that cross-section, and it is balanced (if the beam is strong enough) by the *Moment of Stress* exerted by the particles which the cross-section traverses, called also the *Moment of Resistance*. That moment of stress is due wholly to longitudinal stress, and it is exerted in the following way:—The bending of the beam causes the originally straight layers of particles to become curved; those near the concave side of the beam become shortened; those near the convex side, lengthened; the shortened layers exert longitudinal thrust; the lengthened layers, longitudinal tension; the resultant thrust and the resultant tension are equal and opposite, and compose a couple, whose moment is the moment of stress, equal and opposite to the bending moment.

In problems respecting the transverse strength and stiffness of beams there are four processes: *first*, to determine the shearing load and bending moment produced by the transverse external forces at different cross-sections, and especially at those cross-sections at which they act most severely; *secondly*, to determine the relations between the dimensions and figure of a cross-section of the beam, and the moment of stress which that cross-section is capable of exerting, so that each cross-section, and especially that at which the bending moment is greatest, may have sufficient strength; *thirdly*, to determine the relations between the dimensions and figure of the beam and the deflection produced by the bending moments, in order that the beam may be so designed as to have sufficient stiffness or sufficient flexibility, according to its purpose.

434. **Calculation of Shearing Loads and Bending Moments.**—In the formulæ which follow, the shearing load at a given cross-section will be denoted by F, and the bending moment by M. In British measures it is most convenient to express the bending moment in *inch-lbs.*, because of the transverse dimensions of pieces in machines being expressed in inches.

The mathematical process for finding F and M at any given cross-section of a beam, though always the same in principle, may be varied considerably in detail. The following is on the whole the most convenient way of conducting it:—

Fig. 268 represents a beam *supported* at both ends, and loaded between them. Fig. 269 represents a *bracket;* that is, a beam *supported* and *fixed* at one end, and loaded on a projecting portion. P, Q, represent in each case the supporting forces; in fig. 268, W_1,

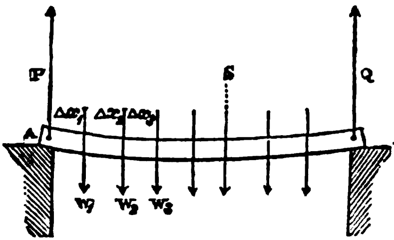

Fig. 268.

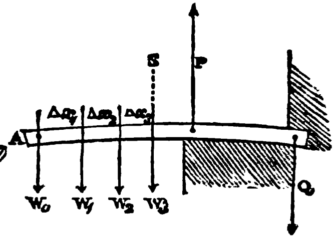

Fig. 269.

W_2, W_3, &c., represent portions of the load; in fig. 269, W_0 represents the endmost portion of the load, and W_1, W_2, W_3, other portions; in both figures, Δx_1, Δx_2, Δx_3, &c., denote the lengths of the intervals into which the lines of action of the portions of the load divide the longitudinal axis of the beam. The forces marked W may be the weights of parts of the beam itself, or of bodies carried by it; or they may be forces exerted by moving pieces in a machine on each other; or, in short, they may be any external transverse forces. If the body called the beam is a shaft, P and Q will be the bearing pressures.

The figures represent the load as applied at detached points; but when it is continuously distributed, the length of any indefinitely short portion of the beam may be denoted by $d\,x$, the intensity of the load upon it *per unit of length* by w, and the amount of the load upon it by $w\,d\,x$.

The process to be gone through will then consist of the following steps:—

STEP I. *To find the Supporting Forces or Bearing Pressures,* P *and* Q.—Assume any convenient point in the longitudinal axis as origin of co-ordinates, and find the distance x_0 of the resultant of the load from it, by Rule IV. of Article 280, pages 324, 325; that is to say,

$$\left.\begin{aligned} x_0 &= \frac{\Sigma \cdot x\,W}{\Sigma \cdot W};\ \text{or} \\ x_0 &= \frac{\int x\,w\,d\,x}{\int w\,d\,x}. \end{aligned}\right\} \qquad (2.)$$

Then, by Rule II. of Article 280, page 323, find the two supporting forces or bearing pressures, P and Q; that is to say, let R be the resultant load, and P R and R Q its distances from the points of support; and make

$$\left.\begin{matrix} PQ : PR : QR \\ :: R : Q : P. \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (3.)$$

STEP II. *To find the Shearing Loads at a Series of Sections.*—In what position soever the origin of co-ordinates may have been during the previous step, assume it now, in a beam supported at both ends, to be at one of the points of support (as A, fig. 268), and in a bracket to be at the loaded point farthest from the fixed end (as A, fig. 269). Consider P as positive and W as negative.

Then the shearing load in any given interval of the length of the beam is the resultant of all the forces acting on the beam from the origin to that interval; so that it has the series of values,

In Fig. 268.	In Fig. 269.
$F_{01} = P;$	$-F_{01} = W_0;$
$F_{12} = P - W_1;$	$-F_{12} = W_0 + W_1;$
$F_{23} = P - W_1 - W_2;$	$-F_{23} = W_0 + W_1 + W_2;$
$F_{34} = P - W_1 - W_2 - W_3;$	$-F_{34} = W_0 + W_1 + W_2 + W_3$
&c.;	&c.;
and generally,	and generally,
$F = P - \Sigma \cdot W; \ldots (4.)$	$-F = \Sigma \cdot W; \ldots\ldots (5.)$

so that the shearing loads which act in a series of intervals of the length of the beam can be computed by successive subtractions or successive additions, as the case may be.

For a continuously distributed load, these equations become respectively,

$$\text{In a beam supported at both ends, } F = P - \int_0^{x'} w\, d\, x; \text{ (6.)}$$

$$\text{In a bracket, } -F = \int_0^{x'} w\, d\, x; \ldots\ldots\ldots\ldots (7.)$$

in which expressions, x' denotes the distance from the origin, A, to the plane of section under consideration.

The positive and negative signs distinguish the two contrary directions of the distortion which the shearing load tends to produce.

The **Greatest Shearing Load** acts in a beam supported at both ends, close to one or other of the points of support, and its value is either P or Q. In a bracket, the greatest shearing load on the projecting part acts close to the outer point of support, and its value is equal to the entire load.

In a beam supported at both ends the **Shearing Load vanishes,** or changes from positive to negative, at some intermediate section, whose position may be found from equation (4) or equation (6), by making $F = 0$. At the second point of support, $F = -Q$.

STEP III. *To find the Bending Moments at a Series of Sections.*—

At the origin A there is no bending moment. Multiply the length of each of the intervals Δx of the longitudinal axis of the beam by the shearing load F, which acts throughout that interval; the first of the products so obtained is the bending moment at the inner end of the first interval; and by adding to it the other products successively, there are obtained successively the bending moments at the inner ends of the other intervals.*

That is to say,—bending moment

at the origin A; $M_0 = 0$;
at the line of action of W_1; $M_1 = F_{01} \cdot \Delta x_1$;
" " " W_2; $M_2 = F_{01} \cdot \Delta x_1 + F_{12} \Delta x_2$;
&c. &c.

$$\text{and generally, } M = \Sigma \cdot F \Delta x \text{..................(8.)}$$

If the divisions Δx are of equal lengths, this becomes

$$M = \Delta x \cdot \Sigma F; \text{....................(9.)}$$

and for a continuously distributed load,

$$M = \int_0^x F\, d x \text{........................(10.)}$$

Substituting for F, equation (10), its values as given by equations (6) and (7) respectively, we obtain the following results:—

For a beam supported at both ends,

$$M = P_1 x' - \int_0^{x'} \int_0^x w\, d x^2$$
$$= P_1 x' - \int_0^{x'} (x' - x)\, w\, d x; \text{.................(11.)}$$

For a beam fixed at one end,

$$- M = \int_0^{x'} \int_0^x w\, d x^2 = \int_0^{x'} (x' - x)\, w\, d x; \text{.........(12.)}$$

in the latter of which equations the symbol $-$ M denotes that the bending moment acts downwards.

The **Greatest Bending Moment** acts, in a bracket, at the outer point of support; and in a beam supported at both ends, at the section where the shearing load vanishes.

STEP IV. *To deduce the Shearing Load and Bending Moment in one Beam from those in another Beam similarly supported and loaded.*—This is done by the aid of the following principles:—

When beams differing in length and in the amounts of the loads upon them are similarly supported, and have their loads similarly distributed, the shearing loads at corresponding sections in them vary as

* See Mr. Herbert Latham's work *On Iron Bridges.*

the total loads, and the bending moments as the products of the loads and lengths.

The length between the points of support of a beam supported at the ends, as in fig. 268, is often called the *span*.

435. **Examples.**—In the following formulæ, which are examples of the application of the principles of the preceding Article to the cases which occur most frequently in practice, W denotes the total load;

w, when the load is distributed, the load per unit of length of the beam;

c, in brackets, the length of the free part of the bracket;

c, in beams either loaded or supported at both ends, the *half span*, between the extreme points of load or support and the middle;

M, the greatest bending moment.

I. Bracket fixed at one end and loaded at the other,............................ $M = c\,W$......(1.)

II. Bracket fixed at one end and uniformly loaded,........................ $M = \frac{c\,W}{2} = \frac{w\,c^2}{2}$ (2.)

III. Beam supported at both ends and loaded at an intermediate point, whose distance from the middle of the span is x,.......................... $M = \frac{(c^2 - x^2)\,W}{2\,c}$ (3.)

IV. Beam supported at both ends and loaded in the middle,................ $(x = 0)$; $M = \frac{c\,W}{2}$ (4.)

V. Beam supported at both ends and uniformly loaded,..................... $M = \frac{c\,W}{4} = \frac{w\,c^2}{2}$ (5.)

VI. If a beam has equal and opposite couples applied to its two ends; for example, if the beam in fig. 270 has the couple of equal and opposite forces P_1 applied at A and B, and the couple of equal and opposite forces P_2 at C and D, and if the opposite moments $P_1 \cdot A\,B = P_2 \cdot C\,D = M$ are equal, then each of the endmost divisions, A B and C D, is in the condition of a bracket fixed at one end and loaded at the other (Example I.); and the middle division, B C, is acted upon by the *uniform bending moment* M, and by no shearing load.

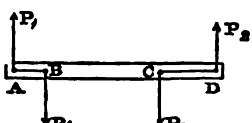

Fig. 270.

VII. Let a beam of the half-span c be loaded with an uniformly distributed load of w units of weight per unit of span; and at a point whose distance from the middle of the span is a, let there be applied an additional load W. It is required to find x, the distance from the middle of the span at which the greatest bending moment is exerted, and M, that greatest moment.

Make

$$\frac{W}{2\,c\,w} = m;$$

then the solutions are as follows :—

CASE 1.—When $\frac{a}{c} =$ or $> \frac{m}{1+m}$; $x = m\,(c - a)$; and

$$M = \frac{w\,c^2}{2}\left(1 + m - \frac{m\,a}{c}\right)^2 . \qquad (6.)$$

CASE 2.—When $\frac{a}{c} =$ or $< \frac{m}{1+m}$; $x = a$; and

$$M = \frac{w\,c^2}{2}(1 + 2\,m)\left(1 - \frac{a^2}{c^2}\right) \qquad (7.)$$

In the following case both sets of formulæ give the same result; when $\frac{a}{c} = \frac{m}{1+m}$; $x = a = m\,(c - a)$; and

$$M = \frac{w\,c^2}{2}\left(\frac{1 + 2\,m}{1+m}\right)^2 . \qquad (8.)$$

436. **Bending Moments produced by Longitudinal and Oblique Forces.**—When a bar is acted upon at a given cross-section by any external force, whose line of action, whether transverse, oblique, or parallel to the axis of the bar, does not traverse the centre of magnitude of that cross-section (see Article 293, page 334), that force exerts a moment upon that cross-section equal to the product of the force into the perpendicular distance of its line of action from the centre of the cross-section, and that moment is to be balanced by the moment of longitudinal stress at the cross-section.

The external force may be resolved into a longitudinal and a transverse component. The longitudinal component is balanced by an uniform longitudinal tension or pressure, as the case may be, exerted at the cross-section, and combined with the stress which resists the bending moment; and the transverse component is resisted by shearing stress.

437. **Moment of Stress—Transverse Strength.**—The bending moment at each cross-section of a beam bends the beam so as to make any originally plane longitudinal layer of the beam, perpendicular to the plane in which the load acts, become concave in the direction towards which the moment acts, and convex in the opposite

direction. Thus, fig. 271 represents a side view of a short portion of a bent beam; C C′ is a layer, originally plane, which is now bent so as to become concave at one side and convex at the other.

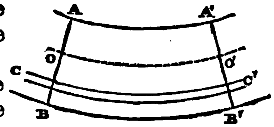

Fig. 271.

The layers at and near the concave side of the beam, A A′, are shortened, and the layers near the convex side, B B′, lengthened, by the bending action of the load. There is one intermediate surface, O O′, which is neither lengthened nor shortened; it is called the "*neutral surface.*" The particles at that surface are not necessarily, however, in a state devoid of strain; for, in common with the other particles of the beam, they are compressed and extended in a pair of diagonal directions, making angles of 45° with the neutral surface, by the shearing action of the load, when such action exists.

The condition of the particles of a beam, produced by the combined bending and shearing actions of the load, is illustrated by fig. 272, which represents a vertical longitudinal section of a rectangular beam, supported at the ends, and loaded at intermediate points. It is covered with a network consisting of two sets of curves cutting each other at right angles. The curves convex upwards are *lines of direct thrust;* those convex downwards are *lines of direct tension*. A pair of tangents to the pair of curves which traverse any particle are the *axes of stress* of that particle. The *neutral surface* is cut by both sets of curves at angles of 45°. At that vertical section of the beam where the shearing load vanishes, and the bending moment is greatest, both sets of curves become parallel to the neutral surface.

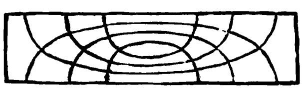
Fig. 272.

When a beam breaks under the bending action of its load, it gives way either by the crushing of the compressed side, A A′, or by the tearing of the stretched side, B B′.

In fig. 273, A represents a beam of a granular material, like cast iron, giving way by the crushing of the compressed side, out of which a sort of wedge is forced. B represents a beam giving way by the tearing asunder of the stretched side.

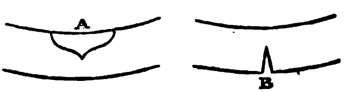

Fig. 273.

The *resistance* of a beam to bending and cross-breaking at any given cross-section is the moment of a couple, consisting of the thrust along the longitudinally-compressed layers, and the equal and opposite tension along the longitudinally-stretched layers.

It has been found by experiment, that in most cases which occur in practice, the longitudinal stress of the layers of a beam may, without material error, be assumed to be *uniformly varying*, its intensity being simply proportional to the distance of the layer from the neutral surface.

Let fig. 274 represent a cross-section of a beam (such as that represented in fig. 271), A the compressed side, B the extended side, C any layer, and O O the *neutral axis* of the section, being the line in which it is cut by the neutral surface. Let p denote the intensity of the stress along the layer C, and y the distance of that layer from the neutral axis. Because the stress is uniformly varying, $p \div y$ is a constant quantity. Let that constant be denoted for the present by a.

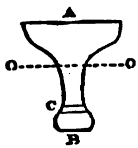

Fig. 274.

Let z be the breadth of the layer C, and $d\,y$ its thickness;
Then the amount of stress along it is

$$p\,z\,d\,y = a\,y\,z\,d\,y;$$

the amount of the stress along all the layers at the given cross-section is

$$a \int y\,z\,d\,y;$$

and this amount must be nothing,—in other words, the total thrust and total tension at the cross-section must be equal,—because the forces applied to the beam are wholly transverse; from which it follows, that

$$\int y\,z\,d\,y = 0, \text{ .... } (1.)$$

and the *neutral axis traverses the centre of magnitude of the cross-section.* This principle enables the neutral axis to be found by the aid of the methods explained in Article 293, page 334.

To find the greatest value of the constant $p \div y$ consistent with the strength of the beam at the given cross-section, let y_a be the distance of the compressed side, and y_b that of the extended side from the neutral axis; f_a the greatest thrust, and f_b the greatest tension, which the material can bear in the form of a beam; compute $f_a \div y_a$, and $f_b \div y_b$, and adopt the *less* of those two quantities as the value of $p \div y$, which may now be denoted by $f \div y_1$; f being f_a or f_b, and y_1 being y_a or y_b, according as the beam is liable to give way by crushing or by tearing.

For the best economy of material, the two quotients ought to be equal; that is to say,

$$\frac{f}{y_1} = \frac{f_a}{y_a} = \frac{f_b}{y_b} = \frac{f_a + f_b}{h}; \text{ .... } (1\text{ A.})$$

and this gives what is called a *cross-section of equal strength.*

The moment relatively to the neutral axis, of the stress exerted along any given layer of the cross-section, is

$$y\,p\,z\,d\,y = \frac{f}{y_1}\,y^2\,z\,d\,y;$$

and the sum of all such moments, being the MOMENT OF STRESS, or MOMENT OF RESISTANCE of the given cross-section of the beam to breaking across, is given by the formula,

$$M = \int p\,y\,z\,d\,y = \frac{f}{y_1}\int y^2\,z\,d\,y; \quad \text{............(2.)}$$

or making $\int y^2\,z\,d\,y = I$,

$$M = \frac{f\,I}{y_1}. \quad \text{........................(2 A.)}$$

When the *breaking* load is in question, the co-efficient f is what is called the MODULUS OF RUPTURE of the material.

When the *proof load* or *working load* is in question, the co-efficient f is the modulus of rupture divided by a suitable *factor of safety*, which, for the working stress in parts of machinery that are made of metal, is usually 6, and for the parts made of wood, 10. Thus, the *working modulus* f is usually 9,000 lbs. on the square inch for wrought iron, 4,500 for cast iron, and from 1,000 to 1,200 for wood.

The factor denoted by I in the preceding equation is what is called the "*geometrical moment of inertia*" of the cross-section of the beam. For sections whose figures are similar, or are parallel projections of each other, the moments of inertia are to each other as the breadths, and as the cubes of the depths of the sections; and the values of y_1 are as the depths. If, therefore, b be the breadth and h the depth of the rectangle circumscribing the cross-section of a given beam at the point where the moment of stress is greatest, we may put

$$I = n'\,b\,h^3, \quad \text{..........................(3.)}$$

$$y = m'\,h, \quad \text{............................(4.)}$$

n' and m' being numerical factors depending on the form of section; and making $n' \div m' = n$, the moment of resistance may be thus expressed,—

$$M = n\,f\,b\,h^2. \quad \text{........................(5.)}$$

Hence it appears that the *resistances of similar cross-sections to cross-breaking are as their breadths and as the squares of their depths.*

Another way of expressing the moment of resistance is as follows:—Let S be the sectional area of the beam, then we have

$$I = k' h^2 S; \quad \text{(3 A.)}$$

in which $k' h^2$ is the *radius of gyration* of the cross-section, k' being a numerical factor depending on the form of section. Then making $k' \div m' = k$, the moment of resistance may be thus expressed:—

$$M = k f h S. \quad \text{(5 A.)}$$

The relation between the load and the dimensions of a beam is found by equating the value of the greatest bending moment in terms of the load and span of the beam, as given in Articles 434, 435, 436, pages 505 to 510, to the value of the moment of resistance of the beam, at the cross-section where that greatest bending moment acts, as given in equation (5) or equation (5 A) of this Article.

The depth h is usually fixed by considerations of stiffness, to be explained further on; and then the unknown quantity is either the breadth, b, or sectional area, S, according as equation (5) or equation (5 A) is made use of. Sometimes, as when the cross-section is circular or square, we have $b = h$; and then we have h^3, instead of $b h^2$ in equation (5), which is solved so as to give h by extraction of the cube root. The following are the formulæ for these calculations:—

$$b = \frac{M}{n f h^2}; \quad \text{(7.)}$$

and when $h = b$,

$$h = \left(\frac{M}{n f}\right)^{\frac{1}{3}}, \quad \text{(7 A.)}$$

$$S = \frac{M}{k f h}. \quad \text{(7 B.)}$$

In finding the value of the geometrical moment of inertia I of cross-sections of complex figure, the following rules are useful:—

If a complex cross-section is made up of a number of simple figures, conceive the centre of magnitude of each of those figures to be traversed by a neutral axis parallel to the neutral axis of the whole section. Find the moment of inertia of each of the component figures relatively to its own neutral axis; multiply its area by the square of the distance between its own neutral axis and the neutral axis of the whole section; and add together all the results so found, for the moment of inertia of the whole section. To express this in symbols, let S′ be the area of any one of the component figures, y' the distance of its neutral axis from the neutral axis of the whole section, I′ its moment of inertia relatively to its own neutral axis; then the moment of inertia of the whole section is

$$I = \Sigma \cdot I' + \Sigma \cdot y'^2 A'. \qquad (8.)$$

When the figure of the cross-section can be made by *taking away* one simpler figure from another, both the area and the moment of inertia of the subtracted figure are to be considered as negative, and so treated, in making use of equation (8).

EXAMPLES OF THE NUMERICAL FACTORS IN EQUATIONS (3), (4), (5), AND (7).

Form of Cross-Sections.	$n' = \frac{I}{b h^3}$.	$m' = \frac{y_1}{h}$	$n = \frac{M}{f b h^2}$.
I. Rectangle $b h$, (including square)	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{1}{6}$
II. Ellipse— Vertical axis h, Horizontal axis b, (including circle)	$\frac{\pi}{64} = \frac{1}{20 \cdot 4}$ $= 0.0491$	$\frac{1}{2}$	$\frac{\pi}{32} = \frac{1}{10 \cdot 2}$ $= 0.0982$
III. Hollow rectangle, $b h - b' h'$; also I-formed section, where b' is the sum of the breadths of the lateral hollows,	$\frac{1}{12}\left(1 - \frac{b' h'^3}{b h^3}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{b' h'^3}{b h^3}\right)$
IV. Hollow square— $h^2 - h'^2$,	$\frac{1}{12}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{1}{6}\left(1 - \frac{h'^4}{h^4}\right)$
V. Hollow ellipse,	$\frac{\pi}{64}\left(1 - \frac{b' h'^3}{b h^3}\right)$	$\frac{1}{2}$	$\frac{\pi}{32}\left(1 - \frac{b' h'^3}{b h^3}\right)$
VI. Hollow circle,	$\frac{\pi}{64}\left(1 - \frac{h'^4}{h^4}\right)$	$\frac{1}{2}$	$\frac{\pi}{32}\left(1 - \frac{h'^4}{h^4}\right)$
VII. Isosceles triangle; base b, height h; y_1 measured from summit,	$\frac{1}{36}$	$\frac{2}{3}$	$\frac{1}{24}$

EXAMPLES OF THE NUMERICAL FACTOR k IN EQUATIONS (5 A) AND (7 B).

FORM OF CROSS-SECTION.	$k = \frac{M}{f h S}$.
I. Rectangle, ...	$\frac{1}{6}$.
II. Ellipse and circle, ...	$\frac{1}{8}$.
III. Hollow rectangle, $S = bh - b'h'$; also I-shaped section, b' being the sum of the depths of the lateral hollows, ...	$\dfrac{1 - \frac{b' h'^3}{b h^3}}{6\left(1 - \frac{b' h'}{b h}\right)}$.
IV. Hollow square, $S = h^2 - h'^2$,...	$\frac{1}{6}\left(1 + \frac{h'^2}{h^2}\right)$.
V. Do., very thin (approx.),	$\frac{1}{3}$.
VI. Hollow ellipse, ...	$\frac{1}{8}\left(1 - \frac{b' h'^3}{b h^3}\right) \div \left(1 - \frac{b' h'}{b h}\right)$.
VII. Hollow circle, ...	$\frac{1}{8}\left(1 + \frac{h'^2}{h^2}\right)$.
VIII. Do., very thin (approx.),	$\frac{1}{4}$.
IX. T-shaped section; flange A, web C; S = A + C (approx.),	$\frac{C(C + 4A)}{6(C + A)(C + 2A)}$.
X. I-shaped section; flanges A, B; web C; S = A + B + C; the beam supposed to give way at the flange A (approx.), ...	$\frac{C(C + 4A + 4B) + 12AB}{6(C + 2B)(A + B + C)}$.
X. A. Do., do., the beam supposed to give way at the flange B (approx.), ...	$\frac{C(C + 4A + 4B) + 12AB}{6(C + 2A)(A + B + C)}$.
XI. I-shaped section, with equal flanges A = B; S = C + 2A (approx.), ...	$\frac{1}{6}\left(1 + \frac{4A}{C + 2A}\right)$.

438. **Longitudinal Sections of Uniform Strength** are those in which the dimensions of the cross-section are varied in such a manner that its safe working moment of resistance is equal to the working bending moment at each section of the beam, and not merely at the section where the bending moment is greatest. That moment of resistance, for figures of the same kind, being proportional to the breadth and to the square of the depth, can be varied either by varying the breadth, the depth, or both. The law of variation depends upon the mode of variation of the moment of flexure of the beam from point to point, and this depends on the distribution of the load and of the supporting forces, in a way which has been stated in previous Articles. When the depth of the beam is made uniform, and the breadth varied, the vertical longitudinal section is rectangular, and the horizontal longitudinal section is of a figure depending on the mode of variation of the breadth. When the breadth of the beam is made uniform, and the depth varied, the horizontal longitudinal section is rectangular, and the vertical longitudinal section is of a figure depending on the mode of variation of the depth. When the beam, or the body which acts as a beam, is of circular cross-section, so that the breadth and depth are equal, each being a diameter of the cross-section, the diameter varies as the cube root of the bending moment. This case occurs in axles which are exposed to a bending moment, and not to a twisting moment. The following are examples of the results of those principles:—

I. Fixed at one end, loaded at the other; $b\,h^2$ varies as the distance from the loaded end.

II. Fixed at one end, uniformly loaded; $b\,h^2$ varies as the square of distance from the free end.

III. Supported at ends, loaded at an intermediate point; $b\,h^2$ varies as the distance from the adjacent point of support.

IV. Supported at ends, uniformly loaded; $b\,h^2$ varies as the product of the distances from the points of support.

In applying the principles of this Article, it is to be borne in mind that they do not take the *shearing load* into account; and that, consequently, the figures described in the above examples may require, at and near the points where the shearing load is greatest, some additional sectional area, to enable them to withstand that load, especially in examples III. and IV.; for in these cases the shearing load is greatest at the points of support, where there is no bending moment.

439. **Deflection of Beams.**—Four sorts of problems occur in connection with the deflection and the stiffness of beams; *first*, to find the *proof*, or *greatest safe deflection*, being the deflection under the proof load; *secondly*, the deflection under any given load, not exceeding the proof load; *thirdly*, to find the dimensions of a beam

which shall have a given deflection under its proof load, or under some other given load; *fourthly*, from the observed deflection it may be required to deduce the intensity of the most severe stress. The following are the rules:—

To find the *curvature* (that is, the reciprocal of the radius of curvature) of an originally straight beam at a given cross-section.

I. The cross-section under its proof stress. Divide the proof stress (f_1) by the distance of the most severely-strained particles from the neutral axis, and by the modulus of elasticity; the quotient will be the *proof curvature;*

$$\frac{1}{r} = \frac{f_1}{\mathrm{E}\, y_1}. \qquad (1.)$$

II. The bending moment given. Divide the bending moment by the moment of inertia of the given cross-section (see Article 437, page 513), and by the modulus of elasticity of the material. In symbols, let r be the radius of curvature; then

$$\frac{1}{r} = \frac{\mathrm{M}}{\mathrm{E\,I}}. \qquad (2.)$$

III. To find the *inclination* of the longitudinal axis of the beam to its original direction at a given point. Divide the length of the beam into small intervals ($d\,x$); multiply the length of each interval by the curvature at its centre $\left(\text{giving the product } \frac{d\,x}{r}\right)$; add together the products for the intervals from a point where the beam continues horizontal to the point where the inclination is required; the sum will be the required inclination; that is,

$$i = \int \frac{d\,x}{r}. \qquad (3.)$$

IV. To find the deflection. Multiply the length of each small interval by its inclination (obtaining the product $i\,d\,x$); add together those products for the intervals extending between the highest and lowest points of the beam; the sum will be the required deflection; that is,

$$v = \int i\,d\,x. \qquad (4.)$$

The preceding is the general method. The following are special rules:—

Let c be the *half-span* of a beam supported at both ends, or the *length* of a bracket fixed at one end; h, the extreme depth: and b, the extreme breadth of the beam; W, any given load; f_1, the proof stress; $m' h$, the distance of the most severely-strained layer from the neutral axis; I, the moment of inertia of the greatest cross-section; m'', n'', m''', n''', numerical factors (see Table below).

V. Greatest inclination under proof load;

$$i_1 = \frac{m'' f_1 c}{E m' h}; \quad \text{.........................(5.)}$$

VI. Proof deflection;

$$v_1 = \frac{n'' f_1 c^2}{E m' h}; \quad \text{.........................(6.)}$$

VII. Greatest inclination under a given load, W;

$$i'_1 = \frac{m''' W c^2}{E I}; \quad \text{.........................(7.)}$$

VIII. Deflection under a given load, W;

$$v'_1 = \frac{n''' W c^3}{E I}. \quad \text{.........................(8.)}$$

Examples.	Proof Load. Factors for Slope. m''	Proof Load. Factors for Deflection. n''	Any Load. Factors for Slope. m'''	Any Load. Factors for Deflection. n'''
A. Uniform Cross-Section.				
I. Constant Moment of Flexure,	1	$\frac{1}{2}$		
II. Fixed at one end, loaded at other,	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{3}$
III. Fixed at one end, uniformly loaded,	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$
IV. Supported at both ends, loaded in middle,	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$
V. Supported at both ends, uniformly loaded,	$\frac{2}{3}$	$\frac{5}{12}$	$\frac{1}{6}$	$\frac{5}{48}$
B. Uniform Strength and Uniform Depth. (The curvature of these is uniform.)				
VI. Fixed at one end, loaded at other,	1	$\frac{1}{2}$	1	$\frac{1}{2}$
VII. Fixed at one end, uniformly loaded,	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
VIII. Supported at both ends, loaded in middle,	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
IX. Supported at both ends, uniformly loaded,	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

EXAMPLES.	Proof Load. Factors for Slope. m''	Proof Load. Factors for Deflection. n''	Any Load. Factors for Slope. m'''	Any Load. Factors for Deflection. n'''
C. UNIFORM STRENGTH AND UNIFORM BREADTH.				
X. Fixed at one end, loaded at other,....................	2	$\frac{2}{3}$	2	$\frac{2}{3}$
XI. Fixed at one end, uniformly loaded,...........	infinite	1	infinite	$\frac{1}{2}$
XII. Supported at both ends, loaded in middle,.......	2	$\frac{2}{3}$	1	$\frac{1}{3}$
XIII. Supported at both ends, uniformly loaded,.......	1·5708	0·5708	0·3927	0·1427

IX. Given, the half-span, c, and the *intended proof deflection*, v_1, of a proposed beam; to find the proper value of the *greatest depth*, h_0; make

$$h_0 = \frac{n'' f_1 c^2}{E m' v_1}; \ldots\ldots\ldots\ldots\ldots\ldots(9.)$$

(taking n'' from the preceding table, and making $m' h_0$, as before, denote the distance from the layer in which the stress is f_1 to the neutral axis).

X. To deduce the greatest stress in a given layer of a beam from the deflection found by experiment.

Let h be the depth of the beam at the section of greatest stress, and y the distance from the neutral axis of that section to that layer of the beam at which the greatest stress is required:—

c, the half-span of a beam supported at both ends, or the length of the loaded part of a beam supported at one end;
n'', the factor for proof deflection, already explained;
E, the modulus of elasticity of the material;
v, the observed deflection;

then the intensity of the required stress is

$$p = \frac{E y v}{n'' c^2}. \ldots\ldots\ldots\ldots\ldots\ldots(10.)$$

XI. To find the deflection of an uniform beam produced by its own weight, or by an uniform load bearing a given proportion, $1 + m$, to the weight of the beam. Let w be the heaviness of the material of which the beam consists; $r^2 = I \div S$, the square of the radius of gyration of its cross-section; n'', as before, the factor for deflection under a given load; then, for a beam supported at both ends,

$$v = \left(1 + m\right)\frac{2\,n''' \,w\,c^4}{\mathrm{E}\,r^2};\ \ldots\ldots\ldots\ldots\ldots\ldots(11.)$$

and for a bracket fixed at one end,

$$v' = \left(1 + m\right)\frac{n'''\,w\,c^4}{\mathrm{E}\,r^2}.\ \ldots\ldots\ldots\ldots\ldots\ldots(12.)$$

A table of values of r^2 will be given at p. 525. The application of this problem to shafts for transmitting power will be explained in the next Chapter.

440. **Beam fixed at the Ends.**—When a beam is not merely supported, but fixed in direction at its two ends, it bends into the form of a curve which has two points of inflection; being convex upwards at the points of support, and concave upwards in the middle. The following are the two most important cases; the cross-section of the beam being supposed uniform in both:—

I. Load concentrated at middle of span. The bending moments at the points of support and at the middle of the span are equal and contrary, and each equal to half of the bending moment upon an equal and similarly loaded beam with ends merely supported; that is, $\mathrm{M} = \frac{\mathrm{W}\,c}{4}$.

Factor for proof deflection, $n'' = \frac{1}{6}$.

Factor for deflection under a given load, $n''' = \frac{1}{24}$.

II. Load uniformly distributed. The bending moment at the middle of the span is one-third, and the contrary bending moment at each point of support two-thirds, of what the bending moment in the middle of the span would be if the ends were merely supported. That is, the most severe bending moment is $\mathrm{M} = -\frac{w\,c^2}{3}$.

Factor for proof deflection, $n'' = \frac{1}{8}$.

Factor for deflection under a given load, $n''' = \frac{1}{48}$.

441. The **Resilience of a Beam** (*A. M.*, 305) is the *work performed* in bending it to the proof deflection;—in other words, the *energy of the greatest shock* which the beam can bear without injury; such energy being expressed by the product of a weight into the height from which it must fall to produce the shock in

question. This, if the load is concentrated at or near one point, is the product of half the proof load into the proof deflection; that is to say, let P be the proof load; then the resilience is

$$\frac{P v_1}{2}. \qquad (1.)$$

Let W be the weight of a mass which is let fall upon the beam from the height z. Then the whole height through which that mass falls, before the beam reaches its proof deflection, is $z + v_1$; and the whole energy of the blow which it gives to the beam is $W (z + v_1)$; which being equated to the resilience, gives the following equation:—

$$W (z + v_1) = \frac{P v_1}{2}; \qquad (2.)$$

an equation which enables any one of the four quantities, W, z, P, v_1, to be calculated when the other three are given.

If the load is distributed, the length of the beam is to be divided into a number of small elements, and half the proof load on each element multiplied by the distance through which that element is depressed. The integral of the products will be the resilience.

SECTION VI.—*Of Resistance to Thrust or Pressure.*

442. **Resistance to Compression and Direct Crushing.**—Resistance to *longitudinal compression*, when the proof stress is not exceeded, is sensibly equal to the resistance to stretching, and is expressed by the same modulus of elasticity, denoted by E (page 493). When that limit is exceeded, it becomes irregular. (See Article 420, page 493.)

The present Article has reference to direct and simple crushing only, and is limited to those cases in which the pillars, blocks, struts, or rods along which the thrust acts are not so long in proportion to their diameter as to have a sensible tendency to give way by bending sideways. Those cases comprehend—

Stone and brick pillars and blocks of ordinary proportions;

Pillars, rods, and struts of cast iron, in which the length is not more than five times the diameter, approximately;

Pillars, rods, and struts of wrought iron, in which the length is not more than ten times the diameter, approximately;

Pillars, rods, and struts of dry timber, in which the length is not more than about five times the diameter.

In such cases the Rules for the strength of ties (page 493) are approximately applicable, substituting *thrust* for *tension*, and using the proper modulus of resistance to direct crushing instead of the tenacity.

Blocks whose lengths are less than about once-and-a-half their diameters offer greater resistance to crushing than that given by the Rules; but in what proportion is uncertain.

The modulus of resistance to direct crushing, as the Tables show, often differs considerably from the tenacity. The nature and amount of those differences depend mainly on the modes in which the crushing takes place. These may be classed as follows:—

I. *Crushing by splitting* (fig. 275) into a number of nearly prismatic fragments, separated by smooth surfaces whose general direction is nearly parallel to the direction of the load, is characteristic of very hard homogeneous substances, in which the resistance to direct crushing is greater than the tenacity; being in many examples about double.

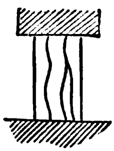

Fig. 275.

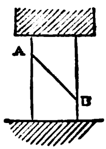

Fig. 276.

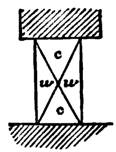

Fig. 277.

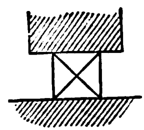

Fig. 278.

II. *Crushing by shearing or sliding* of portions of the block along oblique surfaces of separation is characteristic of substances of a granular texture, like cast iron, and most kinds of stone and brick. Sometimes the sliding takes place at a single plane surface, like A B in fig. 276; sometimes two cones or pyramids are formed, like *c*, *c*, in fig. 277, which are forced towards each other, and split or drive outwards a number of wedges surrounding them, like *w*, *w*, in the same figure. In substances which are crushed by shearing, the resistance to crushing is always much greater than the tenacity; Sometimes the block splits into four wedges, as in fig. 278.

III. *Crushing by bulging*, or lateral swelling and spreading of the block which is crushed, is characteristic of ductile and tough materials, such as wrought iron. Owing to the gradual manner in which materials of this nature give way to a crushing load, it is difficult to determine their resistance to that load exactly. That resistance is in general less, and sometimes considerably less, than the tenacity. In wrought iron, the resistance to the direct crushing of pillars or struts of moderate length, as nearly as it can be ascertained, is from $\frac{2}{3}$ to $\frac{4}{5}$ of the tenacity.

IV. *Crushing by buckling or crippling* is characteristic of fibrous substances, such as wood, under the action of a thrust along the fibres. It consists in a lateral bending and wrinkling of the fibres, sometimes accompanied by a splitting of them asunder. The resistance of such substances to crushing is in general considerably less than their tenacity, especially where the lateral adhesion of the fibres to each other is weak compared with their tenacity. The resistance of most kinds of timber to crushing, when dry, is from $\frac{1}{2}$ to $\frac{2}{3}$ of the tenacity. Moisture in the timber weakens the lateral adhesion of the fibres, and reduces the resistance to crushing to about one-half of its amount in the dry state.

443. **Crushing by Cross-Breaking.**—Long struts and pillars in framework, and rods, bars, or links in machinery, which transmit thrust, give way by bending sideways and breaking across. Let P be the breaking load of such a piece; S, its sectional area; l, its length; r, the *least geometrical radius of gyration* of its cross-section; f and c, two co-efficients depending on the material; then

I. For a piece fixed in direction at both ends;

$$\frac{P}{S} = \frac{f}{1 + \frac{l^2}{c r^2}} \quad \text{(1.)}$$

II. For a piece jointed at both ends (such as a link or connecting-rod in machinery);

$$\frac{P}{S} = \frac{f}{1 + \frac{4 l^2}{c r^2}} \quad \text{(2.)}$$

III. For a piece jointed at one end and fixed in direction at the other (such as a piston-rod);

$$\frac{P}{S} = \frac{f}{1 + \frac{16 l^2}{9 c r^2}} \quad \text{(3.)}$$

The square of the radius of gyration referred to is given by the expression,

$$r^2 = \frac{I}{S}; \quad \text{(4.)}$$

where S is the area of cross-section of the piece, and I the geometrical moment of inertia of that cross-section about a neutral axis perpendicular to the direction in which the piece is most flexible. (See Articles 437, 439, pages 513, 519.)

VALUES OF THE CONSTANTS FOR THE BREAKING LOAD.

	f Lbs. on the Square Inch.	c
Malleable iron,	36,000	36,000
Cast iron,	80,000	6,400
Dry timber, strong kinds,	7,200	3,000

TABLE OF VALUES OF r^2 FOR DIFFERENT FORMS OF CROSS-SECTION.

Solid rectangle; least dimension = h;	$h^2 \div 12.$
Hollow square tube; dimensions, outside, h; inside, h';	$(h^2 + h'^2) \div 12.$
Thin square cell; side = h;	$h^2 \div 6.$
Thin rectangular cell; breadth, b; depth, h;	$\frac{h^2}{12} \cdot \frac{h + 3b}{h + b}.$
Solid cylinder; diameter = h;	$h^2 \div 16.$
Hollow cylinder; diameter, outside, h; inside, h';	$(h^2 + h'^2) \div 16.$
Thin hollow cylinder; diameter = h;	$h^2 \div 8.$
Angle iron of equal ribs; breadth of each = b;	$b^2 \div 24.$
Angle iron of unequal ribs; greater, b; less, h;	$b^2 h^2 \div 12 (b^2 + h^2).$
Cross of equal arms;	$h^2 \div 24.$
H-iron; breadth of flanges, b; their joint area, A; area of web, B;	$\frac{b^2}{12} \cdot \frac{A}{A + B}.$
Channel iron; depth of flanges + ½ thickness of web, h; area of web, B; of flanges, A;	$h^2 \cdot \left\{ \frac{A}{12 (A + B)} + \frac{A B}{4 (A + B)^2} \right\}.$

All the dimensions being in the same units of measure.

444. **Collapsing of Tubes.**—When a thin hollow cylinder, such as an internal boiler flue, is pressed from without, it gives way by *collapsing*, under a pressure whose intensity was found by Mr. Fairbairn (*Philos. Trans.*, 1858) to vary nearly according to the following laws:—

Inversely as the length;

Inversely as the diameter;

Directly as a function of the thickness, which is very nearly the power whose index is 2·19; but which for ordinary practical purposes may be treated as sensibly equal to the *square* of the thickness.

The following formula gives approximately the *collapsing pressure*, p, in lbs. on the square inch, of a plate-iron flue, whose length, l, diameter, d, and thickness, t, are all expressed in *the same units of measure*:—

$$p = 9{,}672{,}000 \frac{t^2}{l \cdot d} \text{ ....(1.)}$$

For kilogrammes on the square millimètre, the constant coefficient becomes 6,800.

Mr. Fairbairn having strengthened tubes by rivetting round them rings of T-iron, or angle iron, at equal distances apart, found that their strength is that corresponding to the length *from ring to ring.*

He also found that the collapsing pressure of a tube of an elliptic form of cross-section is found approximately by substituting for d, in the preceding formula, the diameter of the osculating circle at the flattest part of the ellipse; that is, let a be the greater, and b the lesser *semi-axis* of the ellipse; then we are to make

$$d = \frac{2a^2}{b}. \qquad \text{(2.)}$$

CHAPTER III.

OF SPECIAL PRINCIPLES RELATING TO STRENGTH AND STIFFNESS IN MACHINES.

445. **Subjects of this Chapter.**—In the designing of machines with a view to sufficient strength and stiffness, certain special principles must be kept in view besides those general principles which are applicable to machines in common with structures. The first section of this Chapter gives a summary of those principles; the remaining sections relate to the strength and stiffness of certain special parts of machines.

SECTION I.—*Summary of Principles.*

446. **Load in Machines.**—In most examples of machinery the whole load must be treated as a live load, because of its action being accompanied with vibration; and also in many cases because the straining action of the load operates upon different sets of particles in succession, and comes with more or less suddenness upon such sets of particles. In some of these latter cases the straining action of the load upon a given particle is periodically reversed; for example, the bending moment exerted on a rotating shaft causes alternate tension and thrust to be exerted upon the same particle, as it passes alternately to the stretched and to the compressed side of the axle.

Hence the real factor of safety in machinery is seldom less than 6.

There are exceptional cases in which, owing to the smoothness of the motion and the steadiness of the straining action, the load may be considered as intermediate between a dead load and a live load, so that a smaller factor of safety is sufficient; such, for example, as the transmission of power through bands of such length as to hang in a sensibly curved form.

447. **Straining Actions computed from Power.**—The straining actions on moving pieces can be in some cases wholly, and in others partly, determined from the power transmitted, and from the speed, by methods of calculation which will be described and exemplified further on. The cases in which the straining action can be wholly determined from the power transmitted are those which fulfil the following conditions: uniformity of effort, absence of *lateral components* in the straining forces, and smallness of the straining

actions due to the weight and to the re-action of the piece itself, and of pieces carried by it, so that those parts of the straining action may be treated as insensible.

The rules for computing straining actions from power transmitted are the following:—

I. To compute the effort exerted along a given line of connection; divide the power transmitted, in units of work per second, by the common component along the line of connection of the velocities of the connected points.

If the power is given in horses-power, reduce it in the first place to units of work per second, by multiplying by 550 for foot-lbs., or by 75 for kilogrammètres.

II. To compute the straining moment exerted through a given rotating piece; divide the power transmitted, in units of work in a given time, by the angular motion in the same time: that is, by 2π times the number of turns in that time.

In symbols, let U be the power, in units of work per minute; N, the number of revolutions per minute; M, the straining moment; then

$$M = \frac{U}{2\pi N} = \frac{0{\cdot}159155\ U}{N}. \qquad (1.)$$

This formula gives the moment in the same denomination with the work. If the work is given in foot-lbs. per minute, and the moment is required in inch-lbs., the above expression must be multiplied by 12; that is,

$$M = \frac{12\ U}{2\pi N} = \frac{1{\cdot}91\ U}{N}. \qquad (2.)$$

Let H P denote the number of horses-power transmitted, so that

U in foot-lbs. per minute = 33000 H P; and
U in kilogrammètres per minute = 4500 H P;

then we have

$$\text{M in inch-lbs.} = \frac{63000\ H\ P}{N}; \qquad (3.)$$

$$\text{M in foot-lbs.} = \frac{5250\ H\ P}{N}; \qquad (4.)$$

$$\text{M in kilogrammètres} = \frac{716{\cdot}2\ H\ P}{N}. \qquad (5.)$$

The formula for kilogrammètres is adapted to the French horse-power, which is about one-seventieth part less than the British.

In the cases in which part only of the straining action can be determined from the power transmitted, the causes of additional straining action are the following:—Excess of maximum effort above mean effort; lateral components in straining forces; weight of the piece itself and of pieces carried by it; re-actions of the piece itself and of pieces carried by it, when undergoing acceleration or retardation. It has already been stated in Article 414, page 488, that such additional straining actions are sometimes calculated expressly, and sometimes allowed for by using an apparent factor of safety greater than the mean factor of safety in a suitable proportion.

There are cases in which the best method of calculating the straining action is to determine directly the greatest load, without reference to the power transmitted.

448. **Alternate Strains.**—Pieces are often met with in machinery which are strained alternately in opposite directions, such being especially the case when the motion is reciprocating: for example, the piston-rod and connecting-rod of a steam engine, which are subjected alternately to tension and to thrust; and the beam of a steam engine, which is exposed alternately to bending actions in opposite directions. Such pieces must be adapted to resist efficiently the straining action in either direction, and especially that which is most severe. This principle is applicable to framing as well as to moving pieces.

449. **Straining Effects of Re-action.**—When the particles of a piece undergo changes of speed and direction, their re-actions produce straining effects resembling those produced by their weights; due regard being had to the directions of those re-actions, and to the ratios which they bear to the weights of the particles. For example, if a particle of the weight w undergoes the acceleration $d\,v$, in the time $d\,t$, the re-action of that particle is

$$-\frac{w\,d\,v}{g\,d\,t}, \qquad (1.)$$

and is exerted in a direction opposite to that of the acceleration (Article 287, page 330); and if a particle of the weight w revolves with the angular velocity a, in a circle of the radius r, its re-action (or centrifugal force) is

$$\frac{w\,a^2\,r}{g}, \qquad (2.)$$

and is exerted in a direction away from the centre of the circle (Article 288, page 330).

In many cases of reciprocating motion in machinery, the motion of the reciprocating mass is *harmonic* (as to the meaning of which, see Article 239, page 250); and then its greatest re-action is equal to what its centrifugal force would be if it revolved in a period equal to the time of a double stroke, in a circle of a radius equal to the half-stroke. Let T be the period, or time of a double stroke in seconds; x, the half-stroke; w, the weight of the reciprocating mass; then its greatest re-action is

$$\frac{4\pi^2}{g} \cdot \frac{w x}{T^2} \dots\dots\dots\dots\dots\dots\dots\dots (3.)$$

The co-efficient $\frac{4\pi^2}{g}$ is the reciprocal of $\frac{g}{4\pi^2}$, which is, as already stated in Article 319, page 364, the altitude of a revolving pendulum whose period is one second; that is, nearly, 0·815 foot, or 9·78 inches, or 248 millimètres.

The *moment of re-action* of a mass which undergoes an acceleration of angular velocity, $d\,a$, in the interval of time $d\,t$, is given by the expression

$$-\frac{I\,d\,a}{g\,d\,t}; \dots\dots\dots\dots\dots\dots\dots\dots (4.)$$

in which I denotes the *moment of inertia* of the rotating mass (Article 313, page 358). If the mass has a *rocking* or oscillating motion, following the harmonic law, about its axis, the greatest moment of re-action is as follows:—

$$\frac{4\pi^2}{g} \cdot \frac{I\,\theta}{T^2}; \dots\dots\dots\dots\dots\dots\dots\dots (5.)$$

in which T is the periodic time of a complete or double oscillation, and θ the *semi-amplitude;* that is, the angle in circular measure through which the stroke, or oscillation, extends to each side of the middle position of the rocking body. Values of $\frac{g}{4\pi^2}$ have already been given.

The moments of re-action given by the formulæ (4) and (5) may constitute twisting moments upon shafts, or bending moments upon levers.

450. **Framework.**—The load which strains the framework of a machine consists partly of the weight of that framework itself; but principally of the bearing-pressures exerted by the moving pieces. How those bearing-pressures are to be determined has already been shown in the course of Part II., Chapter IV., Section I. The framework ought to be so designed as to make the bearing-pressures at different points, or the components of those bearing-pressures, as far as possible balance each other. When this principle is perfectly carried out, the pressure exerted by the machine on its foundation will consist simply of its weight; all

the horizontal components of the bearing-pressures, and all the bearing-pressures which act in couples, being mutually balanced. This, however, is possible only when the prime mover, the working machinery, and the material operated upon, are all carried by one connected assemblage of framework. In other cases, all that can be attained is an approximation to the balance of horizontal pressures and of couples. When two bearings occur near each other that are exposed to opposite pressures, or to pressures containing opposite components, it is in general advisable, in designing the frame, to connect those bearings with each other as directly as possible, by means of a strut or of a tie.

451. **Stiffness and Pliability.**—In all cases in which precision of movement is required, stiffness is essential both to the moving pieces and to the framework of a machine. It is ensured, first, by causing the pieces exposed to strain to resist it as far as practicable by direct tension and direct thrust, rather than by twisting or bending stress (Article 420, page 493; and Article 442, page 522); and secondly, where indirect modes of exerting stress are unavoidable, to give the piece such transverse dimensions as are necessary in order to prevent the extent to which it yields from exceeding a certain limit (Article 430, page 502; Article 439, page 517). According to the first of those principles, the framework and the moving pieces of a machine, where rigidity is required, should consist, as far as practicable, of struts and ties; according to the second principle, where beams have to be used, the depth and span, and where shafts have to be used, the diameter and span, are to be so proportioned to each other as to prevent the ratio of the deflection to the span from exceeding a certain limit $\left(\text{usually from } \frac{1}{1,200} \text{ to } \frac{1}{2,000}\right)$. The special rule applicable to shafts will be given further on. As to beams, see page 520.

On the other hand, there are cases in which absolute precision of movement is unnecessary, and in which pliability is an advantage, as giving the power of withstanding shocks. This advantage is possessed by leathern belts, and by raw hide and hempen ropes, because of the great extensibility of the materials. Wire ropes, when stretched tight, possess it to a less degree; but when of a span sufficient to hang visibly in curves, the power of alteration of curvature constitutes a kind of pliability, which enables shocks to be borne; and the same remark applies to chains when hanging slack. Pliability in the shape of compressibility, where thrust has to be resisted, as in connecting-rods, is obtained by using timber, as has already been stated in Article 409, page 474. Beams, and pieces acting as beams, are made flexible to any extent required, by making the depth sufficiently small in comparison with the span, the breadth being at the same time made sufficiently great to give the requisite strength; or by using tough and pliable kinds of timber, such as those mentioned in Article 409, page 473, as possessing those qualities.

452. **Compound Stress.**—Both in moving pieces and in framework, but especially in moving pieces, straining actions of different kinds are sometimes compounded: as direct tension or direct thrust with bending, or bending with twisting. In such cases the resultant stress arising from the combination must be taken into account. The rules applicable to the cases of this sort which commonly occur in practice will be given in the course of the ensuing sections of this Chapter.

SECTION II.—*Special Rules as to Bands, Rods, and Links.*

453. **Belts and Cords at Moderate Speeds.**—The effective working tension required at the driving side of a band is to be found by the rules already given in Article 310 A, pages 351, 352. When the speed at which the band runs is such that the centrifugal tension may be disregarded, and when the band is a belt or cord of organic material, such as leather, raw hide, gutta percha, or hemp, the working tension is to be divided by a suitable co-efficient of working strength, so as to give, according to the nature of the co-efficient employed, either the weight per unit of length, or the sectional area; or, in the case of flat belts of a given thickness, the breadth; or, in the case of cords, the square of the diameter, or the square of the girth. Co-efficients adapted to those different methods of calculation, and to different materials, have already been given in Article 410, pages 474, 475, 476.

454. **Allowance for Centrifugal Tension.**—When the speed is so great that it becomes necessary to allow for centrifugal tension, the co-efficient of working strength to be used is that which is expressed in the form of an equivalent length of the band itself. Let that length be denoted by b. Let v be the velocity with which the band is to run; then the centrifugal tension, expressed in length of band, is $\frac{v^2}{g}$; and this is exerted at every point of the band, in addition to the effective tension required for the transmission of power; so that after deducting the centrifugal tension, the strength which remains available to resist the effective tension is

$$b_0 = b - \frac{v^2}{g}; \qquad (1.)$$

when expressed in length of band. Therefore, let T_1 be the effective working tension required at the driving side of the band; w S, the weight of an unit of length of the band required; then

$$w\,S = \frac{T_1}{b - \frac{v^2}{g}} \quad \text{..........................(2.)}$$

The weight per unit of length is expressed in the form of a product, $w\,S$; in which S denotes the sectional area, and w the heaviness of the material.

455. **Wire Ropes** present a case in which direct tension is combined with an additional stress produced by the bending of the wires round the pulleys. Let D be the diameter of a pulley; d, that of a *single wire;* E, the modulus of elasticity of the wire; then the bending produces a stress which is tensile at one side of the wire, and compressive at the other, and whose intensity, in units of weight on the unit of area, is $\frac{E\,d}{D}$; and in length of the rope, $\frac{E\,d}{w\,D}$; w being the heaviness of the material.

Let b, as before, be the safe working strength expressed in length of rope; v, the velocity at which the rope runs; then the strength in length of rope, available to resist the effective working tension at the driving side, is

$$b_0 = b - \frac{v^2}{g} - \frac{E\,d}{w\,D}; \quad \text{......................(1.)}$$

and the weight per unit of length, w S, of a rope suited to bear the effective working tension T_1, is given by the following equation:—

$$w\,S = \frac{T_1}{b - \frac{v^2}{g} - \frac{E\,d}{w\,D}}. \quad \text{....................(2.)}$$

The most convenient way of using this formula is to fix a minimum value for the ratio $D \div d$, in which the diameter of the pulley is to exceed that of a single wire, and thence to deduce the value of the stress $E\,d \div w\,D$, produced by bending. Then, having calculated the weight, w S, per unit of length, the diameter, d, of a single wire is to be deduced from that weight, and the least proper diameter for a driving pulley, D, by multiplying d by the previously fixed ratio.

An ordinary value of $D \div d$ is 2000.

A wire rope of the ordinary construction consists of six strands spun round a hempen core; and each of the strands consists of six wires spun round a smaller hempen core, so that there are thirty-six wires in all. The diameter of a single wire is given with sufficient accuracy for the present purpose by the formula,

$$d \text{ (in fractions of an inch)} = \sqrt{\left(\frac{w \text{ S in lbs. per foot}}{100}\right)}. \quad (3.)$$

or

$$d \text{ (in millimètres)} = \sqrt{(4\tfrac{1}{3}\ w \text{ S in kilogrammes per mètre})}. \quad (3\text{A.})$$

The following are values of the moduli of elasticity and strength for ropes made of the best charcoal iron wire. Steel wire ropes may be taken as having about the same modulus of elasticity, and as being stronger than iron in the proportion of 4 to 3 nearly.

	Feet of Rope.	Mètres of Rope.
Modulus of Elasticity, $\frac{E}{w}$,.......	7,500,000	2,286,000
Ultimate Tenacity,..............	26,880	8,193
Proof Tension,....................	13,440	4,096
Working Tension with steady action (factor of safety, 3½),*	7,680	2,340
Working Tension with unsteady action (factor of safety, 6),...	4,480	1,365

456. **Deflection and Length of Bands.**—The form in which a band hangs between two pulleys which it connects, is that of a catenary. In cases which occur in practice, the parabola may be used as an approximation to the catenary, without sensible error. This gives the following approximate formula for the deflection of the band at the middle of its span, below a straight line joining its two points of suspension:—

$$y = \frac{c^2}{2\,b_0};\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

in which c is the half-span, measured along the before-mentioned straight line, whether horizontal or sloping; b_0 is the length of rope equivalent to the *available tension* (exclusive of centrifugal tension), and y is the deflection.

Let i be the angle of inclination of the span of the band to the horizon; and s the length of the part of the band which hangs in a curve between the two points of suspension; then

$$s \text{ nearly} = 2\,c + \frac{4}{3} \cdot \frac{y^2}{c} \cos^2 i. \ \ldots\ldots\ldots\ldots (2.)$$

When the span is horizontal, $\cos^2 i = 1$.

The driving and returning parts of the band have different

* This value of the working tension is calculated from the co-efficient of stress given by Reuleaux, as applicable to Hirn's telodynamic transmission. (*Constructionslehre für Maschinenbau*, § 329.)

tensions (see Article 310A, page 352), and therefore different deflections. Their lengths are to be calculated separately, and added together, along with the lengths of the circular parts of the band which pass round the pulleys.

457. **Chains.**—Chains consisting of oval links, when the tendency of each link to collapse is resisted by means of a cross-bar called a *stay* or *stud*, as in fig. 279, have a strength equal to that due to the collective sectional area of the two sides of the link. The tenacity of the iron in the link is reduced by the processes of forging and welding so as to be from $\frac{7}{8}$ to $\frac{3}{4}$ of that of the cable-iron bolt from which it is made; so that, taking the ultimate tenacity of cable-iron bolts at 60,000 lbs. on the square inch, that of a *stud chain* is from 52,500 to 45,000 lbs. on the square inch; and about 7,500 lbs. on the square inch may be taken as a safe working modulus of tension with a live load: the smaller of the preceding co-efficients being divided by 6 as a factor of safety. The *test load* is about half the breaking load, or three times the working load. An unstudded chain has about *two-thirds* of the strength of a studded chain of the same dimensions.

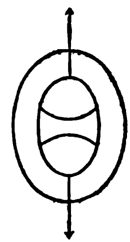

Fig. 279.

The following are the ordinary proportions of the links of a stud-chain, as used for ships' cables and rigging, in terms of the diameter of the bolts from which they are made.

Length: outside, 6 diameters; inside, 4 diameters.

Breadth: outside, $3\frac{1}{2}$ diameters; inside, $1\frac{1}{2}$ diameter.

Thickness of stay: at ends, 1 diameter; at middle, $\frac{6}{10}$ diameter.

The *weight of a stud-chain*, of these proportions, in lbs. per foot, is found by multiplying the square of the diameter of the cable-iron in inches by 9, very nearly; and its weight in kilogrammes per mètre, by multiplying the square of that diameter in millimètres by 0·0208.

In designing chains made of flat links connected by pins, regard must be paid to the principles of Article 424, page 497, so as to give the dimensions of the pins their due proportions to those of the links.

458. **Rods or Links for Tension.**—The following are the rules applicable to the ordinary cases of rods or links for transmitting tension; such as piston-rods in single-acting steam engines.

I. When the resultant tension acts *along the longitudinal axis of the rod*, that is, along a straight line traversing the centres of all the cross-sections, the area of cross-section is to be proportioned to the load according to the rules of Article 420, page 494; the modulus of greatest working stress being taken at 9,000 lbs. on the square inch for wrought iron; 2,500 for cast iron (which, if shocks are to be borne, is not a suitable material for this purpose); and 1,000 for timber of straight-grained and tough kinds.

II. Should the resultant tension act, not along the axis of the rod, but at a distance from it, whose greatest value may be denoted by x, let P denote the load; then the tensile action is combined with a *bending moment*, $P x$.

Let S be the area of cross-section; h, the depth of the rod—that is, its diameter in the plane of the bending moment; k, the numerical factor in equation (5 A) of Article 437, page 514, and in the Table of page 516; the greatest additional intensity of tension produced by the bending moment is

$$\frac{P x}{k h S};$$

and the total intensity of the greatest tension is

$$\frac{P}{S}\left(1 + \frac{x}{k h}\right);$$

and consequently, if f be the modulus of working stress, the proper sectional area is given by the formula,

$$S = \frac{P}{f}\left(1 + \frac{x}{k h}\right). \quad \text{.....................(1.)}$$

III. In a tension rod which is horizontal or inclined, the additional stress produced by the *bending action of its own weight* may require to be taken into consideration. Let w be the heaviness of the material; c, the *half-span* between the points of support measured along the axis of the rod; i, the angle of inclination of that axis to the horizon; then the bending moment is

$$M = \frac{w S c^2 \cos i}{2};$$

and the greatest stress produced by that moment is

$$p = \frac{M}{k h S} = \frac{w c^2 \cos i}{2 k h}. \quad \text{.....................(2.)}$$

The easiest way to make use of this formula in practice is to assume in the first place a convenient value for h; calculate p by equation (2); and then make

$$S = \frac{P}{f - p}. \quad \text{...........................(3.)}$$

IV. If the rod has a *transverse reciprocating motion*, the re-action due to that motion will produce alternate bending actions in opposite directions. Let z be the semi-amplitude of the transverse motion—that is, half its total extent; let n be the number of *double swings* in a second; then make

$$m = \frac{4 \pi^2 n^2 z}{g}; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.)$$

(in which $\frac{g}{4 \pi^2} = 0{\cdot}815$ foot $= 0{\cdot}248$ mètre nearly); then the additional stress produced by this motion is to be found by putting $m\,w$ instead of $w \cos i$ in equation (2), and is to be provided for in the same way with the stress p of that equation. If one end of the rod has a transverse reciprocating motion, while the other has no such motion, or if the two ends have motions of different amplitudes, make z equal to the semi-amplitude of the transverse motion of the centre of the rod; the result will be near enough to the truth for practical purposes.

V. If weight and re-action both take effect in the same vertical plane, make

$$p = \left(m + \cos i\right) \frac{w\,c^2}{2\,k\,h}; \quad\ldots\ldots\ldots\ldots\ldots\ldots(5.)$$

and use this value of p in equation (3).

VI. In designing the *fastenings* for transmitting the tension at the ends of rods or links, regard is to be had to the principles of Article 424, page 497.

VII. The sides of an *eye* at the end of a tension-rod are usually made so as to have a collective sectional area equal to *once-and-a-half* that of the rod; because the uneven distribution of the stress in them diminishes their strength to about two-thirds of what it would be if the stress were uniform; and the same rule is applicable to a *strap* answering the purpose of an eye.

459. **Rods or Links for Reciprocating Stress.**—When a rod transmits alternately tension and thrust of equal amount, the most severe straining action is usually that produced by the thrust; and the proper dimensions are to be found by the rules of Article 443, page 524.

The piston-rods and connecting-rods of double-acting steam engines are examples of this. Piston-rods are to be treated as struts fixed at one end and jointed at the other; connecting-rods as struts jointed at both ends.

The ways of using the rules referred to are as follows:—

I. In ordinary cases of the piston-rods and connecting-rods of steam engines, an approximate sectional area may be calculated in the first place, by multiplying the area of the piston by the greatest intensity of the effective pressure of the steam, so as to find the greatest working load, and dividing it by a modulus of working stress, which may be 2,500 lbs. on the square inch for a piston-rod, and 1,750 lbs. on the square inch for a connecting-rod at the middle of its length. It is usual to give the connecting-rod a swell in the middle; so that at the ends it is of the same area with the piston-rod. The ratio of the greatest to the least diameter is about that of 6 to 5.

This rule may be considered as giving a safe value for the transverse section, when the length of the rod does not exceed about thirty-six times its diameter. When the proportion of length to diameter is greater, the following rule may be applied to wrought-iron rods.

II. Let f be the intensity of the safe working thrust along a short rod—say 6,000 lbs. on the square inch, or 4·2 kilogrammes on the square millimètre, for wrought iron. Let S be the sectional area of the rod; l, its length; h, its diameter; P, the amount of the greatest working thrust; then we have

$$P = \frac{f\,S}{1 + \dfrac{l^2}{c^1\,h^2}}; \qquad \text{(1.)}$$

in which the values of c^1 are as follows:—

For a round rod,	jointed at both ends,	562
,, ,,	fixed at one end,	1,270
,, ,,	fixed at both ends,	2,250
For a square rod,	jointed at both ends,	750
,, ,,	fixed at one end,	1,690
,, ,,	fixed at both ends,	3,000

But in a round rod, we have $h^2 = 1{\cdot}273$ S nearly; and in a square rod, $h^2 = S$; consequently we may make

$$c^1\,h^2 = a\,S; \qquad \text{(2.)}$$

in which a has the following values:—

		a.
For a round rod,	jointed at both ends,	716
,, ,,	fixed at one end,	1,611
,, ,,	fixed at both ends,	2,864
For a square rod,	jointed at both ends,	750
,, ,,	fixed at one end,	1,690
,, ,,	fixed at both ends,	3,000

Thus equation (1) may be made to take the following form:—

$$P\,(a\,S + l^2) = a\,f\,S^2; \qquad \text{(3.)}$$

which quadratic equation, being solved, gives the following formula for the sectional area:—

$$S = \frac{P}{2f} + \sqrt{\left\{ \frac{P^2}{4f^2} + \frac{P\,l^2}{a\,f} \right\}} \dots\dots\dots\dots (4.)$$

III. Sometimes, as has already been stated in page 474, a connecting-rod is made of a square bar of wood to transmit the thrust, bound with a wrought-iron strap to transmit the tension. In this case the transverse area of the strap is to be determined by the principles of Article 458, with a working modulus of tension of about 6,000 lbs. on the square inch (because the stress may be unequally distributed near the ends of the strap, where it bends round the bushes that hold the pins); and that of the square timber bar is to be found by equation (4) of the preceding rule, with the following values for the constants:—

$f =$ 720 lbs. on the square inch;
$=$ 0·5 kilogramme on the square millimètre;
$a =$ 62·5.
$a\,f =$ 45,000 lbs. on the square inch;
$=$ 32 kilogrammes on the square millimètre.

In a compound rod of this kind there is a tendency to slacken the hold of the pins which it connects, through the shortening of the compressed bar; and means must therefore be provided of tightening it from time to time by wedges or screws.

IV. When the form of cross-section chosen is such that the ratio $l^2 \div r^2$, which the length of the rod is to bear to the radius of gyration of its cross-section, can be approximately determined independently of the sectional area, that area is to be found simply by the following formula:—

$$S = \frac{P}{f}\left(1 + \frac{l^2}{c\,r^2}\right) \dots\dots\dots\dots (5.)$$

For values of r^2 and of c, see Article 443, pages 524, 525.

The commonest example of the class of figures to which this method is applicable is the *cross-shaped* section, which is well adapted for transmitting thrust; and for which we have $r^2 = h^2 \div 24$; and consequently

$$c\,r^2 = 1500\,h^2. \dots\dots\dots\dots (6.)$$

V. The *braced* connecting-rod may be looked upon as a modification of the compound rod mentioned in Rule III. of this Article. The thrust-bar occupies the middle of the combination, and may be of timber or of cast iron. From the middle of its length diverge four transverse arms, in the shape of a cross; and four tie-rods extend from end to end of the bar, and at the middle of their length are made to spread asunder by the arms of the cross. The length of those arms is, in a great measure, a matter of practical convenience; about one-twelfth of the length of the compound rod is usual. The effect on the thrust-bar is to increase its strength, so as to be equal to that of a bar of the same section and half the length; that is to say, in equations (4) and (5), for a and c, substitute $4a$ and $4c$. The amount of stress on the tension-rods is increased in the same ratio with their length.

SECTION III.—*Special Rules relating to Axles and Shafts.*

460. **General Explanations as to Shafts, Axles, and Journals.**—The words *axle* and *shaft* are, to a certain extent, used indiscriminately; but it may be held that in most cases the term *shaft* implies the transmission of motive power along the rotating piece denoted by that term, and consequently the exertion of a twisting moment at each cross-section, to be found by the principles of Article 447, page 528; while an *axle* in general is subjected to a bending moment only.

The parts of a shaft which rest on the bearings are called *pivots*, *collars*, *gudgeons*, and *journals*.

Pivots and collars are for bearing end-thrust (see Article 311, page 353). Gudgeons and journals are for bearing transverse pressure. It was proposed by Buchanan, in his *Treatise on Mill-work*, to apply the word *gudgeon* only to the bearing part at the end of a shaft or axle, which is exposed to bending action alone, and not to twisting action; and *journal* to an intermediate bearing part through which a twisting moment is or may be exerted; but the custom of using the word journal in both senses indiscriminately is so prevalent, that it is impracticable to carry out Buchanan's suggestion. The terms *end-journal* and *neck-journal*, or simply *neck*, may serve to distinguish them.

Cast iron may be used for shafts where no shocks are to be borne. In other cases the proper materials are wrought iron and mild steel. The *greatest* proper values of the real modulus of working stress are the following, or nearly so:—

	Lbs. on the square inch.	Kilos. on the square millimètre.
Cast Iron,..........................	4,500	3·16
Wrought Iron,*....................	9,000	6·33
Steel,.................................	13,500	9·5

As to the use and treatment of cast iron for this purpose, as well as for machinery in general, see Article 390, pages 453-455. In the case of wrought-iron shafts, it is important that the

* The working modulus for wrought iron is the result of a series of practical trials of railway carriage axles, extending over many years, by Lieutenant David Rankine and the Author.

continuity of the fibres at and near the surface should be as little broken as possible, and that where the stress is severe there should be no re-entering angles in the outline; for at places where the fibres are interrupted, and at re-entering angles, cracks are apt to commence, which gradually extend inwards, and at length reduce the sound part of the axle to so small a diameter that it snaps in two. This process has been known to occupy two or three years, and sometimes more. (See Article 391, pages 456, 457; and the authorities referred to in the second note to page 456.) Hence, when it is necessary that a journal should be of a diameter materially smaller than the main body of the axle, the parts of different diameters should be connected by a curved and not by an angular shoulder; and the reduction of diameter should as far as possible be produced by forging, rather than by cutting or turning; the process of turning being used only to give precision to the shape.

461. **Gudgeons or End-Journals.**—A gudgeon, or journal at the end of a shaft, exposed to the transverse bending action of the bearing pressure only, has to fulfil two conditions: to have a transverse section sufficient to bear the bending moment safely; and to have a longitudinal section sufficient to prevent the unguent from being forced out by the pressure.

Let W denote the greatest working load, or bearing pressure; x, the length of the gudgeon; h, its diameter. Let p be the proper value of the pressure *per unit of area of longitudinal section* (as to which, see Article 310, page 350); and let f be the modulus of working stress. Then the condition as to proper lubrication is expressed by the following equation:—

$$p\, x\, h = \mathrm{W}; \qquad (1.)$$

and the condition as to strength by

$$\frac{\pi}{32} \cdot f\, h^3 = \mathrm{W}\, x; \qquad (2.)$$

in the latter of which equations provision is made for the contingency of the whole load being concentrated on the outer end of the gudgeon. The fact is, that in well-made and well-fitted machinery, the resultant load acts through the middle of the length of the gudgeon, or very nearly so; and that the bending moment at the shoulder is only about $\frac{\mathrm{W}\, x}{2}$, instead of $\mathrm{W}\, x$; but the error is on the safe side. By elimination from those two equations are obtained the following formulæ:—

$$\text{diameter, } h = \left(\frac{32\, \mathrm{W}^2}{\pi\, p f}\right)^{\frac{1}{4}}; \qquad (3.)$$

$$\text{length, } x = h \cdot \sqrt{\left(\frac{\pi f}{32\, p}\right)} \ldots\ldots\ldots\ldots\ldots (4.)$$

The following is a convenient form of equation (3) for practical use:—Let

$$\sqrt{\left(\frac{\pi p f}{32}\right)} = a; \text{ then } h = \sqrt{\left(\frac{W}{a}\right)} \ldots\ldots (3\text{ A.})$$

The following are examples of the values of the co-efficients. In each case, f = 9,000 lbs. on the square inch, or 6·33 kilogrammes on the square millimètre.

p, lbs. on the square inch, ...	450	300	225	150
p, kilogrammes on the square millimètre,...	0·316	0·211	0·158	0·106
a, lbs. per square inch,	630	514	446	364
a, kilogrammes per square millimètre,...	0·443	0·362	0·314	0·256
$\frac{x}{h}$, ...	1·40	1·71	1·98	2·42

In the case of cast iron, for which the working modulus of stress is one-half of that for wrought iron, each of the values of a and of $x \div h$ in the above table is diminished in the ratio of $1 : \sqrt{2} = 0{\cdot}707 : 1$.

Another method of calculation is to fix an arbitrary value for the ratio of the length to the diameter; say $\frac{x}{h} = m$; then we have, by equation (2),

$$\frac{\pi}{32} f h^3 = W x = m W h;$$

and consequently

$$h = \sqrt{\left(\frac{32\, m\, W}{\pi f}\right)} = \sqrt{\left(\frac{W}{a'}\right)}; \ldots\ldots\ldots\ldots (5.)$$

if a' be taken to denote $\frac{\pi f}{32\, m}$.

The values of the ratio of length to diameter, approved by Fairbairn in his *Treatise on Millwork*, are, for cast iron, 1·5, and for wrought iron, 1·75. The following table shows the corresponding values of a'.

	Cast Iron.	Wrought Iron.
m,	1·5	1·75
f, lbs. on the square inch,	4,500	9,000
f, kilogrammes on the square millimètre,	3·16	6·33
a', lbs. on the square inch,	294	504
a', kilogrammes on the square millimètre,	0·207	0·345

All the rules for end-journals apply also to *crank-pins*.

462. **Bearing Axles** is a term which may be used to distinguish those axles which have to bear a bending action, but not a twisting action.

An axle in this condition is to be treated as a beam; the bending moments at a series of cross-sections being calculated by the rules of Articles 434 to 436, pages 505 to 510. In making those calculations, it is usual to assume, as in the preceding Article, that the bearing-pressures act through the extreme ends of the gudgeons.

The cross-section adopted must be one adapted to resist bending actions in all directions; such as a circle, solid or hollow, a solid octagon, or a cross. In wrought iron and steel, the solid and hollow circular sections are suitable; in cast iron, the hollow circular and the cross-shaped sections; in timber, the solid octagon.

In the case of circular sections, the diameter is to be calculated from the bending moment by equation (7 A) of Article 437, page 514: that is to say, let M be the greatest working bending moment; f, the modulus of working stress; m, the ratio of the diameter of the hollow (if any) to the outside diameter; then

$$h = \left(\frac{32\ \mathrm{M}}{\pi\,(1 - m^4) f}\right)^{\frac{1}{3}} = \left(\frac{\mathrm{M}}{(1 - m^4)\, a''}\right)^{\frac{1}{3}};\ldots\ldots\ldots(1.)$$

if a'' be taken to denote $\pi f \div 32$. The following are examples of the co-efficients:—

	Cast Iron.	Wrought Iron.	Steel.
f, lbs. on the square inch,	4,500	9,000	13,500
f, kilogrammes on the square millimètre,	3·16	6·33	9·5
a'', lbs. on the square inch,	441	882	1,323
a'', kilogrammes on the square millimètre,	0·31	0·62	0·93

It is of course to be understood that, when British measures are used, M is to be expressed in inch-lbs.

The solid octagonal section may be considered as practically equivalent to its inscribed circle.

For the *cross-shaped section*, the outside diameter, h, is in the first place to be fixed; and then the requisite *sectional area* is to be calculated by equation (7 B) of Article 437, page 514, making $k = \frac{1}{12}$ in that equation; that is to say,

$$S = \frac{12 \, M}{f \, h}. \qquad (2.)$$

Bearing shafts are frequently tapered from the place of greatest bending moment towards the points of support, so as to give a *longitudinal section of equal strength* (as to which see Article 438, page 517), or as near an approximation to such a section as is consistent with practical convenience.

463. **Neck-Journals.**—A neck-journal (often called simply a neck, and by Buchanan, a journal) is an intermediate part of a shaft or axle, turned to a smooth and truly cylindrical surface, so as to fit its bearing easily, as stated in Article 371 A, page 424. If it is exposed to bending action only, its diameter is to be determined by the rules of the preceding Article; if to twisting action, or to twisting and bending combined, by rules which will be given in the ensuing Articles.

The lengths of neck-journals are to be calculated so as to give the requisite area for bearing the pressure, according to the principles of Article 310, page 350.

464. **Shafts under Torsion.**—A shaft which transmits motive power is exposed to a twisting moment throughout its whole length. The first step towards determining the proper diameter for the shaft is to calculate that twisting moment from the power to be transmitted and the speed of rotation, according to the principles of Article 447, page 528. From the power and speed, the result obtained in the first place is the mean twisting moment; and the greatest twisting moment may either be deduced from the mean twisting moment directly, or may be provided for, together with various causes of additional stress, by using a sufficiently large *apparent factor of safety*, as already stated in Article 414, page 489.

The greatest values of the real modulus of working stress in shafts under torsion correspond to about 6, as a real factor of safety. The apparent modulus may be considerably less when a greatly increased apparent factor of safety is used. The apparent factor of safety is sometimes as high as 36.

The formulæ to be used in calculating the diameter of a shaft from the twisting moment, or from the power and speed, as the case may be, are as follows. Let M be the moment; H P, the real horses-power to be transmitted; N, the number of revolutions per minute. Then,

$$h = \sqrt[3]{\left(\frac{M}{A}\right)} = \sqrt[3]{\left(\frac{B \cdot H\,P}{N}\right)}; \qquad (1.)$$

The values of the co-efficients A and B being as follows:—

$$A = \frac{M}{h^3} = \frac{\pi}{16} q_1; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.)$$

where q_1 is the modulus of working stress, real or apparent, according to the way in which the moment M is calculated:—

$$B = \frac{h^3 N}{H P} = \frac{C}{2 \pi A} = \frac{8 C}{\pi^2 q_1}; \ldots\ldots\ldots\ldots\ldots (3.)$$

where C denotes the number of units of work per minute in a horse-power.

The following Table shows a series of examples of the constants which occur in these formulæ:—

FACTORS OF SAFETY.							
For Cast Iron,				6*	9	12	18
Wrought Iron,		6*	9	12	18	24	36
Steel,	6*	9	13½	18	27	36	
MODULUS OF STRESS.							
Lbs. on the square inch,	13,770	9,180	6,120	4,590	3,060	2,295	1,530
Kilogrammes on the square millimètre,	9·6	6·4	4·3	3·2	2·1	1·6	1·07
VALUES OF $A = \frac{M}{h^3}$.							
M in inch-lbs.; h in inches,	2,700	1,800	1,200	900	600	450	300
M in foot-lbs.; h in inches,	225	150	100	75	50	37·5	25
M in grammètres; † h in millimètres,	1·89	1·26	0·84	0·63	0·42	0·315	0·21
VALUES OF $B = \frac{N h^3}{H P}$.							
h in inches,	23·3	35	52·5	70	105	140	210
h in centimètres,	380	570	855	1,140	1,710	2,280	3,420

Apparent factors of safety, ranging from 6·6 to 10, are found in the paddle-shafts and propeller-shafts of steam-vessels, the real factor of safety being seldom above 6. Apparent factors of safety ranging from 18 to 36 are met with in the shafting of mills; and they correspond to what are called "light shafting" and "heavy shafting" respectively.

465. **Span between Bearings of Shafts.**—The ordinary rules for

* Real factor of safety.

† A grammètre is one-1,000th part of a kilogrammètre.

fixing the greatest span between the bearings of a line of shafting are based on the principle that the deflection produced by the weight of the shaft itself, and by any additional transverse load which may be applied to it, should not exceed a certain fraction of the span. Different authors give different values for that fraction. In the following formulæ the value adopted is one-2,000th; and the rule obtained agrees with that given by Mr. Molesworth in his *Pocket Book*.

A general formula for the deflection in such cases is given in Article 439, page 521, and is as follows:—

$$v' = (1 + m)\frac{2 n''' w c^4}{E r^2};$$

m being the proportion which the additional load bears to the weight of the shaft; c, the half-span; w, the heaviness of the material; E, its modulus of elasticity; n''', a numerical co-efficient; and r, the radius of gyration of the cross-section of the shaft. In most cases, the load may be treated as uniformly distributed; and as there may be a coupling near each bearing, the shaft is to be treated as simply supported, and not fixed, at each bearing; so that $n''' = \frac{5}{48}$. The values of the square of the radius of gyration, for the figures of cross-section which occur in practice, are:—

For a solid circular section of the diameter h, $r^2 = \frac{h^2}{16}$;

For a hollow circular section; diameter outside, h; inside, h'; $r^2 = \frac{h^2 + h'^2}{16}$;

Cross-shaped section; breadth over arms, h;

$$r^2 = \frac{h^2}{24} \text{ approximately.}$$

The value of $\frac{E}{w}$ may be taken at 7,500,000 feet, or 90,000,000 inches, or 2,286,000 mètres.

Putting for n''' the value already stated, we have

$$v' = (1 + m) \cdot \frac{5 w c^4}{24 E r^2}; \text{ and consequently}$$

$$c^3 = \frac{24 E}{5 (1 + m) w} \cdot \frac{v' r^2}{c}; \qquad (1.)$$

giving for the span between the bearings,

$$2 c = \left(\frac{192 E v' r^2}{5 (1 + m) w c}\right)^{\frac{1}{3}} \qquad (2.)$$

Let $\frac{v'}{c} = \frac{1}{1,000}$; let $\frac{E}{w}$ have the value stated above; and let the section be a solid circle; then

$$2\,c = \left(\frac{3\,E\,h^2}{1,250\,(1+m)\,w}\right)^{\frac{1}{3}} = \left(\frac{a\,h^2}{1+m}\right)^{\frac{1}{3}};\ldots\ldots\ldots\ldots(3.)$$

in which

$$a = 216,000 \text{ inches} = 5,500 \text{ mètres nearly.}$$

When the shaft is loaded with pulleys and with the tensions of their belts, it is usual to assume $m = \frac{1}{3}$.

The result of equation (3) is a span which is not to be exceeded in the construction of a line of shafting of a given diameter.

466. **Shafts under Combined Bending and Twisting Actions.**— When a shaft is strained by bending and twisting actions combined, two cases may be distinguished: first, where the bending moment and twisting moment are both given, and the diameter of the shaft is to be found; for example, where the pressure exerted on a crank-pin produces combined bending and twisting actions on a journal; and, secondly, where the bending moment is produced by the weight and re-action of the shaft itself, and therefore depends on the diameter.

I. When a given twisting moment, M, is combined with a given bending moment, M', make

$$\sqrt{(M^2 + M'^2)} + M' = M'';\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

and find the diameter required in order to bear safely a *twisting moment* equal to M", by means of equation (1) of Article 464, page 543. An example of this problem, solved graphically, is shown in fig. 280, which represents a shaft having a crank at one end. At the centre of the crank-pin, P, is applied the pressure of the connecting-rod; and at the centre of the bearing, S, acts the equal and opposite resistance of that bearing. Representing the common magnitude of those forces by P, they form a couple whose moment is

$$P \cdot S\,P.$$

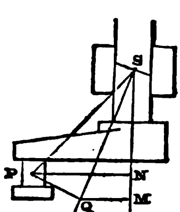

Fig. 280.

Draw S Q bisecting the angle P S M. On S Q let fall the perpendicular P Q. From Q let fall Q M perpendicular to S M.

Calculate the diameter of the shaft, as if to resist a twisting moment, M" = 2 P · S M, and it will be strong enough to resist the combined bending and twisting action of P applied at the point marked P.

The greatest longitudinal stress on the particles of a shaft, produced by the bending action of its own weight, and of any additional load or re-action bearing a given proportion, m, to its own weight, is found by applying equation (2) or equation (5) of Article 458, pages 536 and 537, to the case of a solid cylindrical body; that is to say, when the proper substitutions are made, we have the following value for the intensity of that stress:—

$$p = (\cos i + m) \frac{4 \, w \, c^2}{h}; \text{..................... (2.)}$$

in which c is the half-span between the bearings; h, the diameter; w, the heaviness of the material (being, for iron, 0·278 lbs. per cubic inch, or 7,690 kilogrammes per cubic mètre); i, the inclination of the shaft to the horizon; and m the proportion borne by the additional load and re-action to the weight of the shaft.

Let q be the real intensity of the greatest stress due to twisting. It is given by the formula—

$$q = \frac{16 \, \mathrm{M}}{\pi \, h^3}; \text{.......................... (3.)}$$

where M is the real value of the greatest working twisting moment. $\left(\frac{16}{\pi} = 5{\cdot}1 \text{ nearly.}\right)$ That stress is a shearing stress; and is equivalent to tension and thrust combined, and of the same intensity, q, exerted in two directions perpendicular to each other, tangential to the cylindrical surface of the shaft, and making angles of 45° with the axis.

The *resultant stress* due to the twisting stress, q, and the bending stress, p, combined, is found by the following formula, which has been demonstrated by writers on the internal equilibrium of elastic solids:—

$$s = \sqrt{\left(q^2 + \frac{p^2}{4}\right)} + \frac{p}{2}; \text{................. (4.)}$$

where s denotes the intensity of the resultant stress.

The conditions which the diameter of a shaft ought to fulfil are expressed by the following equation, derived from equation (4):—

$$s^2 - s\,p - q^2 = 0; \text{.................... (4 A.)}$$

in which, for s, is to be put a safe working value of the resultant stress (say 8,000 lbs. on the square inch, or 5·6 kilogrammes on the square millimètre), and for p and q, their values in terms of M, c, w, and h, as given by equations (2) and (3) respectively. The equation then becomes of the *sixth order;* and it is to be solved so as to find h. This can be done by approximation only; and a convenient method of approximation is as follows:—Assume for q an approximate value, q', somewhat less than that of s (say $q' = 0{\cdot}9\ s$). Then calculate an approximate value, h', of the diameter, from equation (3), viz. :—

$$h' = \left(\frac{5{\cdot}1\ \mathrm{M}}{q'}\right)^{\frac{1}{3}} \quad \ldots\ldots(5.)$$

Then calculate, for p, an approximate value, p', from equation (2), viz. :—

$$p' = (\cos i + m)\frac{4\ w\ c^2}{h'}; \quad \ldots\ldots(6.)$$

and from the approximate value of p' calculate a second approximate value of q, as follows:—

$$q'' = \sqrt{(s^2 - s\ p')}. \quad \ldots\ldots(7.)$$

Should this agree with the first approximate value, q', the approximate diameter, h', will answer; and should there be a difference, a second approximation, h'', to the required diameter is to be computed, as follows:—

$$h'' = h'\left(\frac{q'}{q''}\right)^{\frac{1}{3}}. \quad \ldots\ldots(8.)$$

When, as is usually the case, the difference, $q' - q''$, is small compared with q', the following formula for the second approximation is sufficiently near the truth:—

$$h'' = h\left\{1 + \frac{q' - q''}{3\ q'}\right\}. \quad \ldots\ldots(9.)$$

A third approximation might be found by repeating the process; but the second approximation will, in general, be found accurate enough for practical purposes.

467. **Centrifugal Whirling of Shafts.***—Any small deflection of the centre line of a shaft from the straight axis of rotation gives rise, on the one hand, to centrifugal force, tending to make the deflection become greater; and, on the other hand, to elastic stress, resisting the deflection, and tending to straighten the centre line again. The resistance to deflection may be shortly called the *stiffness.* In very small deflections, the centrifugal force and the stiffness both increase according to the same law, being both sensibly propor-

* The substance of this Article first appeared in the *Engineer* of the 9th April, 1869.

tional to the deflection simply; hence, whichever of them is the greater for an indefinitely small deflection, continues to be the greater until some deflection is reached which causes a sufficient difference between their laws of variation. The consequence is, that if, for an indefinitely small deflection, the centrifugal force is equal to or greater than the stiffness, the shaft must go on permanently whirling round in a bent form, to the injury of itself and the adjoining machinery and framing: a kind of motion which may be called *centrifugal whirling*. On the other hand, if for an indefinitely small deflection the stiffness is greater than the centrifugal force, centrifugal whirling is impossible.

For a shaft of a given length, diameter, and material, there is a limit of speed; and for a shaft of a given diameter and material, turning at a given speed, there is a limit of length, below which centrifugal whirling is impossible.

The mathematical expression of the conditions of the problem leads to a linear differential equation of the fourth order, integrable by means of circular and exponential functions. The integrals are (as might have been expected) identical in form with those obtained by Poisson in his investigation of the transverse vibrations of elastic rods (*Traité de Mécanique*, Vol. II., § 528); and some of the numerical results calculated by Poisson are applicable to the present problem. The relation between the limits of length and of speed depends on the way in which the shaft is supported. The only two cases which will here be given are those respectively of a shaft supported on two bearings at its ends, and of an overhanging shaft with one end fixed in direction.

Let $\frac{E}{w}$ be the modulus of elasticity of the material, expressed in units of height of itself (say, for wrought iron, about 7,500,000 feet, or 2,286,000 mètres); r^2, the square of the radius of gyration of the cross-section of the shaft ($= h^2 \div 16$ for a cylindrical shaft of the diameter h); and a, the angular velocity of rotation, ($= 2\pi \times$ number of turns per second). Calculate a certain length, b, as follows:—

$$b = \left(\frac{E g r^2}{w a^2}\right)^{\frac{1}{4}}. \qquad (1.)$$

Then the limit of span, below which centrifugal whirling is impossible, bears a ratio to b depending on the manner in which the shaft is supported; for example,

Shaft supported at the ends; span, $2c$, $\left\{ \begin{array}{l} = \pi b; \\ = 3{\cdot}1416\, b \end{array} \right\}$; (2.)

Shaft overhanging: direction of one end fixed; length, c, $\left\{ \begin{array}{l} = 0{\cdot}595\, \pi b \\ = 1{\cdot}87\, b \end{array} \right\}$. (3.)

In practical calculations it may be convenient to put, instead of $\frac{g}{a^2}$, $\frac{A}{n^2}$; where n is the number of revolutions per second, and $A = \frac{g}{4\pi^2}$ (= 0·815 foot, or 0·248 mètre, nearly) is the altitude of a revolving pendulum which makes one revolution in a second. This gives, for the value of b,

$$b = \left(\frac{E\,A\,r^2}{w\,n^2}\right)^{\frac{1}{4}}. \quad \text{.........................(4.)}$$

It is obvious that r should be expressed in the same units of measure with $\frac{E}{w}$ and A; for example, in feet, if they are expressed in feet.

The inverse formulæ for the limit of speed below which centrifugal whirling is impossible in a shaft of a given length, l, are of course as follows:—

Make $b = 2\,c \times 0·3183$ for a shaft supported at the two ends, and of the half span, c; ....(5.)

Or $b = 0·5347\,c$ for an overhanging shaft; ....(6.)

Then the limit of speed, in revolutions per second, is

$$n = \frac{r}{b^2} \cdot \sqrt{\frac{E\,A}{w}} \quad \text{.........................(7.)}$$

The following are approximate values of $\frac{E\,A}{w}$ and its square and fourth roots, for British and French measures:—

	$\frac{E\,A}{w}$	$\sqrt{\frac{E\,A}{w}}$	$\left(\frac{E\,A}{w}\right)^{\frac{1}{4}}$
Feet, ...	6,100,000	2,470	49·7
Mètres, ...	566,000	752·4	27·4

An additional mass turning along with the shaft, such as a pulley, has little effect on the centrifugal force when it is in the usual position; that is, close to or near to a bearing.

The effect of an additional rotating load distributed uniformly along the shaft may be allowed for by diminishing the height $\frac{E}{w}$

of the modulus of elasticity in the same proportion in which the weight of the shaft itself is less than the gross load.

The effect of an additional rotating load at a point not near a bearing has not yet been investigated. The problem is capable of solution by means of the general integrals already known; but it is not of much practical importance; for when a shaft is so long and so rapid in its rotation as to require precautions against centrifugal whirling, the first precaution is to avoid loading it with rotating masses which are not very near the bearings.

468. **Dimensions of Couplings.**—Couplings have already been referred to in connection with disengaging and re-engaging gear, in Article 260, page 295. An ordinary shaft-coupling may be described as consisting of two discs at the ends of the two lengths of shafting to be coupled, which two discs, when put together, form one cylinder; each of the two discs being cylindrical, and having, on the side which faces the other disc, alternate projecting and receding sectors: the projecting sectors forming claws, which fit into the hollows and between the claws of the opposite disc. When there is only *one projecting sector* or claw on each disc, of a semi-cylindrical figure, the coupling is the *circular half-lap coupling*, introduced by Mr. Fairbairn. It is described by him in the following words (*Treatise on Mills and Millwork*, Part II., page 81):—"It is perfectly round, and consists of two laps, turned to a gauge, and, when put together by a cutting machine, it forms a complete cylinder. . . . . . . A cylindrical box is fitted over these, and fixed by a key, grooved half into the box and half into the shaft. The whole is then turned in the lathe to the same centres as the bearings of the shaft. . . . ."

The following are the proportions given by Mr. Fairbairn for this coupling:—

Area of the coupling,..................	= 2 × area of the shaft.
Or, in other words, diameter of coupling,...........................	= 1·4142 × diameter of shaft.
Length of lap,...........................	= diameter of shaft.
Length of box,..........................	= 2 × diameter of shaft.
To which may be added, outside diameter of box,..................	= $2\frac{1}{4}$ × diameter of shaft.

469. **Bushes and Plumber-Blocks.**—The following rules for the dimensions of the brasses or bushes of bearings for journals, and of the plumber-blocks which carry those bushes, are given on the authority of Mr. Molesworth:—

BUSH: thickness of metal at bottom = 0·15 inch + from 0·09 to 0·12 diameter of journal.

„ thickness of metal at sides = $\frac{3}{4}$ thickness at bottom.

PLUMBER-BLOCK: thickness of sole-plate = 0·3 diameter of journal.
" thickness of cover = 0·4 diameter of journal.
" diameter of bolts, if two in number = 0·25 diameter of journal.
" " if four in number = 0·18 diameter of journal.

SECTION IV.—*Special Rules relating to Pulleys, Wheels, and Teeth.*

470. **Teeth and Rims of Wheels, and Dimensions depending on them.**—The teeth of wheels are made strong enough to resist the bending moment which may arise from the whole force transmitted by a pair of wheels happening to act on one corner of one tooth, such as D in fig. 281.

In fig. 281 A, let the shaded part represent a portion of a cross-section of the rim of the wheel A of fig. 281, and let E H K P be the face of a tooth, on one corner of which, P, acts the force represented by that letter. Conceive any sectional plane, E F, to intersect the tooth from the side E P to the crest P K, and let P G be perpendicular to that plane. Let h be the thickness of the tooth, and let E F $= b$, P G $= l$.

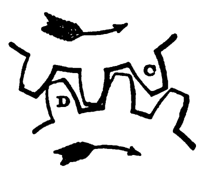

Fig. 281.

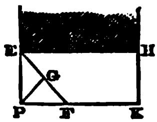

Fig. 281 A.

Then the bending moment at the section E F is P l, and the greatest stress produced by that moment at that section is

$$p = \frac{6\,P\,l}{b\,h^2},$$

which is a maximum when $\angle$ P E F = 45°, and $b = 2\,l$, having then the value,

$$f = \frac{3\,P}{h^2}.$$

Consequently, the proper thickness for the tooth is given by the equation

$$h = \sqrt{\frac{3\,P}{f}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.)$$

This formula is Tredgold's; according to whom the proper value for the greatest working stress, f, is 4,500 lbs. on the square inch, when the teeth are of cast iron; and about 1,125 lbs. on the square inch, when they are wooden cogs; being equivalent respectively to 3·2 and to 0·8 kilogrammes on the square millimètre; so that a wooden cog has twice the thickness of a cast-iron tooth fitted to bear the same pressure.

The thickness having been thus determined, the pitch is to be deduced from it by the following formula:—

$$\text{Pitch} = \frac{h + 0{\cdot}02 \text{ inch}}{0{\cdot}47} = \frac{h + 0{\cdot}5 \text{ millimètre}}{0{\cdot}47}; \ldots\ldots(2.)$$

and all the other dimensions of the teeth, and of the ring which carries them, are to be calculated from the pitch, by means of rules which have already been given in Article 125, page 117.

It has already been stated in the rules referred to, that the depth of the ring, when cast along with the teeth, is equal to the thickness of a tooth at its root. In a *mortise-wheel*, with a cast-iron ring, and wooden cogs fixed into mortises in the ring, the depth of the ring is about equal to the pitch. The thickness of metal in the ring regulates the thickness of metal in other parts of the wheel, which should be the same, or nearly the same.

The hoop-shaped ring which carries the teeth of a wheel is often strengthened by means of a rib or feather in a plane normal to the axis, and sometimes by means of two such ribs, so as to make the cross-section of the rim T-shaped, or trough-shaped, as the case may be. The sectional area of the rim in such cases is made nearly equal to that of the arms, which is determined by means of principles to be stated further on.

The breadth of a wheel is regulated by that of the teeth, which is found by one of the rules already referred to. The breadth of a pulley is regulated by that of the belt which is to run upon it; and the swell (as already stated in Article 170, page 184), is one-24th of the breadth.

471. **Boss and Arms of a Wheel or Pulley.**—When a wheel or pulley is cast in one piece, the nave or boss is usually about twice the diameter of the shaft on which it is fixed, and sometimes a little less. When the arms are cast separate from the boss, and inserted into sockets in it, where they are fixed by means of bolts or of wedges, the boss is from three to four times the diameter of the shaft.

A small wheel or pulley often has instead of arms, a thin flat disc, whose thickness may be made equal to the thickness of metal round the eye of the boss. In rapidly revolving pulleys, the arms are often made of a flat oval section, so as to cleave the air edgewise. The arms of wheels are usually of a T-shaped, cross-shaped, or tubular form of section.

To calculate the bending moment which each of the arms of a wheel has to resist, let M be the greatest moment of the effort transmitted by the wheel; n, the number of arms; r, the geometrical radius of the wheel, from the axis to the pitch-line; x, the length of an arm, from the boss to the rim; M′, the bending moment on each arm; then two cases may be distinguished.

I. If the arms and rim are made in one piece, either by casting or by welding;

$$M' = \frac{M x}{2 n r}; \quad \text{(1.)}$$

and this formula is applicable also to wheels like the paddle wheels of a steamer, in which wrought-iron arms are rigidly bolted or rivetted both to the boss and to the rim.

II. If the arms are cast along with segments of the rim, and fastened into sockets in the boss;

$$M' = \frac{M x}{n r}. \quad \text{(2.)}$$

This second formula is based on the supposition that the joint where an arm is inserted into the boss cannot safely be trusted to bear any part of the bending moment. This is not strictly correct, but it is an error on the safe side.

The transverse section of the arms is to be adapted to bear safely the working moment thus found, by the aid of the rules of Article 437, pages 514 to 516.

In Case I, the greatest bending moment is exerted on each arm at two points, close to the rim and close to the boss respectively; in Case II, the greatest bending moment is exerted close to the rim.

Another way of adjusting the strength of the arms to the moment exerted through them is as follows:—Having fixed the figure and dimensions of an arm according to convenience, calculate the working moment to which it is adapted; let this be denoted by M′; then the number of arms required is given by the following formulæ:—

$$\text{In Case I; } n = \frac{M x}{2 M' r}; \quad \text{(3.)}$$

$$\text{In Case II; } n = \frac{M x}{M' r}; \quad \text{(4.)}$$

The real working modulus of stress for cast iron in these calculations should not exceed 4,500 lbs. on the square inch, or 3·2 kilogrammes on the square millimètre; and for wrought iron, 9,000 lbs. on the square inch, or 6·3 kilogrammes on the square millimètre.

472. **Centrifugal Tension in Wheels and Pulleys.**—The rim of a wheel, moving with the velocity v, is subjected to a centrifugal tension whose amount is equal to the weight of a length $\frac{v^2}{g}$ of that rim (including teeth, if it is a toothed wheel). This is resisted by the tenacity of the rim at its smallest cross-section (or by the fastenings of the rim, if it is made in segments), partly assisted by the tenacity of the arms. Each of the arms has to bear its own centrifugal tension, which, at a point close to the boss, is equal to the weight of a length of the arm itself expressed by $\left(1 - \frac{r'^2}{r^2}\right)\frac{v^2}{2g}$; r being the radius of the wheel, and r' that of the boss; and on the whole, it is an error on the safe side to make the rim strong enough to bear its own centrifugal tension without aid from the arms. This fixes a limit of safety as to speed, for a rim of a given material and construction. Let m be the ratio in which the mean sectional area is greater than the effective sectional area, f the greatest working tensile stress, and let w be the heaviness of the material; then the greatest proper velocity, being that which produces the stress f, is given by the formula :—

$$v = \sqrt{\left(\frac{gf}{mw}\right)} \text{.......................(1.)}$$

The modulus, $\frac{f}{w}$, for cast iron may be taken at 800 feet, or 244 mètres; so that when $m = 1$, as in a pulley, or a fly-wheel without teeth, we have $v = 160$ feet, or 49 mètres, per second nearly. Let a cast-iron spur fly-wheel be so designed that $m = 2$; then $v = 113$ feet, or 34 mètres, per second nearly.

The modulus, $\frac{f}{w}$, for wrought-iron wheel-tyres that are not welded, but rolled out of perforated discs, may be taken at 2,400 feet, or 730 mètres.

473. **Tension-Arms of Vertical Water-Wheels.**—The weight of a great vertical water-wheel, of the construction introduced by Hewes, is hung from a cast-iron boss by means of wrought-iron tension-rods. The load is distributed amongst the rods which, at a given instant, point obliquely or vertically downwards from the boss; and the amount of the tension on each rod is proportional nearly to the square of the cosine of its inclination to the vertical. The mean value of that square is nearly $\frac{1}{2}$; and, at any instant, half the total number of rods point downwards. Hence, let f be

the intensity of the tension on the rods which point vertically downwards; S, the sectional area of a rod; n, the number of rods; W, the load; then

$$W = \frac{n f S}{4}; \quad \text{(1.)}$$

and

$$S = \frac{4 W}{n f}. \quad \text{(2.)}$$

For the working value of f, we may take from 9,000 to 10,000 lbs. on the square inch; or from 6·3 to 7 kilogrammes on the square millimètre. (See *A Manual of the Steam-Engine and other Prime Movers*, Article 155, pages 181, 182.)

474. **Braced Wheels.**—Instead of transmitting power between the boss and the rim of a wheel by means of the resistance of the arms to bending, the arms may be so placed as to transmit power by their direct tension and thrust; and for that purpose they must not be radial, but must lie in the direction of tangents to a circle of a radius somewhat smaller than that of the boss. Let r'' denote the radius of this circle; n, the number of arms; M, the greatest moment transmitted; then the amount of the greatest stress along an arm is given by the following expression:—

$$\frac{M}{n r''}.$$

This is tension for one half of the arms, and thrust for the other half; and their dimensions are to be determined by the rules of Article 459, pages 537 to 539.

475. **Levers, Beams, and Cranks** have usually one or two arms, as the case may be; and each arm is in the condition of a bracket; the greatest bending moment being exerted at that cross-section which traverses the fulcrum, or axis of motion. In the crank of a steam engine, the greatest bending moment is identical with the greatest twisting moment exerted on the shaft to which the crank is fixed.

In ordinary cases it is unnecessary to add anything to the rules which have already been given in Article 434 to 438, pages 504 to 517, for determining bending moments, and the transverse dimensions required in order to resist those moments. Cranks are usually rectangular in section; levers and walking beams are sometimes rectangular and sometimes I-shaped. The bending moment is in most cases exerted in contrary directions alternately, so that the cross-section must be made symmetrical about the neutral axis; and for the modulus of stress must be taken a safe working value of that kind of stress against which the material is weakest; tension for cast iron, thrust for wrought iron; for example:—

	Cast Iron.	Wrought Iron.
Lbs. on the square inch, ...	3000	6000
Kilogrammes on the square millimètre,...	2·1	4·2

Holes made in a lever for the purpose of inserting pins should be as near as possible to the neutral layer; that being the position in which the removal of a given area of material weakens the lever least.

The following rules for the proportionate dimensions of a steam-engine crank, made to be keyed on the end of a shaft, are those deduced by Mr. Bourne from the practice of Messrs. Boulton and Watt:—

	Diameter of Shaft-Journal multiplied by
Crank-web; thickness produced to centre of shaft,...	0·75
" breadth produced to centre of shaft,.....	1·50
Large Eye; breadth,..	1·75
" thickness,..	0·45

	Diameter of Piston-Rod multiplied by
Crank-web; thickness produced to centre of pin,.....	1·10
" breadth produced to centre of pin,.......	1·60
Small Eye; breadth,..	1·87
" thickness,..	0·63
Crank-pin Journal; diameter,..............................	1·40
" " length,..............................	1·60

As to Crank-pins, see pages 541 to 543.

Trussed or *framed levers* are sometimes used; as in the walking beams of American river steamers. A beam of that sort consists mainly of a cast-iron cross, having the ends of its arms tied together by four wrought-iron tie-rods, forming a lozenge-shaped figure. The long arms are from twice to three times the length of the short arms. The long arms are always in a state of thrust; the upper and lower tie-rods alternately are subjected to tension; and the upper and lower short arms of the cross alternately are subjected to a thrust equal to twice the load. The load, the thrust along a long arm, and the tension on a tie-rod, are to each other nearly in the proportions of the length of a short arm, the length of a long arm, and the length of a tie-rod.

CHAPTER IV.

ON THE PRINCIPLES OF THE ACTION OF CUTTING TOOLS.

476. **General Explanations.**—In making the bearing and working surfaces of the parts of a machine, it is only a rough approximation to the required figure that can be obtained by casting, by forging, or by pressure. The precision of form which is essential to smooth motion and efficient working is given by means of cutting tools. The object of the present chapter is to give a brief statement of the principles upon which the action of such tools depends. For detailed information respecting them, reference may be made to the second volume of Holtzapffel's *Treatise on Mechanical Manipulation*, extending from page 457 to page 1025, and to Mr. Northcott's *Treatise on Lathes and Turning;* and for a very clear summary account of their nature and use, to an Essay by Mr. James Nasmyth, published at the end of the later editions of Buchanan's *Treatise on Millwork*.

The appendix to Holtzapffel's volume contains two essays of much value on the general principles of cutting tools—one by Mr. Babbage, and the other by Professor Willis.

477. **Characteristics of Cutting Tools in General.**—The usual material for cutting tools is steel, of a degree of hardness suited to that of the material to be cut. Every cutting tool has at least one cutting edge; and sometimes three or more edges meet and form a point, two or more of those edges being cutting edges; so that the form of the cutting part of a tool is that of a wedge, or of a pyramid, as the case may be. A cutting edge is formed by the meeting of two surfaces, generally plane, and sometimes curved. When a surface forming a cutting edge is oblique to the original surfaces of the bar out of which the tool is made, that surface is called a *chamfer* or *bevel*. The angle at which those surfaces meet may be called the *cutting angle*. It ranges from about 15° to 135°, according to the nature of the material to be cut, and the way in which the tool is to act upon it. Examples of cutting angles of tools for different purposes will be mentioned further on. A narrow cutting edge at the end of a bar-shaped tool is often called the *point* of the tool; the body of the tool is called the *shaft* or the *blade;* the term *shaft* being usually applied to tools with a cutting point or narrow edge at one end, and *blade* to those which have a longitudinal cutting edge; but *blade* is often applied to

both kinds of tools. A bar-shaped shaft is sometimes called a *stem*. The blade or shaft of a tool is sometimes made of iron, and edged or pointed with steel. A larger piece, to which the blade is fixed, is called the *stock;* and in the case of hand-tools, that part of the stock which is grasped by the hand is called the *handle*. The stocks and handles of hand-tools are usually of wood of some strong and tough kind. Where steady pressure is to be exerted, stiff woods are to be chosen, such as beech and mahogany; where blows are to be given, flexible woods, such as ash. Machine-tools are held in *tool-holders* of different sorts, made of cast and wrought iron. A *rest* is a fixed or moveable support for a cutting tool; in machine tools, the rest carries the tool-holder.

The term *machine-tool* is often applied, not merely to the cutting implement itself, but to the whole machine of which it forms part.

The piece of material which is being cut by a tool is commonly called the *work*. A given relative motion of the work and cutting tool may be obtained either by keeping the work fixed and moving the tool, or by keeping the tool fixed and moving the work, or by a combination of both those motions.

478. **Classification of Cutting Tools.**—The following classification is that of Holtzapffel. Cutting tools, according to their mode of action on the bodies to which they are applied, are divided into *Shearing*, *Paring*, and *Scraping* tools; the following being the characters by which those tools are distinguished from each other.

I. A *Shearing Tool* acts by dividing a plate or bar of the material operated on into two parts by shearing; that is to say, by making these two parts separate from each other by sliding at the surface of separation.

II. A *Paring Tool* cuts a thin layer or strip, called a *shaving*, from the surface of the work; thus producing a new surface.

III. A *Scraping Tool* scrapes away small particles from the surface of the work; thus correcting the small irregularities which may have been left by a paring tool.

479. **Shearing and Punching Tools.**—A pair of shears for cutting iron usually consists of two blades; the lower fixed, and the upper moveable in a vertical direction. The inner faces of the blades are plane, and are so fitted as to slide past each other very closely, but without appreciable friction. The ordinary angle for the cutting edges is from 75° to 80°. In shears for cutting plates, the edge of the lower blade is horizontal; that of the upper blade has an inclination of from 3° to 6°, in order that the shearing may begin at one edge of the plate, and go on gradually towards the other edge. Fig. 282 represents a cross-section of the blades of a pair of shears, with their cutting

edges at A and B, and having between them a plate or bar, C C, which is to be shorn through at the plane whose trace is the dotted line A B.

The blades of shears for cutting angle-iron bars have V-shaped edges, to suit the form of cross-section of the bars.

The class of shearing tools comprehends also tools for *punching;* the only difference being in the form of the surface at which shearing takes place; which form determines that of the cutting edges of the tools. The general principle of the action of those tools is shown in fig. 283; in which C C is a plate, lying upon the *die*, represented in vertical section by B B; that is, a flat disc or block of hard steel, having in it a hole a little larger than the hole that is to be punched in the plate. A is the *punch*, of a cylindrical figure, its base being of the size and figure of the hole to be made. The descent of the punch causes the material of the plate C C to separate by shearing at the surface whose traces are marked by dotted lines, and drives out a piece called a *wad*, which drops through the hole in the die. The wad is slightly conical; its upper end being of the size of the punch, and its lower end nearly of the size of the hole in the die.

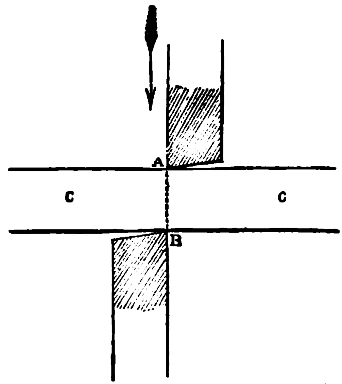

Fig. 282.

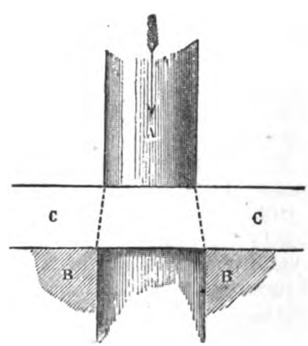

Fig. 283.

The ordinary difference of diameter between the punch and the die ranges from 0·1 to 0·3 of the thickness of the plate. Sometimes the die is made proportionately larger, in order that the holes may be more conical.

The smaller end only of a punched hole is accurate in size and figure, and smooth. The larger end is more or less irregular and ragged. When it is necessary that a punched hole should be made regular and smooth, that is done by scraping with a pyramidal tool called a *rimer* or *broach*.

The *pressure* required for shearing or for punching may be calculated by means of the following formula:—

$$P = f\,c\,t; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.)$$

in which t is the thickness of the plate to be shorn or punched; c, the length of the cut; that is, the breadth of the plate, for shearing, on the circumference of the hole, for punching; so that $c\,t$ is the area of shorn surface; and f, the co-efficient of ultimate resistance to shearing; which may be estimated, for malleable iron, at

50,000 lbs. on the square inch, or
35 kilogrammes on the square millimètre.

The *work* of shearing or punching is estimated by Weisbach (*Ingenieur*, § 129) as equal to that of overcoming the resistance P through *one-sixth of the thickness of the plate;* that is to say, it is expressed by

$$\frac{P\,t}{6} = \frac{f\,c\,t^2}{6}; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.)$$

and this is the net, or useful work. The same author estimates the counter-efficiency of shearing and punching machines at from $1\frac{1}{3}$ to $1\frac{1}{2}$; so that taking the higher of those estimates, the total work of such a machine at one stroke is

$$\frac{P\,t}{4} = \frac{f\,c\,t^2}{4}; \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.)$$

and this formula is to be used in calculating the power required to drive such machines, and the dimensions of their fly-wheels (see page 410).

480. **Paring and Scraping Tools in General.**—The general nature of the modes of action of paring and of scraping tools is illustrated by figs. 284 and 285; paring being represented by fig. 284; scraping by fig. 285. In each figure, the tool is supposed to act upon a cylindrical piece of work in a turning lathe. The arrow in each figure represents the direction of motion of the work relatively to the tool. The plane of each figure is parallel to the direction of the cutting motion, and perpendicular to the face of the work. The point A is the trace of the cutting edge of the tool. F A B is a normal to the face of the work, and is also the trace of a plane normal to the direction of the cutting motion. A C is the trace of a plane tangent to the face of the work. Had the process represented been planing instead of turning, A C would have been the trace of the face of the work itself. A D is the trace of the *face-plane* or *front-plane* of the cutting edge. The angle A C D is called by Babbage the *angle of relief.*

A E is the trace of the *upper plane* of the cutting edge: so called because of its position in the process of turning, when the plane

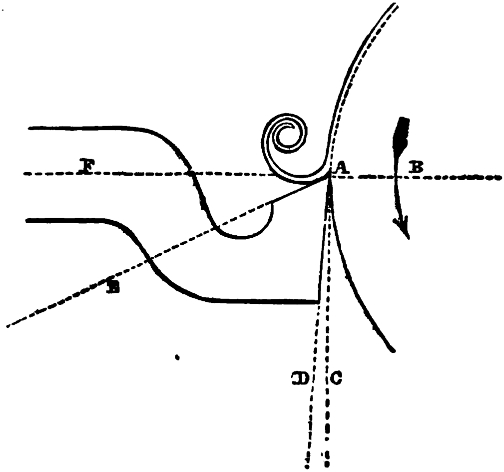

Fig. 284.

A C is vertical, and the tool cuts upwards. In other operations, such as planing and slotting, the plane A E may be turned

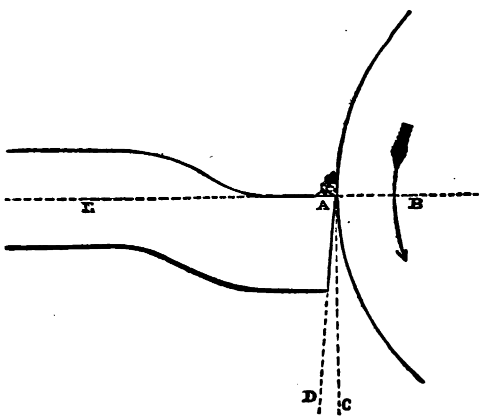

Fig. 285.

forwards or downwards; the fact being, that in every case it is the real front plane as regards the cutting motion; but the custom is to call it the upper plane.

D A E is the *angle of the tool*, or *cutting angle*. In paring tools the cutting angle is always acute, as in fig. 284; having values depending on the nature of the material to be cut, of which examples will be given further on. In scraping tools, the cutting angle may be acute, right, or obtuse; in fig. 285, it is nearly a right angle.

The angle C A E, made by the upper plane A E with the tangent plane A C, may, for the sake of convenience, be called the *working angle*; so that the *working angle is equal to the sum of the cutting angle and the angle of relief*; or in symbols, C A E = D A E + C A D.*

It is upon the working angle that the distinction between properly shaped paring and scraping tools depends. That distinction may be stated as follows:—*Every paring tool has an acute working angle; every right or obtuse working angle makes a scraping tool.*

An acute working angle, if it is too great to suit the material on which it acts, may give a scraping instead of a paring tool.

As regards the effect of the *cutting angle* upon the action of a tool, it is obvious, that as the working angle is greater than the cutting angle, an obtuse or even a right cutting angle gives a scraping tool; and that an acute cutting angle too may give a scraping tool, if the tool is held so as to make the working angle right or obtuse; but that an acute cutting angle is essential to a paring tool.

In well-made paring tools, the difference between the working angle and the cutting angle, or *angle of relief*, C A D, is not made greater than experience has proved to be necessary in order to prevent excessive friction between the face-plane of the tool and the face of the work; that is, about 3°. Any increase of that angle is to be avoided; because it tends to make the tool dig into the work, and to produce "chattering;" and because, by diminishing the cutting angle, it weakens the tool.

The position of the shaft of the tool as held by the tool-holder should in all cases be such that the back and forward motion of adjustment of the cutting edge, A, by which the depth of the cut is regulated, shall take place exactly in the plane F A B, perpendicular to the face of the work; that is to say, in the case of a turning lathe, in a plane traversing the axis of rotation.

The tools shown in the figures have the front ends of their shafts bent, *hooked*, or *cranked*, as it is termed, in such a manner as to

* Mr. Babbage calls E A F the *angle of escape*; so that the angle of escape is the complement of the working angle. In carpenters' hand-planes, the working angle is called the *pitch*.

ensure that the cutting edge, A, shall *not be in advance of the axis of the straight part of the shaft of the tool.* The object of this arrangement is, that any deflection which the tool-shaft may undergo through excessive resistance to the cut, may tend to move the cutting edge, A, away from the work, and not to make it dig into it.

In tools for rough work, and having very stiff shafts, the cranked shape is unnecessary; and then the upper side of the shaft is in the plane B A F; the proper position of the upper plane of the cutting edge being given by means of a hollow or *flute* in the upper side of the tool.

Fig. 286 represents a paring tool designed on the same principles with that shown in fig. 284, but with two cutting edges, and a three-sided pyramidal point. The upper part of the figure, marked with capital letters, is an elevation; the lower part, marked with italic letters, is a plan, or horizontal projection; and corresponding letters mark corresponding points. The cutting edges in the plan are marked *a b* and *a c*; *a b e d* and *a c f d* are the projections of the two face-planes; *a d* is the projection of the *front edge;* and *a b g h c* that of the upper plane. In the elevation are shown one of the cutting edges, A B; one of the face-planes, A B E D; and the front edge, A D. Sometimes the front edge is rounded, together with the angles *c a b* and *f d e*; thus connecting the two straight cutting edges by means of a short curved cutting edge. Sometimes the whole cutting edge is a curve; and then, instead of two face-planes, there is a cylindrical front surface.

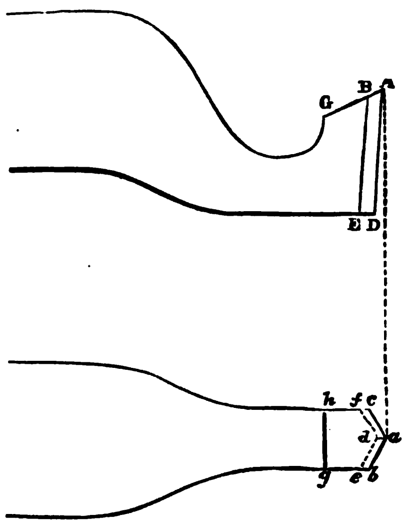

Fig. 286.

The relations amongst the angles made by the planes and edges of such a tool as that shown in fig. 286 may be determined either by spherical trigonometry, or by the rules in descriptive geometry given in this book, Articles 24 to 30, pages 9 to 12. They are treated of also in an essay by Mr. Willis, already mentioned as having been published in the appendix to the second volume of Holtzapffel *On Mechanical Manipulation.*

A tool with one curved or two straight cutting edges may be used to cut a groove, or to pare a layer by successive shavings off the surface of a piece of work. In the latter case the shaft of the tool is to be so formed and held, that one of the straight cutting edges (for example, *a b*), or one side of the curved edge, touches the pared face of the work, and the other (for example, *a c*) cuts into the side of the unpared part of the layer that is being removed; and according as the tool is shaped and placed so as to lie to the right or to the left of the unpared part of the layer, it is called a *right-side* or a *left-side* tool. Thus, in a right-side tool, *a b* touches the pared face; *a c*, the side of the unpared layer; and in a left-side tool, *a c* touches the pared face, and *a b* the side of the unpared layer. A tool with one cutting edge which is parallel to the face of the work, as in fig. 284, or a tool with two cutting edges, as in fig. 286, making equal angles with the face, is called a *face-tool.*

Fig. 287 shows how the principle of having a small angle of relief, with a sufficiently acute cutting angle, is applied to *drills* or *boring bits.* The figure shows a front view, lettered A, B, C, &c.; an edge view, lettered A′, B′, C′, &c.; and an end view, lettered in italics. A A is the axis of rotation; B C, B C, the cutting edges; and the requisite acuteness is given to the cutting angle (marked D′ C′ E′ in the edge view) by means of a *flute* or curved hollow parallel to each of the cutting edges.

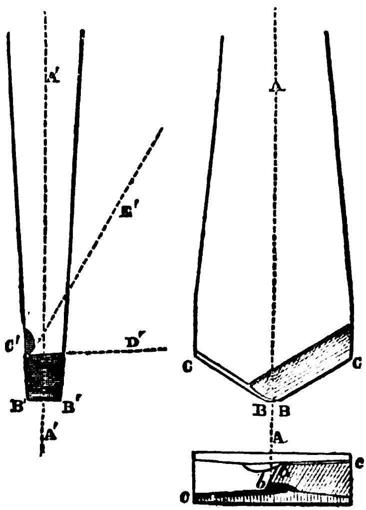

Fig. 287.

481. **Cutting Angles of Tools.**—The best cutting angles for *paring tools* suited to different materials have been ascertained by practical experience. The following are the principal results:—

Materials.	Cutting Angles.
Wood,..................................	from 20° to 45°
Iron and Steel, ...	from 60° to 70°
(The smaller angles for the softer kinds; the greater for the harder.)	
Brass and Bronze,......................	80° and upwards.

In the case of scraping tools, the size of the cutting angle is a question mainly of convenience and strength; for the same tool which is a paring tool, when the working angle is only a little greater than the cutting angle, becomes a scraping tool when the working angle is sufficiently increased.

It may be considered, however, that in order to give sufficient strength to the tool, the *least* cutting angle for a scraping tool should not be less than the cutting angle of a paring tool suited to the same material.

The cutting angle of scraping tools for iron ranges from 60° to nearly 135°; the former value being met with in triangular scrapers for finishing plane surfaces; the latter, in octagonal *broaches* for enlarging and correcting conical holes.

482. **Speed of Cutting Tools.**—The speed of the cutting motion of tools is limited by the heat produced by their action: it must not be so great as to cause that heat to affect the temper of the steel. Hence it is less, the harder the material of the work. The following are examples.

Material.	Speed of Cutting Motion. Feet per Minute.
White Cast Iron, ...	about 5
Steel, and Gray Cast Iron, ...	from 10 to 20
Wrought Iron, ...	from 15 to 25
Brass and Bronze, ...	from 40 to 100
Wood, ...	from 300 to 2000

Higher speeds may be used for planing, and for ordinary turning, where the tool and the cut surface are freely exposed to the air, than for drilling and boring.

483. **Resistance and Work of Paring Tools.**—The following estimate of the resistance to the motion of a tool paring iron, and of the work done, is based on that given by Weisbach:—

Let R denote the resistance to the cutting motion of the tool; b the breadth, and t the thickness of the shaving which it pares off; so that $b\,t$ is the sectional area of the shaving; then

$$\mathrm{P} = f\,b\,t; \quad \text{.........................(1.)}$$

in which f is a co-efficient depending on the hardness and toughness of the material. The value of f for iron is estimated by Weisbach at 50,000 lbs. on the square inch, or 35 kilogrammes on the square millimètre. For steel, it is probably from once-and-half to twice as great. Let l be a given length of shaving; then the work done in paring that length off is

$$\mathrm{P}\,l = f\,l\,b\,t. \quad \text{........................(2.)}$$

Let w be the heaviness of the material; then $w\,l\,b\,t$ is the weight of material pared off; and the work done in that process is evidently equal to *the work of lifting that weight to the height* $\frac{f}{w}$; whose value for iron is

15,000 feet, or
4,570 mètres.

The *counter-efficiency* of the machines in which paring tools are used may be estimated at from 1·3 to 1·5; or say, on an average, 1·4; so that the total work of a machine in paring away a given weight of iron may be estimated as being equal to that of lifting the same weight to a height of

21,000 feet, or
6,400 mètres.

The work done by cutting tools produces heat, which, unless abstracted, tends to injure the tools by raising their temperature. In order to abstract the heat and keep the temperature moderate, the point of the tool, in cutting wrought iron and steel, is moistened with a suitable liquid, such as oil, or a solution of carbonate of soda in water. Pure water should not be used, as it causes iron and steel to rust. Cast iron, brass, and bronze are cut with dry tools.

484. **Combinations of Cutting Tools.**—The boring bit, already mentioned in Article 480, page 566, is in fact a combination of two cutters or paring tools. A combination of several paring tools, working side by side, is seen in the tool sometimes called a *comb*, used in screw-cutting lathes, for cutting several turns of the thread at the same time. Cutters following each other in succession occur in taps and dies, for cutting internal and external screws by hand. A circular cutter, or *rose-cutter*, used in shaping the teeth of wheels, is itself a toothed wheel, each of its teeth being a paring tool. The teeth of a saw form a series of small paring tools or scraping tools, according to their working angles; and by the process of *setting* them—that is, bending them alternately to the right and left—they are made alternately into left-side and right-side tools, so as to cut the two sides of the saw-kerf.

485. **Motions of Machine-Tools in General.**—In most examples of machine-tools, the relative motion of the tool and the work is the resultant of three component motions, usually at right angles to each other, or of two out of those three: the cutting motion, the traversing motion, or transverse feed motion, and the advancing feed motion; the first two taking place parallel to the face of the work, and the third in a direction normal to it.

The *cutting motion* is the most rapid of the three, being that by which the tool acts on the face of the work, leaving a narrow strip or band from which a portion of the material has been pared or scraped away. In many instances, the cutting motion is effected by a motion of the work, the tool remaining fixed, and such is the case especially in turning and screw-cutting lathes, and in almost all planing machines. There are other operations in which the cut is made by a motion of the tool; such as drilling, boring, slotting, and shaping. The speed of the cutting motion has been already mentioned in Article 482, page 567.

The *transverse feed motion* takes place parallel to the face of the work, and at right angles to the cutting motion: it is that motion by which the tool is made to shift its position relatively to the work, so as to make a series of parallel cuts side by side, leaving a series of strips or bands which compose a surface of any required extent. It is sometimes continuous, and sometimes intermittent. The general nature of transverse feed motions has already been described in Article 258, page 293.

The *rate* at which the traversing motion takes place in paring a continuous surface depends on the breadth of the cut; which, in iron, ranges from 0·01 to 0·05 inch (= from 0·25 to 1·25 millimètre). In screw-cutting, the traverse at each revolution is equal to the pitch of the screw.

The *advancing feed-motion* is that by which, after a certain depth of material has been cut away from the face of the work, the tool is advanced so as to cut away an additional depth. This is very often an intermittent motion; and in turning and planing machines it is usually an adjustment, made from time to time by hand. Its extent, at each adjustment, is equal to the depth of the cut; which, in iron, ranges from the smallest appreciable quantity up to 0·75 inch (= 19 millimètres) in ordinary cases.

486. **Making Ruled Surfaces—Planing—Slotting—Shaping.**—A ruled surface is one in which every point is traversed by a straight line lying wholly in the surface. Such a straight line is called a *generating line* of the surface; and the surface may be regarded as generated by the motion of the straight line. A ruled surface may be cut to any required degree of precision, by the successive strokes of a tool, each stroke being made along a straight generating line of the surface.

The class of ruled surfaces comprehends the following different kinds, amongst others:—

I. All *straight surfaces:* that is, surfaces in which the straight generating lines are all parallel to each other. Such surfaces may be either plane or cylindrical; the term cylindrical surfaces embracing not only those whose transverse sections are circular, but those whose transverse sections are curves of any figure, such as the acting surfaces of the teeth of spur-wheels (Article 130, page 120). It has already been stated, in Article 38, page 17, that the bearing surfaces of sliding primary pieces must all be straight.

II. All *conical surfaces:* that is, surfaces in which the straight generating lines all pass through one point. The transverse sections of such a surface may be circular, as in the pitch-cone of a circular bevel wheel (Article 105, page 86); or they may be non-circular curves, as in the acting surfaces of the teeth of such a wheel (Article 144, page 143).

III. *Hyperboloidal or skew-bevel surfaces:* in which the straight generating lines, not intersecting in one point, traverse a series of similar transverse sections of the surface. Such transverse sections may be circular, as in the pitch-surface of a circular skew-bevel wheel (Article 106, page 87); or non-circular, as in the acting surfaces of skew-bevel teeth (Article 145, page 146).

The PLANING MACHINE cuts straight surfaces of all kinds; and is used especially for the cutting of plane surfaces to a certain degree of approximation; that is to say, the *longitudinal straightness* of the surface is perfect; but the transverse straightness is approximate. In a planing machine of ordinary construction, there is a fixed horizontal *bed*, very strongly and stiffly supported: its essential parts being a pair of most accurately made straight, parallel, horizontal guides, of a V-shape in cross-section. In those guides there slides a pair of straight, parallel, horizontal, triangular bars, forming the supports of a stiff and strong horizontal *table*, having in it several *slots* or oblong openings, for the purpose of enabling the *work* to be secured to it. From the two sides of the bed rise a pair of standards, carrying a straight horizontal saddle that bridges across the table. The saddle, by means of straight horizontal guides, supports the tool-holder, which has a transverse traversing motion. The sliding surfaces of the saddle and tool-holder are usually of a dove-tail shape in cross-section.

The cutting action is effected by a longitudinal motion of the table carrying the work: the gearing which communicates that motion ought to be extremely smooth and accurate in its action; such as the rack and helical pinion described at page 289; and the pitch-point of the rack and pinion ought to be as directly as possible below the cutting tool. As to the speed, see Article 482, page 567.

During the return stroke, the tool is lifted clear of the work, and the motion of the rack and pinion is reversed by means of self-acting reversing gear; such, for example, as that mentioned in Article 264, page 299; and the train of wheelwork which produces the return stroke is so proportioned as to give an increased speed of motion to the table, usually about double that of the cutting stroke.

The transverse traversing motion of the tool-holder is intermittent, being made during the return stroke of the table. The combination which directly produces it is usually a transverse horizontal screw driving a nut that is fixed to the tool-holder.

The screw is driven by a suitable train of wheelwork, the first wheel of which is driven by a click, usually of the reversible kind (Article 194, page 207). The extent of traverse after each cutting stroke can be regulated by the help of adjustments of length of stroke (Article 273, page 312), or of change-wheels (Article 271, page 311).

In some large planing machines for very heavy work, the cutting stroke is effected by a longitudinal motion of a strong frame carrying the saddle and the tool-holder; the table with the work being at rest.

For cutting straight surfaces of more or less complex cross-section, and especially for cutting straight grooves and straight rectangular holes, such as key-ways and slots, the SLOTTING-MACHINE is used. In this machine the tool-holder or cutter-bar usually slides vertically in a guiding groove in the *slide-head*, which is carried by a strong overhanging frame. Below the slide-head is a table to which the work is secured, capable of being turned about a vertical axis, and traversed horizontally in two rectangular directions, so as to bring the work into any required position relatively to the cutting-tool.

A SHAPING MACHINE differs from a slotting machine mainly in having a slide-head that is capable of being turned into different positions, so as to cause the tool to make strokes in different directions when required. It is used for cutting ruled surfaces of various kinds. Circular cutters (page 568), driven by suitable shifting trains (Article 228, page 235), are sometimes used in shaping machines.

487. **Scraping Plane Surfaces.**—When the highest practicable degree of accuracy is required in a plane surface, its form may in the first place be given approximately by the planing machine, but it must be finished by the hand-scraper. Scrapers for iron are usually made of very hard steel, with edges of 60°.

When an existing standard plane surface (or *planometer*, as it is sometimes called) is available, it is smeared with a very thin coating of a mixture of red chalk and oil. The new plane, in its approximate condition, is laid face to face on the standard plane, and gently rubbed on it. The prominent places on the new plane pick up the colouring matter, and are marked by it; and thus the workman is guided to the parts that require scraping down. The process is repeated again and again until the new plane fits the standard plane with the required degree of precision.

In the absence of a standard plane, three approximately plane cast-iron plates are made, stiffened at the back by ribs. One pair of those are taken in the first place; and one of them being smeared with a suitable mixture, they are repeatedly rubbed together, so as to mark the prominent places, and *both* are scraped, until they fit each other with a certain degree of accuracy. At this stage of the process, they may be both plane; or both spherical, and of the same radius, one being convex and the other concave. Then the two plates first taken are compared in succession with the third plate, and the operations of rubbing and scraping repeated, with the plates combined by pairs in every possible way, until all three plates accurately fit each other in every combination and position; when they must necessarily be truly plane. This is the process by which standard planes are made; and when a set of three have been made, it is usual to reserve one of them very carefully for testing from time to time the accuracy of the other two, which are employed as standards of planeness and straightness for ordinary use.

488. **Making Surfaces of Revolution — Turning, Drilling, and Boring.**—A turning-lathe usually contains the following principal parts. The *bed*, truly plane and horizontal. The *head-stocks*, or supports for the axis of rotation of the work; one fixed, and the other capable of being shifted longitudinally on the bed to a greater or less distance from the fixed headstock, so as to suit the size of the work. The *saddle*, which slides longitudinally on the bed, carrying the *rest*, which carries the tool-holder. The rest has longitudinal and transverse traversing motions, usually produced by means of screws and nuts, acting on slides with dove-tail-shaped straight bearing surfaces; the position of the tool-holder is adjustable vertically and horizontally.

The longitudinal traversing motion of the saddle is sometimes produced by a pinion driving a rack, like the motion of the table in a planing machine, and sometimes by a strong and very accurately made screw, extending the whole length of the bed; the latter method is used in screw-cutting lathes. Many lathes are provided both with a guide-screw and with a rack-and-pinion motion for traversing, either of which can be used at pleasure. The guide-screw is commonly reserved for screw-cutting, and the rack and pinion used for ordinary purposes.

The moveable headstock carries the *screw-spindle*, which does not rotate, but can be slid back and forward by means of a screw, in order to adjust the position of its point, which forms one of the supports of the work. The fixed or fast headstock carries the *lathe-spindle*, which is a rotating horizontal shaft, driven at a proper speed by means of a suitable belt and pulleys; the speed is capable of adjustment by means of speed-cones, usually of the stepped kind described in Article 171, page 185.

The journals of the lathe-spindle are in most cases made slightly conical, and are tightened in their bearings, when required, by means of screws acting endwise.

The ends of both spindles project inwards from the headstocks: they are capable of being fitted with various contrivances for supporting and holding the work. The screw-spindle usually, and the lathe-spindle sometimes, is fitted with a conical point of steel called a *centre*, the angle at the point ranging from 60° for wood, to 80° or 90° for metal; such points support the work and keep it truly centred on the axis of rotation. The lathe-spindle can also be fitted with *chucks* of different sorts; being discs provided with holes, pins, and other means of holding the work, and causing it to rotate along with the lathe-spindle; or with a *mandril* or cylindrical continuation of the spindle, on which wheels and pulleys, and other pieces of work having eyes in their centres, can be keyed for the purpose of being turned.

A chuck in the form of a large circular disc is called a *face-plate*. Some lathes have face-plates on both spindles; and then the two spindles are driven at the same speed, by means of two pinions on one shaft, gearing with teeth on the rims of the face-plates.

The greatest radius of the work which can be turned in a given lathe is limited by the height of the axis of rotation above the bed; and the lathe is described as a "twelve-inch lathe," a "twenty-four-inch lathe," &c., according to that height.

The tool-holder is adjusted so that the point or cutting part of the tool is exactly in a horizontal plane traversing the axis of rotation. The direction of rotation is such that the surface of the work moves *downwards* at the point of the tool, which accordingly cuts upwards.

The screws and nuts, or the pinions and racks, by which the traversing motions of the tool-holder are produced, are driven from the lathe-spindle through trains containing *change-wheels* (Article 271, page 311); and by means of these the velocity-ratio and directional-relation of the cutting motion and of the traversing motion can be adjusted so as to produce the required resultant or aggregate relative motion. As to the rate of traverse per revolution, see Article 485, page 569.

When the word *traversing* is used without qualification, it generally means that the tool traverses in a direction parallel to the axis of the lathe, so as to turn a cylindrical surface. When the tool is made to move in the direction of a radius perpendicular to the axis, it turns a plane surface; and the process is called *surfacing*. This is very often the means used of making a plane approximately, previous to correcting it by scraping. By combining those two motions, so as to make the tool traverse in a straight line cutting the axis obliquely, a conical surface is turned. When the point of the tool is made to traverse in a circle, one diameter of which coincides with the axis, a spherical surface is turned. A hyperboloidal surface might be turned by making the point of the tool traverse along one of its straight generating lines (see Article 84, page 70; Article 106, page 87).

All the preceding operations are examples of *circular turning*, in which the point of the tool describes, relatively to the work, a circle about the axis, if the traversing motion be neglected, or a helix or spiral of a pitch equal to the traverse per revolution, if this component of the motion be taken into account. In *eccentric turning,* the point is made to describe, relatively to the work, paths of various other kinds, such as eccentric circles, ellipses, epicycloids, and arbitrary curves of various sorts. Such *aggregate paths* are produced, sometimes by epicyclic trains carried by the chuck which holds the work, as in the *eccentric chuck, elliptic chuck,* and *geometric chuck;* sometimes by the action of cams or shaper-plates on the tool-holder. The actions of such combinations have been treated of in Part I., Chapter V., Section IV., pages 261 to 267; and in the Addenda, pages 290, 291.

Drilling and *Boring* may be looked upon as modifications of turning, applied to the making of concave surfaces of revolution, and especially of hollow cylinders. The word boring is often applied to both processes; but when drilling and boring are distinguished from each other, *drilling* means the making of a cylindrical hole by a tool which advances endways, cutting shavings from the flat or conical bottom of the hole (see fig. 287); and *boring,* the enlarging and correcting of a hollow cylindrical surface already made; such as that of a cast-iron steam-engine cylinder.

In *drilling,* the tool or drill usually rotates about a vertical axis; and it is very often driven by a shifting train, carried by a *jib* or train-arm. (As to shifting trains, see Article 228, pages 235 to 238.) This is in order that the position of the drill may be shifted to various parts of the work. The train-arm or jib projects horizontally from a strong hollow standard, containing the vertical shaft that drives the shifting train. The work is supported by a table, which is often made so as to be capable of being turned about a vertical axis, and shifted horizontally on slides in two rectangular directions, in order to bring different points in the work below the drill.

The *feed-motion* is given sometimes by gradually lowering the drill, sometimes by gradually raising the table.

In a *multiple drilling machine* (used for making rows of holes in iron plates) a set of drills are driven from one shaft by means of skew-bevel pinions or other suitable mechanism. The feed motion is given by raising the table. The forms of drilling tools are very various.

In a *boring machine,* the inner surface of a hollow cylinder is pared by means of one or more tools carried by a *cutter-bar* or *cutter-head;* being a cylinder a little smaller than the hollow cylinder to be bored. When the work is a very large cylinder, it is usually fixed; and the rotation and traversing motion are both given to the cutter head.

489. **Screw-Cutting.**—The operation of cutting screws is per-

formed in a lathe; the work rotates, and the tool-holder is made to traverse longitudinally by means of the *guide-screw*, already mentioned at page 572. The nut by means of which the guide-screw drives the saddle is a *clasp-nut* (see Addenda, page 576): which can be thrown into or out of gear with the guide-screw when required. The guide-screw is made with great care and precision. An ordinary value of its pitch is half an inch. The velocity-ratio and directional-relation of the motions of the guide-screw and of the lathe-spindle are adjusted by means of change-wheels to the pitch and direction of the screw to be cut, according to the following principles.

$$\text{I.} \quad \frac{\text{Speed of Rotation of Guide-Screw}}{\text{Speed of Rotation of Lathe-Spindle}} = \frac{\text{Pitch of New Screw}}{\text{Pitch of Guide-Screw}}.$$

II. The direction, right or left-handed, of the new screw, is similar or contrary to that of the guide-screw, according as the directions of rotation of the guide-screw and of the lathe-spindle are similar or contrary.

490. **Wheel-Cutting.**—A wheel-cutting machine, for shaping the teeth of wheels, may be regarded as a special form of the shaping machine, in some cases combined with the turning lathe. The wheel to be cut is fixed on mandrils carried at the end of a rotating spindle, mounted on a head-stock. Sometimes that spindle acts as a lathe-spindle, while the wheel is being turned. When the pitching and tooth-cutting are to be begun, a large worm-wheel, permanently fixed on the spindle, is made to gear with a tangent screw, by means of which it is successively turned through a series of angles, each equal to the pitch-angle; first, for the purpose of pitching the wheel, or marking the pitch-points of the teeth on the pitch-circle, and then for the purpose of changing the position of the wheel after each tooth has been cut, preparatory to cutting the next tooth. The figures of the teeth are given approximately by casting, and finished by cutting.

Each stroke of the cutter is guided so as to take place along a straight line. In spur-wheels that straight line is parallel to the axis; in bevel wheels, it traverses the apex of the pitch-cone; in skew-bevel wheels, it is a generating line of the hyperboloidal surfaces of the teeth. When a single cutter is used, the slide in which it works is guided into the proper positions for the successive strokes by a templet shaped like a tooth or like the space between two teeth. In cutting the teeth of spur-wheels, a rotating circular cutter is used; and the form of the cutting edges of its teeth is the counterpart of that of the space between two teeth.

The cutting of worm-wheels by means of a rotating cutter which is a copy of the screw that is to gear with the wheel, has already been mentioned in Article 157, page 166.

Addendum to Article 263, Page 298.

Clasp-Nut.—In certain machine-tools, the traversing motion of the holder is produced by a screw which rotates about a fixed axis, and dri longitudinally (as described in Article 152, page 157) a nut that fits it tr without pinching it. That nut is made in two halves, capable of be clasped round the screw or unclasped, according as the combination is to thrown into or out of gear. When the combination is about to be thro into gear, either the nut or the screw must, if required, be shifted lon tudinally, so as to bring the threads of the one opposite to the grooves the other.

Addendum to Table V., Page 482.

Strength and Elasticity of Silk and Flax.—

	Silken Thread.	Flaxen Thread.	Flaxen Yarn in Canvas.
Ultimate tenacity—			
In lineal feet of material,	120,000	95,000	from 52,000 to 59,000
In mètres,..................	36,600	29,000	from 15,900 to 18,000
Modulus of elasticity—			
In lineal feet of material,	3,000,000		
In mètres,..................	914,400		

Addendum to Article 465, Page 547; and Article 466, Page 549.

Braced Shaft.—The description given in Article 459, page 539, of a braced connecting-rod, applies also to the general construction of a braced shaft. The object of that construction is to diminish the deflection and the bending stress of a shaft of long span, through the support given to the shaft at the middle of the span by the bracing. The following is the easiest way of computing the dimensions:—The diameter of the shaft, h, is to be calculated so as to bear safely the twisting moment, combined with a bending moment due to half the actual span. Let c be that half-span; and y the length of one of the arms of the bracing cross, which is to be fixed according to convenience. Let $\frac{f}{w}$ be the safe working tension of the bracing rods, in length of themselves; say, 3,000 feet, or 900 mètres. Let d be the proper diameter for a tension-rod. Then the proper ratio for the diameter of each of the four tension-rods to the diameter of the shaft is given by the following formula:—

$$\frac{d}{h} = \sqrt{\left\{\frac{\frac{2wc^2}{fy}}{1 - \frac{2wc^2}{fy}}\right\}}$$

The tension-rods should be tightened by means of screws just sufficiently to prevent any perceptible slackness of the rods which successively come uppermost as the shaft rotates.

Additional Authorities.

Dynamics of Machines:—Moseley's *Mechanics of Engineering*.

Construction of Machines:—D. K. Clark *On the Exhibited Machinery of* 1862 (especially Part III., pages 128 to 210).

Comparative Table of French and British Measures.

	No.	Log.	Log.	No.	
Grains in a gramme,...	15·43235	1·188432	$\bar{2}$·811568	0·064799	Gramme in a grain.
Pounds avoird. in a kilogramme,	2·20462	0·343334	$\bar{1}$·656666	0·453593	Kilog. in a lb. avoirdupois.
Ton in a tonne,....................	0·984206	$\bar{1}$·993086	0·006914	1·01605	Tonnes in a ton.
Feet in a mètre, ...	3·2808693	0·515989	$\bar{1}$·484011	0·30479721	Mètres in a foot.
Inch in a millimètre,............	0·03937043	$\bar{2}$·595170	1·404830	25·39977	Millimètres in an inch.
Mile in a kilomètre, ...	0·621377	$\bar{1}$·793355	0·206645	1·60933	Kilomètre in a mile.
Square feet in a square mètre,..	10·7641	1·031978	$\bar{2}$·968022	0·0929013	Square mètre in a square foot.
Square inch in a square millimètre, ...	0·00155003	$\bar{3}$·190340	2·809660	645·148	Square millim. in a square inch.
Cubic feet in a cubic mètre,.....	35·3156	1·547967	$\bar{2}$·452033	0·0283161	Cubic mètre in a cubic foot.
Foot-pounds in a kilogrammètre,	7·23308	0·859323	$\bar{1}$·140677	0·138254	Kilogrammètre in a foot-pound.
Pounds-to-the-foot in a kilogramme-to-the-mètre, ...	0·671963	$\bar{1}$·827345	0·172655	1·48818	Kilogrammes-to-the-mètre in a pound-to-the-foot.
Pounds-to-the-square-foot in a kilogramme-to-the-square-mètre, ...	0·204813	$\bar{1}$·311356	0·688644	4·88252	Kilogrammes-to-the-square-mètre in a pound-to-the-square-foot.
Pounds-to-the-square-inch in a kilog.-to-the-square-millimètre,........................	1422·31	3·152994	$\bar{4}$·847006	0·000703083	Kilog.-to-the-square-millimètre in a pound-to-the-square-inch.
Pounds-to-the-cubic-foot in a kilogramme-to-the-cubic-mètre, ...	0·062426	$\bar{2}$·795367	1·204633	16·019	Kilogrammes-to-the-cubic-mètre in a pound-to-the-cubic-foot.
Fahrenheit-degrees in a centigrade-degree, ...	1·8	0·255273	$\bar{1}$·744727	0·5555$\dot{5}$	Centigrade-degree in a Fahrenheit degree.
British units of heat in a French unit,..................	3·96832	0·598607	$\bar{1}$·401393	0·251996	French units of heat in a British unit.

	No.	Log.	Log.	No.	
Cubic inch in a cubic millimetre,...............	0·000061025	$\bar{5}$·785511	4·214489	16,387	Cubic millimetres in a cubic inch.
Yards in a metre,.............	1·0936231	0·038868	$\bar{1}$·961132	0·91439180	Metre in a yard.
Square yards in a sq. metre,	1·19601	0·077735	$\bar{1}$·922265	0·836112	Square metre in a square yard.
Cubic yards in a cubic metre,	1·30799	0·116603	$\bar{1}$·883397	0·764534	Cubic metre in a cubic yard.
Sq. miles in a sq. kilometre,	0·386109	$\bar{1}$·586710	0·413290	2·589941	Square kilometre in a sq. mile.
Acres in a hectare,............	2·4711	0·392889	$\bar{1}$·607111	0·4046782	Hectare in an acre.
Mean geographical mile in a kilometre, nearly,......	0·54	$\bar{1}$·73236	0·26764	1·852	Kilometres in a mean geographical mile, nearly.
Gallon in a litre,...............	0·220215	$\bar{1}$·342847	0·657153	4·54102	Litres in a gallon.
£ sterling in a franc, ...	0·039651	$\bar{2}$·598255	1·401745	25·22	Francs in a £ sterling.
Shilling in a franc, ...	0·79302	$\bar{1}$·899285	0·100715	1·261	Franc in a shilling.
Penny in a centime, ...	0·09516	$\bar{2}$·978466	1·021534	10·508	Centimes in a penny.
Horse-power in a force de cheval,....................	0·98633	$\bar{1}$·99402	0·00598	1·01386	Force de cheval in a horse-power.
£ per foot in a franc per metre, ...	0·012086	$\bar{2}$·082266	1·917734	82·74	Francs per metre in a £ per foot.
£ per square foot in a franc per square metre,........	0·0036836	$\bar{3}$·566277	2·433723	271·48	Francs per square metre in a £ per square foot.
£ per cubic foot in a franc per cubic metre,..........	0·00112276	$\bar{3}$·050288	2·949712	890·66	Francs per cubic metre in a £ per cubic foot.
£ per lb. avoirdupois in a franc per kilogramme,...	0·017986	$\bar{2}$·254921	1·745079	55·61	Francs per kilogramme in a £ per lb. avoirdupois.
£ per acre in a franc per hectare,..................	0·016046	$\bar{2}$·205365	1·794635	62·321	Francs per hectare in a £ per acre.
£ per gallon in a franc per litre,........................	0·18006	$\bar{1}$·255408	0·744592	5·5538	Francs per litre in a £ per gallon.

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